A NOTION OF REDUNDANCY FOR INFINITE FRAMES

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ABSTRACT

Bodmann, Casazza and Kutyniok [8] introduced a quantitative notion of redundancy for finite frames - which they called *upper and lower redundancies* - that match better with an intuitive understanding of redundancy for finite frames in a Hilbert space. The objective of this paper is to see how much of this theory generalizes to infinite frames.

***Keywords*—** Frames, Erasures, Linearly Independent Sets, Noise, Redundancy, Redundancy Function, Spanning Sets, Sparse Approximation.

1. INTRODUCTION

The customary notion of redundancy for a finite frame $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}_n$ is to use $\frac{N}{n}$. Many people have felt for a long time that this was not really satisfactory since it assigns *redundancy 2* to each of the following frames (where $\{e_i\}_{i=1}^n$ is an orthonormal basis for $\mathcal{H}_n$):

$$\Phi_1 = \{e_1, e_1, e_2, e_2, \dots, e_n, e_n\};$$

$$\Phi_2 = \{e_1, \dots, e_1, e_2, e_3, \dots, e_n\},$$

where e_1 occurs $(n+1)$-times,

$$\Phi_3 = \{e_1, 0, e_2, 0, \dots, e_n, 0\}.$$

The frame Φ_1 has redundancy 2 and is a disjoint union of two spanning sets and a disjoint union of two linearly independent sets. This description of redundancy is informative. But for Φ_2, the frame is heavily concentrated in one dimension of the space. In particular, this frame is made up of just one spanning set and it requires $(n+1)$ linearly independent sets to represent it. Finally, the frame Φ_3 is made up of one orthonormal basis and a collection of zero vectors. Assigning this frame redundancy 2 is quite misleading. Although zero vectors are important in some areas of frame theory, such as filter bank theory, counting them in redundancy gives no useful information. What is important, is to keep track of the number of zero vectors while not letting them artificially increase redundancy.

The authors were supported by AFOSR F1ATA00183G003: NSF 1008183: DTRA/NSF 1042701

In this paper, we generalize the results of [8] by applying their quantitative notion of redundancy for finite frames in a Hilbert space $\mathcal{H}$, to infinite frames. Most of the results carry over easily, but a few fail in this setting .

Concerning infinite-dimensional Hilbert spaces, much work has been done on the idea of *deficits, excesses and redundancy* [1, 2, 3, 4, 5, 6, 7]. Our approach is slightly different in that we are interested in how many spanning sets or linearly independent sets are in the frame. However, our notion does not capture much information about infinite frames whose frame vectors are not bounded.

1.1. Review of Frames

We start by fixing our terminology while briefly reviewing the basic definitions related to frames. Let $\mathcal{H}$ denote a finite or infinite dimensional Hilbert space. A *frame* for a Hilbert space $\mathcal{H}$ is a family of vectors $\{\phi_i\}_{i\in I}$ (with $|I|$ finite or infinite) for which there exists constants $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

When A is chosen as the largest possible value and B as the smallest for these inequalities to hold, then we call them the *(optimal) frame constants*. If A and B can be chosen as $A = B$, then the frame is called A*-tight*, and if $A = B = 1$ is possible, Φ is a *Parseval frame*. A frame is called *equal-norm*, if there exists some $c > 0$ such that $\|\varphi_i\| = c$ for all $i = 1, \dots, N$, and it is *unit-norm* if $c = 1$.

Apart from providing redundant expansions, frames can also serve as an analysis tool. In fact, they allow the analysis of data by studying the associated *frame coefficients* $(\langle x, \varphi_i\rangle)_{i\in I}$, where the operator T_Φ defined by $T_\Phi : \mathcal{H} \to \ell_2(I)$, $x \mapsto (\langle x, \varphi_i\rangle)_{i\in I}$ is called the *analysis operator*. The adjoint T_Φ^* of the analysis operator is typically referred to as the *synthesis operator* and satisfies $T_\Phi^*((c_i)_{i\in I}) = \sum_{i\in I} c_i\varphi_i$. The main operator associated with a frame, which provides a stable reconstruction process, is the *frame operator*

$$S_\Phi = T_\Phi^* T_\Phi : \mathcal{H} \to \mathcal{H}, \quad x \mapsto \sum_{i\in I} \langle x, \varphi_i\rangle \varphi_i,$$

a positive, self-adjoint invertible operator on $\mathcal{H}$. In the case of a Parseval frame, we have $S_\Phi = \mathrm{Id}_{\mathcal{H}}$. In general, S_Φ allows reconstruction of a signal $x \in \mathcal{H}$ through the reconstruction formula

$$x = \sum_{i \in I} \langle x, S_\Phi^{-1} \varphi_i \rangle \varphi_i. \quad (1)$$

The sequence $(S_\Phi^{-1}\varphi_i)_{i=1}^N$, which can be shown to form a frame itself, is often referred to as the *canonical dual frame*.

For a more extensive introduction to frame theory, we refer the interested reader to the books [14, 17, 13] as well as to the survey papers [15, 16].

2. DEFINING REDUNDANCY

2.1. Definitions

As explained before, we first introduce a local redundancy given in [8], which encodes the concentration of frame vectors around one point. Since the norms of the frame vectors do not matter for concentration, we normalize the given frame and also consider only points on the unit sphere $\mathbb{S} = \{x \in \mathcal{H} : \|x\| = 1\}$ in $\mathcal{H}$. Hence another way to view local redundancy is by considering it as some sort of density function on the sphere. A consequence of normalizing the frame vectors, is that our new set may no longer be Bessel.

Throughout the paper, we let $\langle y \rangle$ denote the span of some $y \in \mathcal{H}$ and $P_{\langle y \rangle}$ the orthogonal projection onto $\langle y \rangle$.

Definition 2.1. *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame for $\mathcal{H}$. For each $x \in \mathbb{S}$, the* redundancy function $\mathcal{R}_\Phi : \mathbb{S} \to \mathbb{R}^+$ *is defined by*

$$\mathcal{R}_\Phi(x) = \sum_{i \in I} \|P_{\langle \varphi_i \rangle}(x)\|^2.$$

We might think about the function $\mathcal{R}_\Phi$ as a redundancy pattern on the sphere, which measures redundancy at each single point. Also notice that this notion is reminiscent of the fusion frame condition [12], here for rank-one projections.

In contrast to [8], this redundancy function may not assume its maximum or minimum on the unit sphere and in general both the max and min of this function could be infinite.

Definition 2.2. *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame for $\mathcal{H}$. Then the* upper redundancy of Φ *is defined by*

$$\mathcal{R}_\Phi^+ = \sup_{x \in \mathbb{S}} \mathcal{R}_\Phi(x)$$

and the lower redundancy of Φ *by*

$$\mathcal{R}_\Phi^- = \inf_{x \in \mathbb{S}} \mathcal{R}_\Phi(x).$$

Moreover, Φ has a uniform redundancy, *if*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+.$$

This notion of redundancy hence equals the upper and lower frame bound of the normalized version of the frame - which could now be infinite.

3. THE CASE OF INFINITE REDUNDANCY

3.1. Main Result

With the previously defined quantitative notion of upper and lower redundancy, we can now check the properties from [8] which hold (and those which do not hold) in the infinite dimensional setting.

Theorem 3.1. *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame for an ∞-dimensional real or complex Hilbert space $\mathcal{H}$ and assume that $\mathcal{R}^+ < \infty$.*

[D1] Generalization. *If Φ is an equal-norm Parseval frame, then*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+.$$

[D2] Nyquist Property. *The following conditions are equivalent:*

(i) *We have $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$.*

(ii) *The normalized version of Φ is tight.*

Also the following conditions are equivalent.

(i') *We have $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = 1$.*

(ii') *Φ is orthogonal.*

[D3] Upper and Lower Redundancy. *We have*

$$0 < \mathcal{R}_\Phi^- \le \mathcal{R}_\Phi^+ < \infty.$$

[D4] Additivity. *For each orthonormal basis $(e_i)_{i=1}^n$,*

$$\mathcal{R}^\pm_{\Phi \cup (e_i)_{i=1}^n} = \mathcal{R}_\Phi^\pm + 1.$$

Moreover, for each frame Φ' in $\mathcal{H}$,

$$\mathcal{R}^-_{\Phi \cup \Phi'} \ge \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^- \quad \textit{and} \quad \mathcal{R}^+_{\Phi \cup \Phi'} \le \mathcal{R}_\Phi^+ + \mathcal{R}_{\Phi'}^+.$$

In particular, if Φ and Φ' have uniform redundancy, then

$$\mathcal{R}^-_{\Phi \cup \Phi'} = \mathcal{R}_\Phi + \mathcal{R}_{\Phi'} = \mathcal{R}^+_{\Phi \cup \Phi'}.$$

[D5] Invariance. *Redundancy is invariant under application of a unitary operator U on $\mathcal{H}$, i.e.,*

$$\mathcal{R}^\pm_{U(\Phi)} = \mathcal{R}^\pm_\Phi,$$

under scaling of the frame vectors, i.e.,

$$\mathcal{R}^\pm_{(c_i\varphi_i)_{i=1}^N} = \mathcal{R}^\pm_\Phi, \quad c_i \textit{ scalars},$$

and under permutations, i.e.,

$$\mathcal{R}^\pm_{(\varphi_{\pi(i)})_{i=1}^N} = \mathcal{R}^\pm_\Phi, \quad \pi \in S_{\{1,\dots,N\}},$$

[D6] Spanning Sets. *In the finite setting, Φ contains $\lfloor \mathcal{R}_\Phi^- \rfloor$ disjoint spanning sets. In the infinite dimensional setting, this property fails as we will show with an example.*

[D7] Linearly Independent Sets. Φ *can be partitioned into* $\lceil \mathcal{R}_\Phi^+ \rceil$ *linearly independent sets.*

Proof. The property [D1] is true because redundancy is the upper and lower frame bounds of the normalized version of the frame.

The first part of [D2] is true by definition and for the second part of [D3], it is well known that a unit norm Parseval frame must be an orthonormal basis.

Property [D4] follows easily from the argument in [8].

Property [D5] is ovbious.

Property [D6] fails as we will see in the next section.

[D7] follows from Theorem 4.2 of [11] which states that: Every Bessel sequence $\{\varphi_i\}_{i\in I}$ with Bessel bound B and $\|\varphi_i\| \geq c$ for all $i \in I$ (in our case $c = 1$), can be decomposed into $\lceil B/c^2 \rceil$ linearly independent sets. □

The redundancy function gives little information near the extreme cases - as was true in the finite dimensional case, as the following example shows.

Example 3.2. We add an example in which the frame is not merely composed of vectors from the unit basis $\{e_1, \ldots, e_n\}$. Letting $0 < \varepsilon < 1$, we choose $\Phi_4 = (\varphi_i)_{i\in I}$ as

$$\varphi_i = \begin{cases} e_1 & : \quad i = 1, \\ \sqrt{1-\varepsilon^2} e_1 + \varepsilon e_i & : \quad i \neq 1 \\ e_i & : \quad i > N. \end{cases}$$

This frame is strongly concentrated around the vector e_1. A direct calculation yields:

$$1 + (N-1)(1-\varepsilon^2) \leq \mathcal{R}_{\Phi_4}^+ < N,$$

and

$$0 < \mathcal{R}_{\Phi_4}^- \leq \varepsilon^2.$$

The frame Φ_3 shows that the new redundancy notion gives little information near the extreme cases: $\mathcal{R}^- \approx 0$ and $\mathcal{R}^+$ large, but becomes increasingly more accurate as $\mathcal{R}^-$ and $\mathcal{R}^+$ become closer to one another. By [D2], the frame Φ_4 is not orthogonal, nor is it tight. [D6] is not applicable for this frame, since $\lfloor \mathcal{R}_{\Phi_4}^- \rfloor = 0$ although there does exist a partition into one spanning set. Now, [D7] implies that this frame can be partitioned into $N-1$ linearly independent sets. Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent. As before, we observe that [D7] is not sharp for large values of $\mathcal{R}^+$. However, these become increasingly accurate as $\mathcal{R}^-$ and $\mathcal{R}^+$ approach each other.

4. INFINITE PARSEVAL FRAMES

It is easy to construct infinite equal norm Parseval frames for which the norms of the vectors are aribitrarily close to one.

Theorem 4.1. *For any* $r \leq 1$*, there is an equal norm Parseval frame* $\{\varphi_i\}_{i\in\mathbb{N}}$ *for* ℓ_2 *with* $\|\varphi_i\|^2 = r$*, for all* $i \in \mathbb{N}$.

Proof. Given the orthonormal basis $\{e^{2\pi int}\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$, let $E \subset [0,1]$ be a measurable set for which $|E| = r$. Then

$$\left\{e^{2\pi int}\chi_E\right\}_{n\in\mathbb{Z}}$$

is a Parseval frame of norm r vectors. □

Since *linear independence* is so weak in the infinite dimensional setting, we will now see that (equal norm Parseval) frames can have some surprising properties. This will affect our work in this area.

Example 4.2. *For every natural number* $k \in \mathbb{N}$ *there is an equal norm Parseval frame for* ℓ_2 *which can be written as* j*-linearly independent and disjoint spanning sets, for all* $j = 1, 2, \ldots, k$.

Proof. It is straightforward to choose families of vectors $\{f_{ij}\}_{i,j=1}^\infty$ satisfying:

(1) The vectors $\{f_{ij}\}_{i,j=1}^\infty$ are linearly independent.

(2) For each $j = 1, 2, \ldots$, we have that span $\{f_{ij}\}_{i=1}^\infty$ is dense in ℓ_2.

It follows that if we apply Gram-Schmidt to $\{f_{ij}\}_{i=1}^\infty$ for $j = 1, 2, \ldots$, we get a sequence of orthonormal bases $\{g_{ij}\}_{i=1}^\infty$ for ℓ_2, with the property that $\{g_{ij}\}_{i,j=1}^\infty$ is a linearly independent set. Fix $k \in \mathbb{N}$ and consider the family:

$$\left\{\frac{1}{\sqrt{k}} f_{ij}\right\}_{i=1,j=1}^{\infty,\ k}.$$

This family clearly has the desired properties. □

Example 4.3. *There is a Parseval frame for* ℓ_2 *which can be written as* j*-linearly independent and disjoint spanning sets, for all* $j = 1, 2, \ldots, \infty$*. (Note that* $j = \infty$ *is included here).*

Proof. We use the family $\{g_{ij}\}_{i,j=1}^\infty$ from Example 4.2 and form the Parseval frame

$$\left\{\frac{1}{2^j} g_{ij}\right\}_{i,j=1}^{\infty}.$$

This is the required family. □

5. MORE ON THE INFINITE VERSION OF PROPERTY [D6]

For infinite dimensional spaces we have a stronger version of linear independence.

Definition 5.1. *A family of vectors* $\{\varphi_i\}_{i=1}^\infty$ *is* ω-**independent** *if whenever*

$$\sum_{i=1}^\infty a_i \varphi_i = 0,$$

it follows that $a_i = 0$*, for all* $i = 1, 2, \ldots$*. If we have this property only for all* $\{a_i\}_{i=1}^\infty \in \ell_2$*, we say the family of vectors is* ℓ_2-**independent**.

Now we give an more direct proof of Corollary 2.4 of [9].

Theorem 5.2. *Let* $\{Pe_i\}_{i=1}^{\infty}$ *be a Parseval frame in* $\mathcal{H}$*. If* $I \subset \mathbb{N}$*, the following are equivalent:*

1. *The family* $\{Pe_i\}_{i\in I}$ *spans* $P(\mathcal{H})$*.*
2. *The family* $\{(I-P)e_i\}_{i\in I^c}$ *is* ℓ_2*-independent.*

Proof. (1) $\Rightarrow$ (2): Assume that $\{(I-P)e_i\}_{i\in I^c}$ is not ℓ_2-independent. Then there exists scalars $\{b_i\}_{i\in I^c} \in \ell_2$ so that

$$\sum_{i\in I^c} b_i(I-P)e_i = 0.$$

It follows that

$$f = \sum_{i\in I^c} b_i e_i = \sum_{i\in I^c} b_i Pe_i \in P(\mathcal{H}).$$

Thus,

$$\langle f, Pe_j\rangle = \langle Pf, e_j\rangle = \sum_{i\in I^c} b_i\langle e_i, e_j\rangle = 0, \text{ if } j \neq i. \text{ i.e. if } j \in I.$$

So $f \perp span\ \{Pe_i\}_{i\in I}$, and this family is not spanning for $P(\mathcal{H})$.

(2) $\Rightarrow$ (1): First assume there is an $f \in P(\mathcal{H})$ so that $f \perp span\ \{Pe_i\}_{i\in I}$. Then, $f = \sum_{i\in I} a_i Pe_i$. Also,

$$\langle f, Pe_i\rangle = \langle Pf, e_i\rangle = \langle f, e_i\rangle = 0, \text{ for all } i \in I.$$

Hence, $f = \sum_{i\in I^c} b_i e_i$, with not all $b_i = 0$ and $\{b_i\}_{i\in I^c} \in \ell_2$. Thus,

$$\sum_{i\in I^c} b_i e_i = f = Pf = \sum_{i\in I^c} b_i Pe_i.$$

i.e.

$$\sum_{i\in I^c} b_i(I-P)e_i = 0.$$

That is, $(I-P)e_i\}_{i\in I^c}$ is not ℓ_2-independent. □

We now have a result from [10], which gives the case where we can partition our frame into ℓ_2-independent spanning sets.

Theorem 5.3. *Let* $\{f_i : i \in \mathbb{N}\}$ *be a Parseval frame for a separable Hilbert space* $\mathcal{H}$*. If for every finite set* $I \subset \mathbb{N}$*,* $\{f_i : i \in I\}$ *is linearly independent and* $\{f_i : i \notin I\}$ *is complete, then for each* $2 \le R < \infty$*, there exists a partition* $\{I_1, \ldots, I_R\}$ *of* $\mathbb{N}$ *such that for each* $1 \le r \le R$*,* $\{f_i : i \in I_r\}$ *is* ℓ^2 *independent and complete.*

6. CONCLUDING REMARKS

In the case of infinite redundancy, it is possible that our upper frame bound is infinity. It is not clear at this time if anything can be concluded in this case.

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