A QUANTITATIVE NOTION OF REDUNDANCY FOR FINITE FRAMES

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ABSTRACT. The objective of this paper is to improve the customary definition of redundancy by providing quantitative measures in its place, which we coin *upper and lower redundancies*, that match better with an intuitive understanding of redundancy for finite frames in a Hilbert space. This motivates a carefully chosen list of desired properties for upper and lower redundancies. The means to achieve these properties is to consider the maximum and minimum of a redundancy function, which is interesting in itself. The redundancy function is defined on the sphere of the Hilbert space and measures the concentration of frame vectors around each point. A complete characterization of functions on the sphere which coincide with a redundancy function for some frame is given. The upper and lower redundancies obtained from this function are shown to satisfy all of the intuitively desirable properties. In addition, the range of values they assume is characterized.

1. INTRODUCTION

The theory of frames is nowadays a very well established field in which redundancy appears both as a mathematical concept and as a methodology for signal processing. Frames ensure, for instance, resilience against noise, quantization errors and erasures in signal transmissions [19]. Recently, the ability of redundant systems to provide sparse representations has been extensively exploited [5]. Hence, it is fair to say that frames – or redundant systems – have become a standard notion in applied mathematics, computer science, and engineering.

Therefore one would expect that the 'redundancy' of a frame, at least in finite dimensions, is a well-explored concept. However, to the best of the authors' knowledge, there does not exist a precise quantitative notion of redundancy other than the number of frame vectors per dimension. From a scholarly point of view, this is a rather unsatisfactory definition. It does not distinguish between the two toy examples of frames $\{e_1, e_1, e_1, e_2\}$ and $\{e_1, e_1, e_2, e_2\}$ in $\mathbb{R}^2$ (e_1 and e_2 being the canonical orthonormal basis vectors), as one might wish, nor does it provide much insight into properties of the frame. Its main advantage is that for unit-norm tight frames it equals the value of the frame bound. Hence, although the idea of redundancy is the crucial property in various applications and thus the foundation of frame theory, a mathematically precise, meaningful definition is missing.

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In this paper, we take a systematic approach to the problem of introducing a quantitative notion of redundancy for finite frames in a Hilbert space $\mathcal{H}$, say, by first establishing a list of desiderata that such a quantity should satisfy. We then propose a definition of a redundancy function on the unit sphere in $\mathcal{H}$, which is shown to satisfy all the postulated conditions, thereby immediately supplying us with a detailed list of properties of a frame that its redundancy reveals.

1.1. **Review of Finite Frames.** We start by fixing our terminology while briefly reviewing the basic definitions related to frames. Let $\mathcal{H}$ denote an n-dimensional real or complex Hilbert space. In this finite-dimensional situation, $\Phi = (\varphi_i)_{i=1}^N$ is called a *frame* for $\mathcal{H}$, if it is a – typically, but not necessarily linearly dependent – spanning set. This definition is equivalent to ask for the existence of constants $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_{i=1}^N |\langle x, \varphi_i\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

When A is chosen as the largest possible value and B as the smallest for these inequalities to hold, then we call them the *(optimal) frame constants*. If A and B can be chosen as $A = B$, then the frame is called A-*tight*, and if $A = B = 1$ is possible, Φ is a *Parseval frame*. A frame is called *equal-norm*, if there exists some $c > 0$ such that $\|\varphi_i\| = c$ for all $i = 1, \ldots, N$, and it is *unit-norm* if $c = 1$.

Apart from providing redundant expansions, frames can also serve as an analysis tool. In fact, they allow the analysis of data by studying the associated *frame coefficients* $(\langle x, \varphi_i\rangle)_{i=1}^N$, where the operator T_Φ defined by $T_\Phi : \mathcal{H} \to \ell_2(\{1, 2, \ldots, N\})$, $x \mapsto (\langle x, \varphi_i\rangle)_{i=1}^N$ is called the *analysis operator*. The adjoint T_Φ^* of the analysis operator is typically referred to as the *synthesis operator* and satisfies $T_\Phi^*((c_i)_{i=1}^N) = \sum_{i=1}^N c_i\varphi_i$. The main operator associated with a frame, which provides a stable reconstruction process, is the *frame operator*

$$S_\Phi = T_\Phi^* T_\Phi : \mathcal{H} \to \mathcal{H}, \quad x \mapsto \sum_{i=1}^N \langle x, \varphi_i\rangle\varphi_i,$$

a positive, self-adjoint invertible operator on $\mathcal{H}$. In the case of a Parseval frame, we have $S_\Phi = \mathrm{Id}_{\mathcal{H}}$. In general, S_Φ allows reconstruction of a signal $x \in \mathcal{H}$ through the reconstruction formula

$$x = \sum_{i=1}^N \langle x, S_\Phi^{-1}\varphi_i\rangle\varphi_i. \tag{1}$$

The sequence $(S_\Phi^{-1}\varphi_i)_{i=1}^N$ which can be shown to form a frame itself, is often referred to as the *canonical dual frame*.

We note that the choice of coefficients in the expansion (1) is generally not the only possible one. If the frame is linearly dependent – which is typical in applications – then there exist infinitely many choices of coefficients $(c_i)_{i=1}^N$ leading to expansions of $x \in \mathcal{H}$ by

$$x = \sum_{i=1}^N c_i\varphi_i. \tag{2}$$

This fact, for instance, ensures resilience to erasures and noise. The particular choice of coefficients displayed in (1) is the smallest in ℓ_2 norm [11], hence contains the least energy. A different paradigm has recently received rapidly increasing attention, namely to choose the coefficient sequence to be sparse in the sense of having only few non-zero entries, thereby allowing data compression while preserving perfect recoverability (see, e.g., [5] and references therein).

For a more extensive introduction to frame theory, we refer the interested reader to the books [12, 20, 11] as well as to the survey papers [17, 18].

1.2. **Problems with the Customary Notion of Redundancy.** The previous subsection illustrated the fact that frame theory is entirely based on the notion of redundancy. So far, the redundancy of a frame $\Phi = (\varphi_i)_{i=1}^N$ for an n-dimensional Hilbert space $\mathcal{H}$ was generally understood as the quotient $\frac{N}{n}$, a customary, but somewhat crude measure as we will illustrate below. If the finite frame is unit-norm and tight, then this quotient coincides precisely with the frame bound, which in this case might indeed serve as a redundancy measure. But let us consider the two frames $\Phi_{1,s}$ and Φ_2 for $\mathcal{H}$ defined by

$$\Phi_{1,s} = \{e_1, \ldots, e_1, e_2, e_3, \ldots, e_n\}, \quad \text{where } e_1 \text{ occurs } s \text{ times}, \tag{3}$$

and

$$\Phi_2 = \{e_1, e_1, e_2, e_2, e_3, e_3, \ldots, e_n, e_n\}, \tag{4}$$

with $\{e_1, \ldots, e_n\}$ being an orthonormal basis for $\mathcal{H}$. If $s = n + 1$, then the crude, customary measure of redundancy coincides for $\Phi_{1,s}$ and Φ_2. However, intuitively the redundancy of $\Phi_{1,s}$ seems to be very localized, whereas the redundancy of Φ_2 seems to be quite uniform. The fact that Φ_2 can be split into two spanning sets, but $\Phi_{1,s}$ cannot, gives further support to this intuition. Also, the frame Φ_2 is robust with respect to any one erasure, whereas $\Phi_{1,s}$ does not have this property. Neither of these facts can be read from the customary redundancy measure, which makes it rather unsatisfactory.

Concerning infinite-dimensional Hilbert spaces, research has already progressed, and we refer to the recent publication [2] (see also [1]). In this paper, the authors provide a meaningful quantitative notion of redundancy which applies to general infinite frames. In their work, redundancy is defined as the reciprocal of a so-called frame measure function, which is a function of certain averages of inner products of frame elements with their corresponding dual frame elements. More recently, in [1], it is shown that ℓ_1-localized frames satisfy several properties intuitively linked to redundancy such as that any frame with redundancy greater than one should contain in it a frame with redundancy arbitrarily close to one, the redundancy of any frame for the whole space should be greater than or equal to one, and that the redundancy of a Riesz basis should be exactly one, were proven for this notion. However, as stated, these notions of redundancy only apply to infinite frames.

1.3. **An Intuition-Driven Approach to Redundancy.** Concluding from the previous subsection, there does not exist a satisfactory notion of redundancy for finite frames, which forces us to first build up intuition on what we expect of such a notion. In order to properly define a quantitative notion of redundancy, we will agree on a list of desiderata that our notion is required to satisfy.

Inspired by the two examples (3) and (4), which our notion of redundancy shall certainly distinguish, we realize that the local concentration of frame vectors should play an essential role. Hence a local notion of redundancy is desirable. Based on this, suitable global redundancy measures should be the minimal and maximal possible local redundancy attained. This philosophy also coincides with the philosophy of density considerations upon which the notion of redundancy for infinite frames is built [2, 1], since lower and upper densities have been studied multiple times giving suitable measures for local concentrations (see [15]). Coming back to (3) and (4), ideally, the upper redundancy of (3) should be s and the lower 1, whereas the upper and lower redundancies of (4) should coincide and equal 2. More generally, if a frame consists of orthonormal basis vectors which are individually repeated several times, then the lower redundancy should be the smallest number of repetitions and the upper redundancy the largest. This should still hold true if the single frame vectors are arbitrarily scaled, since resilience against erasures as one main aspect of redundancy should intuitively be invariant under this operation.

But even more, for general frames, redundancy should give us information, for instance, about orthogonality and tightness of the frame, about the maximal number of spanning sets and the minimal number of linearly independent sets our frame can be divided into, and about robustness with respect to erasures. Ideally, in the case of a unit-norm tight frame, upper and lower redundancy should coincide and equal the customary measure of redundancy, which seems the appropriate description for this very particular class of frames.

1.4. Desiderata.

Summarizing and analyzing the requirements we have discussed, we state the following list of desired properties for an upper redundancy $\mathcal{R}_\Phi^+$ and a lower redundancy $\mathcal{R}_\Phi^-$ of a frame $\Phi = (\varphi_i)_{i=1}^N$ for an n-dimensional real or complex Hilbert space $\mathcal{H}$.

- [D1] *Generalization.* If Φ is an equal-norm Parseval frame, then in this special case the customary notion of redundancy shall be attained, i.e., $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \frac{N}{n}$.
- [D2] *Nyquist Property.* The condition $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$ shall characterize tightness of a normalized version of Φ, thereby supporting the intuition that upper and lower redundancy being different implies 'non-uniformity' of the frame. In particular, $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = 1$ shall be equivalent to orthogonality as the 'limit-case'.
- [D3] *Upper and Lower Redundancy.* Upper and lower redundancy shall be 'naturally' related by $0 < \mathcal{R}_\Phi^- \le \mathcal{R}_\Phi^+ < \infty$.
- [D4] *Additivity.* Upper and lower redundancy shall be subadditive and superadditive, respectively, with respect to unions of frames. They shall be additive provided that the redundancy is uniform, i.e., $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$.
- [D5] *Invariance.* Redundancy shall be invariant under the action of a unitary operator on the frame vectors, under scaling of single frame vectors, as well as under permutation, since intuitively all these actions should have no effect on, for instance, robustness against erasures, which is one property redundancy shall intuitively measure.
- [D6] *Spanning Sets.* The lower redundancy shall measure the maximal number of spanning sets of which the frame consists. This immediately implies that the lower redundancy is a measure for robustness of the frame against erasures in the sense that any set of a particular number of vectors can be deleted yet leave a frame.

[D7] *Linearly Independent Sets.* The upper redundancy shall measure the minimal number of linearly independent sets of which the frame consists.

It is straightforward to verify that for the special type of frames consisting of orthonormal basis vectors, each repeated a certain number of times, the upper and lower redundancies given by the maximal or minimal number of repetitions satisfy these conditions. The challenge is now to extend this definition to all frames in such a way that the properties are preserved.

1.5. **Contribution of this paper.** The contribution of this paper is two-fold. Firstly, we introduce a list of desiderata that a rationally defined quantitative notion of redundancy for a finite frame shall satisfy. Secondly, we propose a notion of upper and lower redundancy which will be shown to, in particular, satisfy all properties advocated in those desiderata.

2. Defining Redundancy and Main Result

2.1. **Definitions.** As explained before, we first introduce a local redundancy, which encodes the concentration of frame vectors around one point. Since the norms of the frame vectors do not matter for concentration, we normalize the given frame and also consider only points on the unit sphere $\mathbb{S} = \{x \in \mathcal{H} : \|x\| = 1\}$ in $\mathcal{H}$. Hence another way to view local redundancy is by considering it as some sort of density function on the sphere.

We now define a notion of local redundancy. For this, we remark that throughout the paper, we let $\langle y \rangle$ denote the span of some $y \in \mathcal{H}$ and $P_{\langle y \rangle}$ the orthogonal projection onto $\langle y \rangle$.

Definition 2.1. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}$. For each $x \in \mathbb{S}$, the* redundancy function $\mathcal{R}_\Phi : \mathbb{S} \to \mathbb{R}^+$ *is defined by*

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2.$$

We might think about the function $\mathcal{R}_\Phi$ as a redundancy pattern on the sphere, which measures redundancy at each single point. Also notice that this notion is reminiscent of the fusion frame condition [6], here for rank-one projections.

The following observation is rather trivial, but useful.

Lemma 2.2. *If $\Phi = (\varphi_i)_{i=1}^N$ is a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}$, then the redundancy function $\mathcal{R}_\Phi$ assumes its maximum and its minimum on the unit sphere in $\mathcal{H}$.*

Proof. By definition, the function $\mathcal{R}_\Phi$ is continuous. Moreover, since the unit sphere in finite dimensions is compact, the function attains its extrema. □

This consideration allows us to define the maximal and minimal value the redundancy function attains as upper and lower redundancy.

Definition 2.3. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a finite-dimensional real or complex Hilbert space $\mathcal{H}$. Then the* upper redundancy of Φ *is defined by*

$$\mathcal{R}_\Phi^+ = \max_{x \in \mathbb{S}} \mathcal{R}_\Phi(x)$$

and the lower redundancy of Φ *by*

$$\mathcal{R}_\Phi^- = \min_{x\in\mathbb{S}} \mathcal{R}_\Phi(x).$$

Moreover, Φ *has a* uniform redundancy*, if*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+.$$

This notion of redundancy hence equals the upper and lower frame bound of the normalized frame. Surprisingly, this notion will be shown to satisfy all desiderata we phrased.

2.2. **Main Result.** With the previously defined quantitative notion of upper and lower redundancy, we can now verify the desired properties from Subsection 1.4 explicitly in the following theorem, whose proof will be given in Section 5.

Theorem 2.4. *Let* $\Phi = (\varphi_i)_{i=1}^N$ *be a frame for an* n*-dimensional real or complex Hilbert space* $\mathcal{H}$*.*

[D1] Generalization. *If* Φ *is an equal-norm Parseval frame, then*

$$\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = \frac{N}{n}.$$

[D2] Nyquist Property. *The following conditions are equivalent:*

(i) *We have* $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+$*.*

(ii) *The normalized version of* Φ *is tight.*

Also the following conditions are equivalent.

(i') *We have* $\mathcal{R}_\Phi^- = \mathcal{R}_\Phi^+ = 1$*.*

(ii') Φ *is orthogonal.*

[D3] Upper and Lower Redundancy. *We have*

$$0 < \mathcal{R}_\Phi^- \leq \mathcal{R}_\Phi^+ < \infty.$$

[D4] Additivity. *For each orthonormal basis* $(e_i)_{i=1}^n$*,*

$$\mathcal{R}^\pm_{\Phi\cup(e_i)_{i=1}^n} = \mathcal{R}_\Phi^\pm + 1.$$

Moreover, for each frame Φ' *in* $\mathcal{H}$*,*

$$\mathcal{R}^-_{\Phi\cup\Phi'} \geq \mathcal{R}^-_\Phi + \mathcal{R}^-_{\Phi'} \quad \textit{and} \quad \mathcal{R}^+_{\Phi\cup\Phi'} \leq \mathcal{R}^+_\Phi + \mathcal{R}^+_{\Phi'}.$$

In particular, if Φ *and* Φ' *have uniform redundancy, then*

$$\mathcal{R}^-_{\Phi\cup\Phi'} = \mathcal{R}_\Phi + \mathcal{R}_{\Phi'} = \mathcal{R}^+_{\Phi\cup\Phi'}.$$

[D5] Invariance. *Redundancy is invariant under application of a unitary operator* U *on* $\mathcal{H}$*, i.e.,*

$$\mathcal{R}^\pm_{U(\Phi)} = \mathcal{R}^\pm_\Phi,$$

under scaling of the frame vectors, i.e.,

$$\mathcal{R}^\pm_{(c_i\varphi_i)_{i=1}^N} = \mathcal{R}^\pm_\Phi, \quad c_i \textit{ scalars},$$

and under permutations, i.e.,

$$\mathcal{R}^\pm_{(\varphi_{\pi(i)})_{i=1}^N} = \mathcal{R}^\pm_\Phi, \quad \pi \in S_{\{1,\ldots,N\}},$$

[D6] Spanning Sets. Φ *contains* $\lfloor \mathcal{R}_\Phi^- \rfloor$ *disjoint spanning sets. In particular, any set of* $\lfloor \mathcal{R}_\Phi^- \rfloor - 1$ *vectors can be deleted yet leave a frame.*

[D7] Linearly Independent Sets. *If* Φ *does not contain any zero vectors, then it can be partitioned into* $\lceil \mathcal{R}_\Phi^+ \rceil$ *linearly independent sets.*

2.3. **Examples.** Let us now analyze the running examples (3) and (4) from Subsection 1.4, as well as introduce and analyze one additional frame. We will show that the upper and lower redundancies precisely equal those values, which we intuitively anticipated a reasonable notion to attain. We will further exploit Theorem 2.4 to derive additional information about the frames.

Example 2.5. $\Phi_{1,s}$ satisfies

$$\mathcal{R}_{\Phi_{1,s}}^- = 1 \quad \text{and} \quad \mathcal{R}_{\Phi_{1,s}}^+ = s.$$

This can be seen as follows. By definition,

$$\mathcal{R}_{\Phi_{1,s}}(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2 = s|\langle x, e_1 \rangle|^2 + \sum_{i=2}^n |\langle x, e_i \rangle|^2.$$

Hence $\mathcal{R}_{\Phi_{1,s}}(e_1) = s$. For $x \neq e_1$, letting $c_i = \langle x, e_i \rangle$,

$$\mathcal{R}_{\Phi_{1,s}}(x) = sc_1^2 + \sum_{i=2}^n c_2^2 = (s-1)c_1^2 + 1 < (s-1) + 1 = s.$$

This implies $\mathcal{R}_{\Phi_{1,s}}^+ = s$. Moreover,

$$\mathcal{R}_{\Phi_{1,s}}(e_2) = 1 \leq \mathcal{R}_{\Phi_{1,s}}(x) \quad \text{for all } x \neq e_2,$$

which implies $\mathcal{R}_{\Phi_{1,s}}^- = 1$.

Exploiting [D2], the frame $\Phi_{1,s}$ is neither orthogonal nor is it tight. [D6] tells us that $\Phi_{1,s}$ can be split into 1 spanning set, which is indeed the maximal number, since, for instance, e_2 occurs one time and is orthogonal to all other elements from the frame. Concluding from [D7], $\Phi_{1,s}$ can be partitioned into s linearly independent sets, which can be chosen as $\{e_1\}$ $s-1$ times and $\{e_1, \ldots, e_n\}$. It is also evident that this is the minimal possible number, since the s vectors e_1 need to be placed into separate linearly independent sets. Naturally, the frame $\Phi_{1,s}$ can be normalized to become a tight frame. The upper and lower redundancies however remain the same, since only the system of normalized vectors is considered.

Example 2.6. Φ_2 possesses a uniform redundancy. More precisely,

$$\mathcal{R}_{\Phi_2}^- = 2 \quad \text{and} \quad \mathcal{R}_{\Phi_2}^+ = 2.$$

This follows from

$$\mathcal{R}_{\Phi_2}(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2 = 2\sum_{i=1}^n |\langle x, e_i \rangle|^2 = 2,$$

and taking the max and the min over the sphere.

Notice that Φ_2 is a 2-tight frame. Hence the uniform redundancy coincides with the customary notion of redundancy as the quotient $(2n)/n = 2$ by [D1]. Further, by [D6] and [D7], Φ_2 can be partitioned into 2 spanning sets and also into 2 linearly independent sets.

Those partitions can here in fact be chosen to be the same, more precisely, can be chosen to be the two orthonormal bases of which Φ_2 is composed.

Example 2.7. We add a third example in which the frame is not merely composed of vectors from the unit basis $\{e_1, \ldots, e_n\}$. Letting $0 < \varepsilon < 1$, we choose $\Phi_3 = (\varphi_i)_{i=1}^N$ as

$$\varphi_i = \begin{cases} e_1 & : \; i = 1, \\ \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i & : \; i \neq 1. \end{cases}$$

This frame is strongly concentrated around the vector e_1. We first observe that

$$\mathcal{R}_{\Phi_3}(e_1) = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(e_1)\|^2 = 1 + \sum_{i=2}^N |\langle e_1, \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i\rangle|^2 = 1 + (N-1)(1-\varepsilon^2).$$

However, this is not the maximum, which is in fact attained at the average point of the frame vectors. But in order to avoid clouding the intuition by technical details, we omit this analysis, and observe that

$$1 + (N-1)(1-\varepsilon^2) \leq \mathcal{R}^+_{\Phi_3} < N.$$

Since

$$\mathcal{R}_{\Phi_3}(e_2) = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(e_2)\|^2 = \sum_{i=2}^N |\langle e_2, \sqrt{1-\varepsilon^2}e_1 + \varepsilon e_i\rangle|^2 = \varepsilon^2,$$

we can conclude similarly, that

$$0 < \mathcal{R}^-_{\Phi_3} \leq \varepsilon^2.$$

The frame Φ_3 shows that the new redundancy notion gives little information near the extreme cases: $\mathcal{R}^- \approx 0$ and $\mathcal{R}^+ \approx N$, but becomes increasingly more accurate as $\mathcal{R}^-$ and $\mathcal{R}^+$ become closer to one another (cf. the discussion after Theorem 4.1). By [D2], the frame Φ_3 is not orthogonal, nor is it tight. [D6] is not applicable for this frame, since $\lfloor \mathcal{R}^-_{\Phi_3} \rfloor = 0$ although there does exist a partition into one spanning set. Hence, [D6] is not sharp, but becomes telling, if $\mathcal{R}^- \geq 2$. Now, [D7] implies that this frame can be partitioned into $N-1$ linearly independent sets. Again, we see that we can do better than this by merely taking the whole frame which happens to be linearly independent. As before, we observe that [D7] is not sharp for large values of $\mathcal{R}^+$. However, these become increasingly accurate as $\mathcal{R}^-$ and $\mathcal{R}^+$ approach N/n (again we refer to the discussion following Theorem 4.1).

3. Characterization of Redundancy Functions

As we just saw, the notion of redundancy is built upon the notion of a redundancy function. Hence to analyze the notion of redundancy more closely, we will investigate characteristic properties of redundancy functions.

Interestingly, the redundancy function itself is a useful object to exploit. Assume we use a frame Φ to encode a vector x into its frame coefficients to prevent data loss if some of the coefficients are erased (lost or impractically delayed). Indeed, any input vector can still be perfectly recovered if the set of frame vectors belonging to coefficients, which are not erased, form a frame Φ', or equivalently, if the associated redundancy function $\mathcal{R}_{\Phi'}$ is strictly positive. However, $\mathcal{R}_{\Phi'}$ contains useful information even if this is not the case: For any input vector x and any residual set Φ', the projection of x onto the orthogonal complement of the zero set of $\mathcal{R}_{\Phi'}$ can be recovered. In practice, the input and the erasures are typically random, and only some information about their distribution is known. To achieve a small distortion of the transmitted vector, it is then desirable to choose the frame in such a way that the input vectors are concentrated near the orthogonal complement of the (random) zero set of $\mathcal{R}_{\Phi'}$. This is a new type of frame design problem arising from the redundancy function, which we will investigate in detail elsewhere.

The first observation we make in this context is that a frame is not uniquely specified by its redundancy function, since we can scale the single frame vectors arbitrarily, yet the frame is still associated with the same normalized frame. Thus, searching for a frame which possesses a predefined redundancy function is in fact searching for the following equivalence class.

Proposition 3.1. *Let $\mathbb{F}$ be the set of frames for a finite-dimensional real or complex Hilbert space $\mathcal{H}$. Then the relation $\sim$ on $\mathbb{F}$ defined by*

$$\Phi \sim \Psi \quad :\Longleftrightarrow \quad \mathcal{R}_\Phi = \mathcal{R}_\Psi$$

is an equivalence relation on $\mathbb{F}$.

Proof. All three conditions, reflexivity, symmetry, and transitivity are immediate by definition of the relation. □

The just introduced equivalence relation can be described in a different way by linking the redundancy function with the associated frame operator. For this, we require the following notion: For a frame $\Phi = (\varphi_i)_{i=1}^N$ in $\mathcal{H}$, we let $\tilde{S}_\Phi$ denote the frame operator with respect to the normalized version of Φ, i.e.,

$$\tilde{S}_\Phi(x) = \sum_{i=1}^{N} P_{\langle\varphi_i\rangle}.$$

Further, we denote the associated quadratic form by

$$\mathcal{Q}_\Phi(x) = \langle \tilde{S}_\Phi x, x\rangle,$$

and note that$\mathcal{Q}_\Phi$ extends $\mathcal{R}_\Phi$ to all $x \in \mathcal{H}$.

We recall that indeed, there is a one-one correspondence between positive (semi-)definite operators and quadratic forms.

Theorem 3.2. [21, Theorem 3.5] *Each positive (semi-)definite, bounded operator A on a real or complex Hilbert space $\mathcal{H}$ is uniquely determined by the associated quadratic form $\mathcal{Q}_A(x) = \langle Ax, x\rangle$, $x \in \mathcal{H}$.*

The essence of the proof is the so-called polarization identity. For real Hilbert spaces, we have

$$\langle Ax, y\rangle = \frac{1}{4}(\mathcal{Q}_A(x+y) - \mathcal{Q}_A(x-y))$$

and for the complex case

$$\langle Ax, y\rangle = \frac{1}{4}(\mathcal{Q}_A(x+y) - \mathcal{Q}_A(x-y)) + \frac{i}{4}(\mathcal{Q}_A(x+iy) - \mathcal{Q}_A(x-iy)).$$

We use this fact to establish when two frames belong to the same equivalence class.

Corollary 3.3. *If Φ, Ψ are two frames for a finite-dimensional Hilbert space $\mathcal{H}$, then the following conditions are equivalent.*

(i) $\mathcal{R}_\Phi = \mathcal{R}_\Psi$ *on* $\mathbb{S}$.
(ii) $\tilde{S}_\Phi = \tilde{S}_\Psi$ *on* $\mathcal{H}$.

Proof. The redundancy function of a frame Υ extends to all $x \in \mathcal{H}$ by the quadratic scaling

$$\mathcal{Q}_\Upsilon(x) = \|x\|^2 \cdot \mathcal{R}_\Upsilon(x/\|x\|),$$

which defines a quadratic form and thus is equal to $\mathcal{Q}_\Upsilon(x) = \langle \tilde{S}_\Upsilon x, x\rangle$. Since the quadratic form $\mathcal{Q}_\Upsilon$ and the operator $\tilde{S}_\Upsilon$ are in one-to-one correspondence by Theorem 3.2, equality of the redundancy function for two frames Φ and Ψ is equivalent to the normalized frame operators being identical. □

It follows from Corollary 3.3 that equivalent frames Φ and Ψ, i.e., $\mathcal{R}_\Phi \sim \mathcal{R}_\Psi$, must have the same number of non-zero frame vectors. That is, the number of non-zero frame vectors is the sum of the eigenvalues of the *equal* frame operators $\tilde{S}_\Phi = \tilde{S}_\Psi$.

Since tight frames are in some sense the most natural generalization of orthonormal bases, one might ask whether each equivalence class contains at least one tight frame. It is easily seen that, for each $N \geq n$, one of the equivalence classes contains all the unit norm tight frames with N vectors plus non-zero multiples of their frame vectors. i.e. $(c_i\Phi_i)_{i=1}^N$, with $c_i \neq 0$. Other classes *may* contain tight frames. For example, the equivalence class of the frame $\Phi_{1,s}$ contains the Parseval frame

$$\{s^{-1/2}e_1, \ldots, s^{-1/2}e_1, e_2, \ldots, e_n\}, \quad \text{where } e_1 \text{ occurs } s \text{ times.}$$

In general, an equivalence class need not contain any tight frames at all. For example, consider the frame Φ_3. This is a unit-norm linearly independent set, whose frame operator is not a multiple of the identity, but the sum of its eigenvalues is n. Now assume towards a contradiction that there exists a tight frame $\Psi = (\psi_i)_{i=1}^m$ in its equivalence class. Then – as just mentioned – we must have that $m = n$, hence it must be an equal-norm orthogonal set. This implies $\mathcal{R}_\Psi = \mathrm{Id}_\mathcal{H} \neq \mathcal{R}_{\Phi_3}$, a contradiction.

Given a positive self-adjoint rank-n operator S on $\mathcal{H}$, we next characterize when it coincides with the frame operator of a normalized frame.

Proposition 3.4. *Let T be a positive, invertible operator on a real or complex Hilbert space $\mathcal{H}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the following conditions are equivalent.*

(i) *There exists a frame $\Phi = (\varphi_i)_{i=1}^N$ with $\varphi_i \neq 0$ for all i such that $T = \tilde{S}_\Phi$.*
(ii) *There exists some $N \in \mathbb{N}$, $N \geq n$ such that*

$$\sum_{i=1}^n \lambda_i = N.$$

Proof. (i) $\Rightarrow$ (ii). By (i), T is the frame operator associated with the normalized version of Φ. Hence the trace of T satisfies

$$\mathrm{tr}[T] = N = \sum_{i=1}^{N} \|\varphi_i/\|\varphi_i\|\|^2 = \sum_{j=1}^{n} \lambda_j \,,$$

and the number of frame vectors has to satisfy $N \geq n$ because Φ is spanning.

(ii) $\Rightarrow$ (i). Let $(e_i)_{i=1}^n$ be the orthonormal eigenbasis of T such that

$$Te_i = \lambda_i e_i, \quad i = 1, \dots, n \,,$$

and assume the sum of the eigenvalues is an integer $N \geq n$. By a result from [13, 9], there exists an equal-norm frame $\Phi = (\varphi_i)_{i=1}^N$ having T as its frame operator, with $\|\varphi_i\| = 1$ for all $i \in \{1, 2, \dots n\}$. This implies $S_\Phi = \tilde{S}_\Phi$, thus $T = \tilde{S}_\Phi$ as required. □

We remark that Proposition 3.4 is constructive, because the inductive proof in [13, 9] provides such a frame.

To prepare the characterization of redundancy functions further, we recall Gleason's theorem.

Theorem 3.5. [14] *Let $\mathcal{H}$ be a Hilbert space of dimension $n \geq 3$, and let $g : \mathbb{S} \to \mathbb{R}_0^+$ be chosen such that*

$$\sum_{i=1}^{n} g(e_n) = 1$$

for any orthonormal basis $\{e_1, e_2, \dots e_n\}$. Then there exists a trace-normalized positive definite operator T such that $\langle Tx, x\rangle = g(x)$ for all $x \in \mathbb{S}^n$.

We need a slight generalization of this theorem.

Corollary 3.6. *Let $\mathcal{H}$ be a Hilbert space of dimension $n \geq 3$, let $N > 0$, and let $g : \mathbb{S} \to \mathbb{R}_0^+$ be chosen such that*

$$\sum_{i=1}^{n} g(e_n) = N$$

for any orthonormal basis $\{e_1, e_2, \dots e_n\}$. Then there exists a positive definite operator T with $\mathrm{tr}[T] = N$ such that $\langle Tx, x\rangle = g(x)$ for all $x \in \mathbb{S}$. Moreover, g is strictly positive if and only if T is invertible.

Proof. The first part is a simple scaling argument. The second part of this corollary follows from the first one because the minimum of g is the smallest eigenvalue of T. □

We are now ready to state and prove the main result of this section which provides a complete characterization of all functions on the sphere which are redundancy functions of an equivalence class of frames.

Theorem 3.7. *Let $f : \mathbb{S} \to \mathbb{R}_0^+$, $\mathcal{H}$ be an n-dimensional real or complex Hilbert space with $n \geq 3$, and let q be the extension of f to $\mathcal{H}$ given by $q(0) = 0$ and $q(x) = \|x\|^2 f(x/\|x\|)$ for any $x \neq 0$. Let ω denote the probability measure on the unit sphere which is invariant under all unitaries. Then the following conditions are equivalent.*

(i) *There exists a frame* Φ *for* $\mathcal{H}$ *such that*

$$f(x) = \mathcal{R}_\Phi(x) \quad \textit{for all } x \in \mathbb{S}.$$

(ii) *The function* f *is strictly positive on* $\mathbb{S}$*, its extension* q *satisfies the parallelogram identity*

$$q(x+y) + q(x-y) = 2(q(x) + q(y)) \quad \textit{for all } x, y \in \mathbb{S}$$

and f *integrates to*

$$\int_{\mathbb{S}} f(x) d\omega(x) = N/n$$

with some integer $N \geq n$.

(iii) *The function* f *is strictly positive on* $\mathbb{S}$ *and there exists an integer* $N \geq n$ *such that for any orthonormal basis* $\{e_1, e_2, \dots e_n\}$,

$$\sum_{i=1}^{n} f(e_i) = N\,.$$

Also the following conditions are equivalent.

(i') *There exists a unit-norm tight frame* Φ *for* $\mathcal{H}$ *such that*

$$f(x) = \mathcal{R}_\Phi(x) \quad \textit{for all } x \in \mathbb{S}.$$

(ii') *There exists some integer* $N \geq n$ *such that*

$$f(x) = N/n \quad \textit{for all } x \in \mathbb{S}.$$

Proof. We first focus on the equivalence of (i) and (ii). For this, notice that, by Corollary 3.3, (i) is equivalent to the existence of a frame Φ for $\mathcal{H}$ such that $f(x) = \|\tilde{S}_\Phi^{1/2} x\|^2$ for all $x \in \mathbb{S}$, where $\tilde{S}_\Phi^{1/2}$ is positive and invertible. Since, by [10], each positive, invertible operator is the frame operator of an equal-norm frame, (i) is equivalent to the existence of a positive, invertible operator S satisfying $f(x) = \|S^{1/2}x\|^2$ for all $x \in \mathbb{S}$. Hence (i) is equivalent to f defining a new norm

$$\|S^{1/2}x\|^2 = f(x) =: |||x|||^2$$

on $\mathcal{H}$ which, by Lemma 3.3, satisfies

$$|||x|||^2 = \langle Sx, x\rangle.$$

Since this defines a new inner product on $\mathcal{H}$ by $(x, y) = \langle Sx, y\rangle$, (i) is equivalent to the fact that the norm defined by f is induced by an inner product. Hence, by the Jordan-von Neumann theorem [16], condition (i) is equivalent to the parallelogram identity. Moreover, the operator associated with the quadratic form q can be written as a sum of orthogonal projection operators if and only if its trace is a positive integer. This amounts to

$$\int_{\mathbb{S}} f(x) d\omega(x) = N/n, \quad \text{for some positive integer } N\,,$$

see, for example, the proof in [3, Proposition 3.2]. Finally we observe that f is strictly positive if and only if $N \geq n$, because otherwise the sum of projections would not be invertible and thus would not yield a frame operator.

To verify the equivalence of (i) and (iii), we note that given a frame Φ we can always remove vanishing vectors from it without changing $\mathcal{R}_\Phi$. Since $\mathcal{R}_\Phi$ extends to the quadratic form of the normalized frame operator $\tilde{S}_\Phi$, its trace satisfies $\mathrm{tr}[\tilde{S}_\Phi] = N$, N being the number of (non-zero) projections summed to obtain $\tilde{S}_\Phi$. This trace can be computed in any orthonormal basis, so

$$\mathrm{tr}[\tilde{S}_\Phi] = \sum_{i=1}^{n} \mathcal{R}_\Phi(e_i) = N.$$

Conversely, we recall that the version of Gleason's theorem stated in Corollary 3.6 yields that any f satisfying the summation condition extends to the quadratic form of an operator T. Moreover, T is invertible because f is assumed to be strictly positive on $\mathbb{S}$, and thus T is the frame operator for some Φ. Again invoking [13], the frame can be assumed to be unit norm, thus $T = \tilde{S}_\Phi = S_\Phi$.

The equivalence of (i') and (ii') follows from Lemma 4.2. □

We remark that the hypothesis $n \geq 3$ is only relevant for the equivalence with (iii), since the proof of these equivalences exploits Gleason's theorem.

We further remark that the previous theorem is in fact constructive, since the Jordan-von Neumann theorem together with the polarization identity can be used to explicitly compute the frame operator $\tilde{S}_\Phi$ associated with a function $f : \mathbb{S} \to \mathbb{R}_0^+$ which extends to a quadratic form. Then, using the technique in [13, 8], an associated unit-norm frame can be explicitly constructed.

It is not clear to us, what the correct equivalent condition would be, if we drop the word 'unit-norm' in (i'). This concerns the question of a characterization of normalized frames which come from tight frames. This in turn is closely related to the still open question of when the frame vectors can be scaled so that a tight frame is generated as well as to the question of which equivalence classes contain a tight frame.

4. Upper and Lower Redundancy

Having studied and characterized redundancy functions, we now focus on the notion of redundancy itself and will provide more insight into it, in addition to Theorem 2.4.

When introducing a new notion, one of the first questions should concern its range. The following result will provide precise information about the range of the upper and lower redundancy, and even characterize when there does exist a frame such that a particular pair of values for upper and lower redundancy can be attained.

To avoid inessential complications, we again exclude the case of frames which contain zero vectors.

Theorem 4.1. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a real or complex Hilbert space $\mathcal{H}$ having dimension $n \geq 2$ and let $\varphi_i \neq 0$ for all $i \in \{1, 2, \ldots, N\}$. The upper and lower redundancies of Φ then satisfy the inequalities*

$$0 < \mathcal{R}_\Phi^- \leq \frac{N}{n} \leq \mathcal{R}_\Phi^+ < N. \tag{5}$$

Moreover, if $\mathcal{R}_\Phi^- = \frac{N}{n}$ or $\mathcal{R}_\Phi^+ = \frac{N}{n}$, then the normalized version of Φ is a tight frame.

Finally, let $n \leq N$, $r_1 \in (0, \frac{N}{n}]$, *and* $r_2 \in [\frac{N}{n}, N)$. *Then the following conditions are equivalent.*

(i) *There exists a frame* $\Phi = (\varphi_i)_{i=1}^N$ *for* $\mathcal{H}$, $n \geq 2$, *such that*

$$\mathcal{R}_\Phi^- = r_1 \quad \textit{and} \quad \mathcal{R}_\Phi^+ = r_2.$$

(ii) *We have*

$$(n-1)r_1 + r_2 \leq N.$$

In particular, for every $r_1 \in (0, \frac{N}{n}]$ *and every* $r_2 \in [\frac{N}{n}, N)$, *we can find unit-norm frames* $\Phi = (\varphi_i)_{i=1}^N$ *and* $\Psi = (\psi_i)_{i=1}^N$ *with*

$$\mathcal{R}_\Phi^- = r_1 \quad \textit{and} \quad \mathcal{R}_\Psi^+ = r_2.$$

Proof. For the proof of (5), we recall from the proof of Theorem 3.7 that N/n is the mean value of f with respect to the probability measure ω on the sphere $\mathbb{S}$, which implies

$$\min_{x \in \mathbb{S}} \mathcal{R}_\Phi(x) \leq \int_{\mathbb{S}} \mathcal{R}_\Phi(x) d\omega(x) = \frac{N}{n} \leq \max_{x \in \mathbb{S}} \mathcal{R}_\Phi(x). \tag{6}$$

Furthermore, since the vectors in Φ span the finite dimensional space $\mathcal{H}$ for each x, there exists some $i \in \{1, \ldots, N\}$ such that $\langle x, \varphi_i \rangle \neq 0$, and hence $\|P_{\langle \varphi_i \rangle}(x)\|^2 > 0$. This yields

$$0 < \mathcal{R}_\Phi^-. \tag{7}$$

Finally, we have

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle \varphi_i \rangle}(x)\|^2 \leq \sum_{i=1}^N \|x\|^2 = N \quad \text{for all } x \in \mathbb{S}. \tag{8}$$

Now assume that we have equality in (8) for some $x \in \mathbb{S}$. This implies that

$$\|P_{\langle \varphi_i \rangle}(x)\|^2 = \langle P_{\langle \varphi_i \rangle}(x), x \rangle = \|x\|^2 \quad \text{for all } i \in \{1, 2, \ldots n\},$$

and thus x is an eigenvector of eigenvalue one for all $P_{\langle \varphi_i \rangle}$. Since each $P_{\langle \varphi_i \rangle}$ is rank one and projects on the span of φ_i, either $x = 0$ or all φ_i are collinear. However, if all φ_i are collinear, they cannot span $\mathcal{H}$ if its dimension is $n \geq 2$. Hence, it follows that

$$\mathcal{R}_\Phi^+ < N. \tag{9}$$

Combining (6), (7), and (9) proves (5).

For the *moreover*-part, we notice that if $\mathcal{R}_\Phi^- = \frac{N}{n}$, then the average of $\mathcal{R}_\Phi$ equals its minimum. This implies $\omega(\{x \in \mathbb{S} : \mathcal{R}_\Phi(x) > N/n\}) = 0$. Now the continuity of $\mathcal{R}_\Phi$ ensures that it is constant.

Next we study the equivalence between (i) and (ii). To prove (i) $\Rightarrow$ (ii), let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for $\mathcal{H}$ which without loss of generality we can assume to be equal-norm. Then assume that the frame operator for $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ has eigenvalues

$$\mathcal{R}_\Phi^- = r_1 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = r_2 = \mathcal{R}_\Phi^+.$$

From this,

$$(n-1)r_1 + r_2 \leq \sum_{j=1}^n \lambda_i = N,$$

hence (ii) follows directly.

For the converse direction, we observe that (ii) implies the existence of real numbers $r_1 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = r_2$ satisfying

$$\sum_{j=1}^{n} \lambda_j = N.$$

By Proposition 3.4, we can find an equal-norm frame $(\varphi_i)_{i=1}^N$ whose frame operator has eigenvalues $\lambda_1, \ldots, \lambda_n$. Since

$$\sum_{i=1}^{N} \|\varphi_i\|^2 = \sum_{j=1}^{n} \lambda_j = N,$$

the frame $(\varphi_i)_{i=1}^N$ is even unit-norm. This proves (i).

It remains to prove the *in particular*-part. On the one hand, given $r_1 \in (0, \frac{N}{n}]$, we choose $r_2 = \frac{N}{n}$. Hence (ii) is satisfied, and by the equivalence of (i) and (ii) there exists a frame $\Phi = (\varphi_i)_{i=1}^N$ for $\mathcal{H}$ so that $\mathcal{R}_\Phi^- = r_1$. If, on the other hand, we are given $r_2 \in [\frac{N}{n}, n)$, then we may choose $r_1 \in (0, \frac{N}{n}]$ small enough such that (i) is satisfied. Again, arguing as before, there then exists a frame $\Phi = (\varphi_i)_{i=1}^N$ for $\mathcal{H}$ so that $\mathcal{R}_\Phi^+ = r_2$.

The proof of the theorem is complete. □

We remark that Theorem 4.1 is constructive, since already Proposition 3.4 – which was employed to show existence of an equal-norm frame in the equivalence of (i) and (ii) – was constructive.

Let us now, for a moment, analyze the previous result in light of the interpretation of $\mathcal{R}_\Phi^-$ provided by [D6] in terms of partitioning Φ into spanning sets and of $\mathcal{R}_\Phi^+$ provided by [D7] in terms of partitioning Φ into linearly independent sets. If the redundancy is not uniform, both partitions might not coincide. However, if the redundancy is uniform, hence the values of $\mathcal{R}_\Phi^-$ and $\mathcal{R}_\Phi^+$ both equal N/n, the partitions suddenly can be chosen to be the same. In fact, [4] shows that in this case we can partition our frame into $\lfloor \frac{N}{n} \rfloor$ linearly independent spanning sets plus a linearly independent set. We remind the reader that Examples 2.5, 2.6, and 2.7 already gave a hint of the fact that [D6] and [D7] become sharper as they approach the value N/n.

In case of an equal-norm frame, the upper and lower redundancy is immediately computed from the frame bounds.

Lemma 4.2. *Let $\Phi = (\varphi_i)_{i=1}^N$ be an equal-norm frame for a Hilbert space $\mathcal{H}$, having frame bounds A and B. Set $c = \|\varphi_i\|^2$ for all $i = 1, \ldots, N$. Then*

$$\mathcal{R}_\Phi^- = \frac{A}{c} \quad \text{and} \quad \mathcal{R}_\Phi^+ = \frac{B}{c}.$$

Proof. By definition,

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^{N} \|P_{\langle \varphi_i \rangle}(x)\|^2 = c^{-1} \sum_{i=1}^{N} |\langle x, \varphi_i \rangle|^2.$$

The claim now follows from the characterization of the frame bounds

$$A = \min_{x\in\mathbb{S}} \sum_{i=1}^N |\langle x, \varphi_i\rangle|^2 \quad \text{and} \quad B = \max_{x\in\mathbb{S}} \sum_{i=1}^N |\langle x, \varphi_i\rangle|^2.$$

□

Another question concerns the change of redundancy once an invertible operator is applied to a frame. This, in particular, relates the upper and lower redundancies of a frame to those of its canonical dual.

Lemma 4.3. *Let $\Phi = (\varphi_i)_{i=1}^N$ be a frame for a real or complex Hilbert space $\mathcal{H}$. For any invertible operator T on $\mathcal{H}$,*

$$\kappa(T)^{-2}\mathcal{R}_\Phi^\pm \le \mathcal{R}_{T(\Phi)}^\pm \le \kappa(T)^2\mathcal{R}_\Phi^\pm\,, \tag{10}$$

where $\kappa(T) = \|T\|\|T^{-1}\|$ denotes the condition number of T.

In particular, if S_Φ denotes the frame operator associated with Φ, and $\widetilde{\Phi}$ denotes the canonical dual frame of Φ, then

$$\kappa(S_\Phi)^{-1}\mathcal{R}_\Phi^\pm \le \mathcal{R}_{\widetilde{\Phi}}^\pm \le \kappa(S_\Phi)\mathcal{R}_\Phi^\pm.$$

Proof. For each $x \in \mathbb{S}$, assuming without loss of generality that $\varphi_i \neq 0$ for all i,

$$\mathcal{R}_{T(\Phi)}(x) = \sum_{i=1}^N \|P_{\langle T(\varphi_i)\rangle}(x)\|^2 \le \sum_{i=1}^N \frac{\|T\|^2|\langle x, \varphi_i\rangle|^2}{\|\|T^{-1}\|^{-1}\varphi_i\|^2} = \|T\|^2\|T^{-1}\|^2\mathcal{R}_\Phi(x).$$

Now maximizing or minimizing over $x \in \mathbb{S}$ implies $\mathcal{R}_{T(\Phi)}^\pm \le \|T\|^2\|T^{-1}\|^2\mathcal{R}_\Phi^\pm$. The lower bound $\|T\|^{-2}\|T^{-1}\|^{-2}\mathcal{R}_\Phi^\pm \le \mathcal{R}_{T(\Phi)}^\pm$ follows similarly.

The *in particular*-part follows by recalling that $\widetilde{\Phi} = S_\Phi^{-\frac{1}{2}}\Phi$, by the identity $\|S_\Phi^{\pm 1/2}\| = \|S_\Phi^{\pm 1}\|^{1/2}$, and by applying (10). □

5. Proof of Theorem 2.4

[D1] If $\Phi = (\varphi_i)_{i=1}^N$ is an equal-norm tight frame for an n-dimensional real or complex Hilbert space $\mathcal{H}$, then

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 = \|\varphi_1\|^{-2}\sum_{i=1}^N |\langle x, \varphi_i\rangle|^2 = \|\varphi_1\|^{-2} = \frac{N}{n}.$$

[D2] (i) ⇔ (ii). This follows immediately from Lemma 4.2.
(i') ⇒ (ii'). Towards a contradiction, assume that Φ is not orthogonal. Without loss of generality, $\varphi_1 \not\perp \langle\varphi_2, \ldots, \varphi_N\rangle$, in particular, $\varphi_1 \neq 0$. Hence, choosing $x = \varphi_1/\|\varphi_1\|$, we obtain

$$\mathcal{R}_\Phi(x) = 1 + \sum_{i=2}^N \|\varphi_1\|^{-2}\|P_{\langle\varphi_i\rangle}(x)\|^2 > 1.$$

Thus $\mathcal{R}_\Phi^+ > 1$, a contradiction to (i').
(ii') $\Rightarrow$ (i'). Let $x \in \mathbb{S}$. Since $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ is an orthonormal basis, we obtain

$$\mathcal{R}_\Phi(x) = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 = \|x\|^2 = 1.$$

This implies

$$\mathcal{R}_\Phi^- = \max_{x\in\mathbb{S}} \mathcal{R}_\Phi(x) = 1 = \min_{x\in\mathbb{S}} \mathcal{R}_\Phi(x) = \mathcal{R}_\Phi^+.$$

[D3] By definition, $\mathcal{R}^- \leq \mathcal{R}^+$. Moreover, since $\Phi = (\varphi_i)_{i=1}^N$ is a frame for $\mathcal{H}$, also $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ is spanning $\mathcal{H}$, where without loss of generality we assume that $\varphi_i \neq 0$ for all i. Hence $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ forms a frame for $\mathcal{H}$, and thus possesses a positive lower frame bound, i.e.,

$$\mathcal{R}_\Phi^- = \inf_{x\in\mathbb{S}} \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 > 0,$$

as well as a finite upper frame bound, i.e.,

$$\mathcal{R}_\Phi^+ = \sup_{x\in\mathbb{S}} \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 < \infty.$$

[D4] The first claim follows from the basic observation that

$$\sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 + \sum_{i=1}^n \|P_{\langle e_i\rangle}(x)\|^2 = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 + 1,$$

which implies

$$\mathcal{R}^\pm_{\Phi\cup(e_i)_{i=1}^n} = \mathcal{R}_\Phi^\pm + 1.$$

Next, let $\Phi' = (\varphi_i')_{i=1}^M$. Then, for each $x \in \mathbb{S}$, we have

$$\mathcal{R}_{\Phi\cup\Phi'}^-(x) = \sum_{i=1}^N \|P_{\langle\varphi_i\rangle}(x)\|^2 + \sum_{i=1}^M \|P_{\langle\varphi_i'\rangle}(x)\|^2.$$

Hence

$$\mathcal{R}_{\Phi\cup\Phi'}^- = \min_{x\in\mathbb{S}} \mathcal{R}_{\Phi\cup\Phi'}^-(x) \geq \min_{x\in\mathbb{S}} \mathcal{R}_\Phi^-(x) + \min_{x\in\mathbb{S}} \mathcal{R}_{\Phi'}^-(x) = \mathcal{R}_\Phi^- + \mathcal{R}_{\Phi'}^-.$$

as well as

$$\mathcal{R}_{\Phi\cup\Phi'}^+ = \max_{x\in\mathbb{S}} \mathcal{R}_{\Phi\cup\Phi'}^+(x) \leq \max_{x\in\mathbb{S}} \mathcal{R}_\Phi^+(x) + \max_{x\in\mathbb{S}} \mathcal{R}_{\Phi'}^+(x) = \mathcal{R}_\Phi^+ + \mathcal{R}_{\Phi'}^+.$$

The *in particular*-part follows immediately from here.

[D5] For each $x \in \mathbb{S}$,

$$\mathcal{R}_{U(\Phi)}(x) = \sum_{i=1}^N \|P_{\langle U(\varphi_i)\rangle}(x)\|^2 = \mathcal{R}_\Phi(U^*(x)).$$

Since $\|U^*(x)\| = 1$, we conclude that $\mathcal{R}^{\pm}_{U(\Phi)} = \mathcal{R}^{\pm}_{\Phi}$.
Invariance under scaling and under permutation of the frame vectors is immediate from the definition of upper and lower redundancies.

[D6] Without loss of generality, we can assume that each frame element φ_i is non-zero. Since $\mathcal{R}^-_\Phi$ is the lower frame bound of the frame $(\varphi_i/\|\varphi_i\|)_{i=1}^N$, it follows from [4] that $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ can be partitioned into $\lfloor \mathcal{R}^-_\Phi \rfloor$ spanning sets. Hence, $(\varphi_i)_{i=1}^n$ can also be partitioned into $\lfloor \mathcal{R}^-_\Phi \rfloor$ spanning sets.
The *in particular*-part follows automatically from here.

[D7] Let S be the frame operator of the frame $(\varphi_i/\|\varphi_i\|)_{i=1}^N$, where we assume that $\varphi_i \neq 0$ for all i. Then $(S^{-1/2}(\varphi_i/\|\varphi_i\|))_{i=1}^N$ is a Parseval frame and

$$\frac{1}{\mathcal{R}^+}\mathrm{Id}_{\mathcal{H}} \le S^{-1} \le \frac{1}{\mathcal{R}^-}\mathrm{Id}_{\mathcal{H}}.$$

Hence, for all $i = 1, 2, \ldots, N$,

$$\left\|S^{-1/2}\left(\varphi_i/\|\varphi_i\|\right)\right\|^2 \ge \frac{1}{\mathcal{R}^+}. \tag{11}$$

Next we check the Rado-Horn condition (see [7]). For this, let $I \subset \{1, 2, \ldots, N\}$, and let P be the orthogonal projection of $(S^{-1/2}(\varphi_i/\|\varphi_i\|))_{i=1}^N$ onto span $(S^{-1/2}(\varphi_i/\|\varphi_i\|))_{i\in I}$. Employing the fact that $(S^{-1/2}(\varphi_i/\|\varphi_i\|))_{i=1}^N$ is Parseval as well as the estimate (11), we obtain

$$\begin{aligned}
\dim\langle S^{-1/2}\left(\varphi_i/\|\varphi_i\|\right) : i \in I\rangle &= \sum_{i=1}^N \left\|P(S^{-1/2}\left(\varphi_i/\|\varphi_i\|\right))\right\|^2 \\
&\ge \sum_{i\in I} \left\|P(S^{-1/2}\left(\varphi_i/\|\varphi_i\|\right))\right\|^2 \\
&= \sum_{i\in I} \left\|S^{-1/2}\left(\varphi_i/\|\varphi_i\|\right)\right\|^2 \\
&\ge \frac{|I|}{\mathcal{R}^+}.
\end{aligned}$$

Summarizing,

$$\frac{|I|}{\dim\langle S^{-1/2}(\varphi_i/\|\varphi_i\|) : i \in I\rangle} \le \mathcal{R}^+. \tag{12}$$

By the Rado-Horn theorem [7], condition (12) implies that $(S^{-1/2}(\varphi_i/\|\varphi_i\|))_{i=1}^N$ can be partitioned into $\lceil \mathcal{R}^+ \rceil$ linearly independent sets. Since $S^{-1/2}$ is an invertible operator, it follows that $(\varphi_i/\|\varphi_i\|)_{i=1}^N$ – and hence also $(\varphi_i)_{i=1}^N$ – can be partitioned into $\lceil \mathcal{R}^+ \rceil$ linearly independent sets.

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