Approximation of the inverse frame operator and applications to Gabor frames.

Peter G. Casazza and Ole Christensen *

June 24, 1998

Abstract

A frame allows every element in a Hilbert space $\mathcal{H}$ to be written as a linear combination of the frame elements, with coefficients called frame coefficients. Calculation of the frame coefficients requires inversion of an operator S on $\mathcal{H}$. We show how the inverse of S can be approximated as close as we want using finite-dimensional linear algebra. In contrast with previous methods, our approximation can be used for any frame. Various consequences for approximation of the frame coefficients or approximation of the solution to a moment problem are discussed. We also apply the results to Gabor frames and frames consisting of translates of a single function.

1 Introduction

A frame $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ has the property that every element $f \in \mathcal{H}$ has a representation as $f = \sum_{i=1}^{\infty} a_i f_i$ for a set of square-summable coefficients $\{a_i\}_{i=1}^{\infty}$. Frame theory gives a canonical choice for $\{a_i\}_{i=1}^{\infty}$, the so-called frame coefficients. From the mathematical point of view this is gratifying, but for applications it is a problem that calculation of the frame

*The first author was supported by NSF grant DMS 970618 and the second author by the Danish Research Council. The second author also wants to thank University of Charlotte, NC, and University of Missouri-Columbia for providing good working conditions.

coefficients require inversion of an operator S on $\mathcal{H}$, which is usually infinite-dimensional.

In the present paper we introduce a new method to approximate the inverse of S using finite subsets of the frame. This means that S^{-1} can be approximated using finite-dimensional methods for any frame $\{f_i\}_{i=1}^{\infty}$, which is very important for applications.

The present work is strongly motivated by the fact that the projection method discussed in [4] does not allow one to approximate the inverse frame operator corresponding to a Gabor frame. We discuss this important observation in section 2, along with discussing basic properties of frames.

The new method is presented in section 3. We show how the inverse frame operator corresponding to any frame can be approximated as close as we want in the strong operator topology, by operators that can be found using only finite-dimensional linear algebra. The speed of convergence is estimated, and some consequences for approximation of the frame coefficients are discussed.

In section 4 we apply the results to a moment problem. Section 5 is devoted to applications to Gabor frames and frames consisting of translates of a single function. We formulate a similar result for wavelets.

2 Preliminaries

In all that follows, $\mathcal{H}$ denotes a separable Hilbert space with the inner product $< \cdot, \cdot >$ linear in the first entry; I denotes a countable index set.

A family of elements $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* if

$$\exists A, B > 0 : \quad A||f||^2 \leq \sum_{i\in I} | < f, f_i > |^2 \leq B||f||^2, \quad \forall f \in \mathcal{H}.$$

The numbers A, B are called frame bounds.

We say that $\{f_i\}_{i\in I}$ is a *Riesz frame* if every subfamily of $\{f_i\}_{i\in I}$ is a frame for its closed linear span, with the same frame bounds A, B for each subfamily. Observe, that if $\{f_i\}_{i\in I}$ is known to be a frame, we only need to check the existence of a common lower bound (which is, however, always the more difficult part).

If $\{f_i\}_{i\in I}$ is a frame, the *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in I} < f, f_i > f_i.$$

The series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded, invertible, and self-adjoint operator. This leads to the *frame decomposition*:

$$f = SS^{-1}f = \sum_{i \in I} < f, S^{-1}f_i > f_i, \quad \forall f \in \mathcal{H}.$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{< f, S^{-1}f_i >\}_{i \in I}$ are called *frame coefficients.* For more general information about frames we refer to [8, 11].

Frames can equally well be considered in finite-dimensional spaces. It is easy to see that every finite collection of elements in $\mathcal{H}$ is a frame for its span. For convenience we index our frames by the natural numbers in the rest of the section. Given a frame $\{f_i\}_{i=1}^{\infty}$, we let $n \in N$ and consider the family $\{f_i\}_{i=1}^{n}$, which is a frame for $\mathcal{H}_n := span\{f_i\}_{i=1}^{n}$ with frame operator

$$S_n : \mathcal{H}_n \to \mathcal{H}_n, \quad S_n f = \sum_{i=1}^{n} < f, f_i > f_i$$

and frame decomposition $f = \sum_{i=1}^{n} < f, {S_n}^{-1} f_i > f_i, \quad f \in \mathcal{H}_n$. It can be shown that the orthogonal projection P_n of $\mathcal{H}$ onto $\mathcal{H}_n$ is given by

$$P_n f = \sum_{i=1}^{n} < f, {S_n}^{-1} f_i > f_i, \quad f \in \mathcal{H}.$$

It is very natural to ask whether

$$(1) \qquad < f, {S_n}^{-1} f_i > \to < f, S^{-1} f_i >, \quad \forall f \in \mathcal{H}, \forall i \in N.$$

Since (1) concerns the limit $n \to \infty$, the question makes sens even thought $< f, {S_n}^{-1} f_i >$ is only defined for $n \geq i$.

The question is important for practical implementations of frames: whereas calculation of $< f, S^{-1} f_i >$ requires inversion of S (which can be difficult when $\mathcal{H}$ is infinite-dimensional) calculation of $< f, {S_n}^{-1} f_i >$ can be done using finite-dimensional linear algebra.

The question above is studied in [4, 6, 1, 2]. In particular it is shown in [6] that the answer is yes if $\{f_i\}_{i=1}^{\infty}$ is a Riesz frame. Unfortunately, the answer is usually no for a Gabor frame, as we show now. We want to discuss this in some detail. The reader, who is mainly interested in a method that works for any frame, can skip the rest of the introduction and continue with the next section.

Remember that a Gabor frame for $L^2(R)$ has the form

$$\{f_{k,l}(x)\}_{k,l\in Z} = \{e^{ikbx}f(x-la)\}_{k,l\in Z},$$

where $a, b > 0$ and $f \in L^2(R)$ are fixed. Note that i denotes the complex unit number here!

Gabor frames where the function f has compact support play a special role. It is well known, cf. [9], that $\{f_{k,l}(x)\}_{k,l\in Z}$ is a frame for $L^2(R)$ if f has support in an interval of length $\frac{2\pi}{b}$ and there exist constants $A, B > 0$ such that $A \leq \sum_{l\in Z} |f(x-la)|^2 \leq B$, a.e.. For a frame $\{f_{k,l}(x)\}_{k,l\in Z}$ of this type we will now give a quick argument showing that (1) is not satisfied unless $\{f_{k,l}(x)\}_{k,l\in Z}$ is a Riesz basis.

Proposition 2.1: *Suppose that $f \in L^2(R)$ has compact support and that $\{f_{k,l}(x)\}_{k,l\in Z}$ is a frame for $L^2(R)$. If (1) is satisfied for an indexing $\{f_i\}_{i=1}^{\infty}$ of the frame elements, then $\{f_{k,l}(x)\}_{k,l\in Z}$ is a Riesz basis.*

Recently, Heil, Ramanathan and Topiwala [10] showed that the functions $\{f_{k,l}(x)\}_{k,l\in Z}$ are linearly independent (meaning that every finite collection of the elements $\{f_{k,l}(x)\}_{k,l\in Z}$ is linearly independent) if $f \neq 0$ has compact support. Proposition 2.1 now follows from Lemma 2.2 below, which is due to Kim and Lim [12]. For the readers convenience we include a short new proof:

Lemma 2.2:*Let $\{f_i\}_{i=1}^{\infty}$ be a frame and suppose that $\{f_i\}_{i=1}^{\infty}$ is linearly independent. If (1) is satisfied, then $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis.*

Proof: Let $n \in N$. The linear independence of $\{f_i\}_{i=1}^{\infty}$ implies that $\{f_i\}_{i=1}^{n}$ is a Riesz basis for $\mathcal{H}_n$. The dual basis is $\{S_n{}^{-1}f_i\}_{i=1}^{n}$, so

$$< f_i, S_n{}^{-1}f_j >= \delta_{i,j}, \quad i, j = 1, 2, ..., n,$$

where $\delta_{i,j} = 1$ whenever $i = j$, and $\delta_{i,j} = 0$ otherwise. By letting $n \to \infty$ and using (1), we obtain that

$$< f_i, S^{-1}f_j >= \delta_{i,j}, \quad \forall i, j \in N,$$

which means that $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis. **Q.E.D.**

Remark: In [10], the authors actually conjecture that $\{f_{k,l}(x)\}_{k,l\in Z}$ is linearly independent whether or not f has compact support. If the conjecture holds, we can remove the assumption about f having compact support from Proposition 2.1.

For a Riesz basis $\{f_i\}_{i=1}^{\infty}$, there exist easier ways to calculate $< f, S^{-1}f_i >$ than to use (1), so Proposition 2.1 is a serious shortcoming for Gabor frames. Furthermore, a frame of translates is automatically linearly independent, so the same trouble appears. For wavelets the question is still open.

In the next section, we introduce a new method for approximation of the inverse frame operator using finite subsets of the frame. In particular we obtain a way of approximation of the frame coefficients which is similar to (1), but which can be used for any frame. We are convinced that the new method will be very useful in many applications where frames appear.

3 Approximation of S^{-1}.

In the whole section we let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B.

Lemma 3.1: *Given $n \in N$, there exists a number $m(n)$ such that*

$$\frac{A}{2}||f||^2 \leq \sum_{i=1}^{n+m(n)} |< f, f_i >|^2, \quad \forall f \in \mathcal{H}_n.$$

Proof: Let $n \in N$. Given $\epsilon > 0$, choose a finite set of elements $g_k \in \mathcal{H}_n$ such that $||g_k|| = 1, \forall k$, and such that the balls

$$B(g_k, \epsilon) := \{f \in \mathcal{H}_n \mid ||f - g_k|| \leq \epsilon\}$$

cover the compact set $\{f \in \mathcal{H}_n \mid ||f|| = 1\}$. Since $A \leq \sum_{i=1}^{\infty} |< g_k, f_i >|^2$ for all k, we can choose $m(n)$ such that

$$A\frac{2}{3} \leq \sum_{i=1}^{n+m(n)} |< g_k, f_i >|^2, \quad \forall k.$$

Now let $f \in \mathcal{H}_n, ||f|| = 1$. Choose k such that $f \in B(g_k, \epsilon)$. By the opposite triangle inequality applied to

$$\{< f, f_i >\}_{i=1}^{n+m(n)} = \{< g_k, f_i > - < g_k - f, f_i >\}_{i=1}^{n+m(n)},$$

we have

$$[\sum_{i=1}^{n+m(n)} | < f, f_i > |^2]^{1/2} \geq [\sum_{i=1}^{n+m(n)} | < g_k, f_i > |^2]^{1/2} - [\sum_{i=1}^{n+m(n)} | < g_k - f, f_i > |^2]^{1/2}$$

$$\geq \sqrt{A\frac{2}{3}} - \sqrt{B}||g_k - f|| \geq \sqrt{A\frac{2}{3}} - \sqrt{B}\epsilon.$$

By choosing ϵ small enough, $\sqrt{A\frac{2}{3}} - \sqrt{B}\epsilon \geq \sqrt{\frac{A}{2}}$, from which the result follows.
Q.E.D.

The next lemma shows that for every frame $\{f_i\}_{i=1}^{\infty}$ we can construct a family of frames " approaching $\{f_i\}_{i=1}^{\infty}$", which have common frame bounds. Remember that the approximation (1) works for every Riesz frame; the lemma below turns out to be the key to an improved method that works for every frame.

Lemma 3.2: *For any $n \in N$, choose $m(n)$ as in Lemma 3.1. $\{P_n f_i\}_{i=1}^{n+m(n)}$ is a frame for $\mathcal{H}_n$ with bounds $\frac{A}{2}, B$. The frame operator is $P_n S_{n+m(n)} : \mathcal{H}_n \to \mathcal{H}_n$, and*

$$||P_n S_{n+m(n)}|| \leq B, \quad ||(P_n S_{n+m(n)})^{-1}|| \leq \frac{2}{A}.$$

Proof: Let $f \in \mathcal{H}_n$. Then

$$\sum_{i=1}^{n+m(n)} | < f, P_n f_i > |^2 = \sum_{i=1}^{n+m(n)} | < f, f_i > |^2 \geq \frac{A}{2}||f||^2.$$

Also,

$$\sum_{i=1}^{n+m(n)} | < f, P_n f_i > |^2 = \sum_{i=1}^{n+m(n)} | < f, f_i > |^2 \leq \sum_{i=1}^{\infty} | < f, f_i > |^2 \leq B||f||^2.$$

So $\{P_n f_i\}_{i=1}^{n+m(n)}$ is a frame for $\mathcal{H}_n$ with the claimed bounds. The frame operator is given by

$$f \longmapsto \sum_{i=1}^{n+m(n)} < f, P_n f_i > P_n f_i = P_n S_{n+m(n)} f, \quad f \in \mathcal{H}_n.$$

The norm estimates now follow from Proposition 3.4 in [3], where it is proved that the norm of a frame operator is at most equal to the upper frame bound, and that the norm of the inverse frame operator is at most equal to the reciproce of the lower frame bound. **Q.E.D.**

We are now ready to prove that S^{-1} can be approximated arbitrarily closely in the strong operator topology using the operators $(P_n S_{n+m(n)})^{-1} P_n : \mathcal{H}_n \to \mathcal{H}_n, \ \ n \in N$. Observe that $(P_n S_{n+m(n)})^{-1} P_n$ can be found using finite-dimensional methods!

Theorem 3.3: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B. For $n \in N$, choose $m(n)$ as in Lemma 3.1. Then*

$$(P_n S_{n+m(n)})^{-1} P_n f \to S^{-1} f \textit{ for } n \to \infty, \quad \forall f \in \mathcal{H}.$$

Proof: Let $f \in \mathcal{H}$ and define

$$\phi_n := S^{-1} f - (P_n S_{n+m(n)})^{-1} P_n f$$

$$= P_n S^{-1} f - (P_n S_{n+m(n)})^{-1} P_n f + (I - P_n) S^{-1} f.$$

Since $(I - P_n) S^{-1} f \to 0$ as $n \to \infty$, it is enough to show that

$$\psi_n := P_n S^{-1} f - (P_n S_{n+m(n)})^{-1} P_n f \to 0.$$

Since $\psi_n \in \mathcal{H}_n$ we can apply the operator $P_n S_{n+m(n)}$ to get

$$\psi_n = (P_n S_{n+m(n)})^{-1} (P_n S_{n+m(n)} P_n S^{-1} f - P_n f).$$

Consequently,

$$||\psi_n|| \leq ||(P_n S_{n+m(n)})^{-1}|| \cdot ||P_n S_{n+m(n)} P_n S^{-1} f - P_n f||$$

$$\leq \frac{2}{A}||S_{n+m(n)}P_nS^{-1}f - f|| \to 0 \text{ for } n \to \infty.$$

Q.E.D.

The proof of Theorem 3.3 gives an estimate for how fast $(P_nS_{n+m(n)})^{-1}P_nf$ converges to $S^{-1}f$:

$$\begin{aligned} & ||S^{-1}f - (P_nS_{n+m(n)})^{-1}P_nf|| = ||\phi_n|| \leq ||\psi_n|| + ||(I-P_n)S^{-1}f|| \\ \leq & \frac{2}{A}(||f - S_{n+m(n)}S^{-1}f|| + ||S_{n+m(n)}(I-P_n)S^{-1}f||) + ||(I-P_n)S^{-1}f|| \\ \leq & \frac{2}{A}||\sum_{i=n+m(n)+1}^{\infty} < S^{-1}f, f_i > f_i|| + (\frac{2B}{A}+1)||(I-P_n)S^{-1}f|| \\ \leq & \frac{2\sqrt{B}}{A}[\sum_{i=n+m(n)+1}^{\infty} |< S^{-1}f, f_i >|^2]^{1/2} + (\frac{2B}{A}+1)||(I-P_n)S^{-1}f||. \end{aligned}$$

This is, however, not good for applications since the estimate involves S^{-1}. The next theorem shows that we can obtain more useful estimates for the speed of convergence by replacing the condition on $m(n)$ by a stronger one. First, we need a lemma. The proof is very similar to the proof of Lemma 3.1, so we omit it.

Lemma 3.4: *Let $\mathcal{H}, \mathcal{L}$ be Hilbert spaces and let $T_k : \mathcal{H} \to \mathcal{L}$, $k \in N$, be a sequence of bounded operators such that $T_kf \to 0$ for $k \to \infty$, $\forall f \in \mathcal{H}$. Given $\epsilon > 0$ and a finite dimensional subspace $\mathcal{K}$ of $\mathcal{H}$, there exists a number k_0 such that for $k \geq k_0$,*

$$||T_kf|| \leq \epsilon||f||, \ \forall f \in \mathcal{K}.$$

Lemma 3.4 is needed in order to ensure that the hypothesis of the theorem below can be satisfied. Consider a fixed $n \in N$ and let $\mathcal{K} := \mathcal{H}_n$. The family of operators

$$T_k : \mathcal{H} \to \ell^2, \ \ T_kf = \{< f, f_i >\}_{i=k}^{\infty}$$

satisfies the condition in Lemma 3.4. Thus, given $\epsilon > 0$, there exists $k_0 \in N$ such that

$$||T_{k_0}f||^2 \leq \sum_{i=k_0}^{\infty} |< f, f_i >|^2 \leq \epsilon||f||^2, \ \forall f \in \mathcal{H}_n.$$

Denote the restriction of an operator T to a subspace $\mathcal{K}$ by $T_{|\mathcal{K}}$.

Theorem 3.5: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B and let $\{\epsilon_n\}_{n=1}^{\infty} \subseteq]0, A[$ be a decreasing sequence of real numbers converging to 0. Given $n \in N$, chose $m(n)$ such that*

$$\sum_{i=n+m(n)+1}^{\infty} |< f, f_i >|^2 \leq \frac{\epsilon_n{}^2}{B}||f||^2, \ \forall f \in \mathcal{H}_n.$$

Consider $S_{n+m(n)}$ as an isomorphism from $\mathcal{H}_n$ onto $\mathcal{K}_n := S_{n+m(n)}\mathcal{H}_n$ and let Q_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}_n$. Then

$$||S^{-1}f - (S_{n+m(n)|\mathcal{H}_n})^{-1}Q_n f|| \leq \frac{1}{A}(||(I - Q_n)f|| + \frac{\epsilon_n}{A - \epsilon_n}||Q_n f||), \quad \forall f \in \mathcal{H}.$$

Proof: By assumption,

$$\begin{aligned} & ||(S - S_{n+m(n)})_{|\mathcal{H}_n}||^2 \\ = \ & sup_{f \in \mathcal{H}_n, ||f||=1} ||(S - S_{n+m(n)})f||^2 \\ = \ & sup_{f \in \mathcal{H}_n, ||f||=1} || \sum_{i=n+m(n)+1}^{\infty} < f, f_i > f_i||^2 \\ \leq \ & sup_{f \in \mathcal{H}_n, ||f||=1} B \sum_{i=n+m(n)+1}^{\infty} |< f, f_i >|^2 \leq \epsilon_n{}^2. \end{aligned}$$

Thus $||(S - S_{n+m(n)})_{|\mathcal{H}_n}|| \leq \epsilon_n$. It follows that for $f \in \mathcal{H}_n$,

$$||S_{n+m(n)}f|| = ||Sf - (Sf - S_{n+m(n)}f)||$$

$$\geq ||Sf|| - ||(S - S_{n+m(n)})f|| \geq (A - \epsilon_n)||f||.$$

Therefore

$$||(S_{n+m(n)|\mathcal{H}_n})^{-1}|| \leq \frac{1}{A - \epsilon_n}.$$

Now, for $f \in \mathcal{H}$ we have

$$\begin{aligned} & ||S^{-1}f - (S_{n+m(n)|\mathcal{H}_n})^{-1}Q_n f|| \\ \leq \ & ||S^{-1}f - S^{-1}Q_n f|| + ||S^{-1}Q_n f - (S_{n+m(n)|\mathcal{H}_n})^{-1}Q_n f|| \end{aligned}$$

$$
\begin{aligned}
&= ||S^{-1}(I-Q_n)f|| + ||S^{-1}(Q_nf - S(S_{n+m(n)|\mathcal{H}_n})^{-1})Q_nf|| \\
&\leq \frac{1}{A}(||(I-Q_n)f|| + ||(S_{n+m(n)} - S)(S_{n+m(n)|\mathcal{H}_n})^{-1}Q_nf||) \\
&\leq \frac{1}{A}(||(I-Q_n)f|| + ||(S - S_{n+m(n)})_{|\mathcal{H}_n}|| \cdot ||(S_{n+m(n)|\mathcal{H}_n})^{-1}|| \cdot ||Q_nf||) \\
&\leq \frac{1}{A}(||(I-Q_n)f|| + \frac{\epsilon_n}{A-\epsilon_n}||Q_nf||).
\end{aligned}
$$

Q.E.D.

The condition in Theorem 3.5 implies that for all $f \in \mathcal{H}_n$ we have

$$\sum_{i=1}^{n+m(n)} |<f, f_i>|^2 \geq (A - \frac{{\epsilon_n}^2}{B})||f||^2.$$

By comparing this to the condition on $m(n)$ in Theorem 3.3, namely

$$\sum_{i=1}^{n+m(n)} |<f, f_i>|^2 \geq \frac{A}{2}, \quad \forall f \in \mathcal{H}_n,$$

we see that as soon as $\epsilon_n \leq \sqrt{\frac{AB}{2}}$, the condition in Theorem 3.5 forces us to chose a bigger value for $m(n)$ than in Theorem 3.3. Thus, in the following we will specify carefully which condition we refer to. We will use both results, partly for the above reason, and partly because the condition in Theorem 3.5 can not be verified using linear algebra as is the case with the condition in Theorem 3.3.

A problem of particular interest is that of approximation of the frame coefficients $<f, S^{-1}f_i>$, $f \in \mathcal{H}$. Theorem 3.3 shows that we can approximate $<f, S^{-1}f_i>$ as close as we want using finite-dimensional methods, since

$$<f, (P_nS_{n+m(n)})^{-1}P_nf_i> \to <f, S^{-1}f_i> \text{ for } n \to \infty, \ \forall f \in \mathcal{H}.$$

Actually, much more is true: the *sequence* of coefficients $\{<f, (P_nS_{n+m(n)})^{-1}P_nf_i>\}_{i=1}^{n+m(n)}$ converges to $\{<f, S^{-1}f_i>\}_{i=1}^{\infty}$ in ℓ^2-sense as $n \to \infty$, i.e.,

$$\sum_{i=1}^{n+m(n)} |<f, (P_nS_{n+m(n)})^{-1}P_nf_i> - <f, S^{-1}f_i>|^2$$

$$+\sum_{i=n+m(n)+1}^{\infty} |< f, S^{-1}f_i >|^2 \to 0 \text{ for } n \to \infty.$$

This is the content of the following Theorem. Observe that the second term above trivially converges to 0 as $n \to \infty$, so we can concentrate on the first term.

Theorem 3.6: *For $n \in N$, choose $m(n)$ as in Lemma 3.1. Then*

$$\sum_{i=1}^{n+m(n)} |< f, (P_n S_{n+m(n)})^{-1} P_n f_i > - < f, S^{-1} f_i >|^2 \to 0 \; for\, n \to \infty, \quad \forall f \in \mathcal{H}.$$

Proof:

$$\begin{aligned} & \sum_{i=1}^{n+m(n)} |< f, (P_n S_{n+m(n)})^{-1} P_n f_i > - < f, S^{-1} f_i >|^2 \\ = & \sum_{i=1}^{n+m(n)} |< (P_n S_{n+m(n)})^{-1} P_n f - S^{-1} f, f_i >|^2 \\ \leq & \; B||S^{-1} f - (P_n S_{n+m(n)})^{-1} P_n f||^2 \to 0 \text{ for } n \to \infty. \end{aligned}$$

Q.E.D.

A similar proof shows that under the assumption in Theorem 3.5, also

$$\sum_{i=1}^{n+m(n)} |< f, (S_{n+m(n)|\mathcal{H}_n})^{-1} Q_n f_i > - < f, S^{-1} f_i >|^2 \to 0 \text{ for } n \to \infty, \quad \forall f \in \mathcal{H}.$$

The fact that the inverse frame operator and the frame coefficients can be approximated arbitrarily closely does not make it a trivial matter to use the results in concrete applications. For big values of n, the dimension of $\mathcal{H}_n$ is large, making it computationally expensive to compute e.g. $(P_n S_{n+m(n)})^{-1}$. Application of our result is simplified drastically in cases where S_n has a special structure that makes the inversion easy. Recently it has been discovered [13] that the frame operator S for a finite discrete Gabor expansion (i.e., Gabor analysis on a finite subset of $\ell^2(Z)$) has a rich mathematical structure which reduces the computational cost in inverting S. In [13] Theorem 8.4.3

Strohmer estimates the number of operations needed. Thus our results has a great potential for application in that case. For a different approach to this special case we refer to [14].

It is not known whether the frame operator for a finite Gabor system in $L^2(R)$ also has a structure that makes inversion easier. It is an interesting open question for future work.

4 Approximation of the solution to a moment problem.

The principle of approximation using finite subsets of the frame can be used in many other contexts, of which we present one here. Let again $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ and let $\{a_i\}_{i=1}^{\infty} \in \ell^2(N)$. We ask whether there exists $f \in \mathcal{H}$ such that

$$< f, f_i >= a_i, \quad \forall i \in N.$$

A problem of this type is called a *moment problem.* For the general theory we refer to [15]. It is easy to find examples where there is no solution (this is for instance the case if there is a linear dependence between some elements in $\{f_i\}_{i=1}^{\infty}$ that is not reflected in $\{a_i\}_{i=1}^{\infty}$) but as shown in [5] there always exists a unique element in $\mathcal{H}$ minimizing $\sum_{i=1}^{\infty} |a_i - < f, f_i > |^2$; this element is $f = \sum_{i=1}^{\infty} a_i S^{-1} f_i$. We call $f = \sum_{i=1}^{\infty} a_i S^{-1} f_i$ the *best approximation solution* to the moment problem.

In light of Theorem 3.3, a natural question is whether

$$\sum_{i=1}^{n+m(n)} a_i (P_n S_{n+m(n)})^{-1} P_n f_i \to \sum_{i=1}^{\infty} a_i S^{-1} f_i, \quad \forall \{a_i\}_{i=1}^{\infty} \in \ell^2(N).$$

The next Theorem shows that the answer is yes. Again, this means that the best approximation solution to the moment problem can be approximated as close as we want using finite-dimensional methods.

Theorem 4.1: *For $n \in N$, choose $m(n)$ as in Lemma 3.1. Then*

$$\sum_{i=1}^{n+m(n)} a_i (P_n S_{n+m(n)})^{-1} P_n f_i \to \sum_{i=1}^{\infty} a_i S^{-1} f_i \text{ for } n \to \infty, \quad \forall \{a_i\}_{i=1}^{\infty} \in \ell^2(N).$$

Proof: Let $\{a_i\}_{i=1}^\infty \in \ell^2(N)$. By Theorem 3.3 applied to $\sum_{i=1}^\infty a_i f_i$,

$$(P_n S_{n+m(n)})^{-1} P_n \sum_{i=1}^\infty a_i f_i \to S^{-1} \sum_{i=1}^\infty a_i f_i = \sum_{i=1}^\infty a_i S^{-1} f_i \text{ for } n \to \infty.$$

Since

$$(P_n S_{n+m(n)})^{-1} P_n \sum_{i=1}^\infty a_i f_i$$

$$= \sum_{i=1}^{n+m(n)} a_i (P_n S_{n+m(n)})^{-1} P_n f_i + \sum_{i=n+m(n)+1}^\infty a_i (P_n S_{n+m(n)})^{-1} P_n f_i,$$

it is enough to show that

$$\sum_{i=n+m(n)+1}^\infty a_i (P_n S_{n+m(n)})^{-1} P_n f_i \to 0 \text{ for } n \to \infty :$$

$$\begin{aligned}
& \Big|\Big| \sum_{i=n+m(n)+1}^\infty a_i (P_n S_{n+m(n)})^{-1} P_n f_i \Big|\Big|^2 \\
= \; & sup_{||f||=1} \Big| < \sum_{i=n+m(n)+1}^\infty a_i (P_n S_{n+m(n)})^{-1} P_n f_i, f > \Big|^2 \\
= \; & sup_{||f||=1} \Big| \sum_{i=n+m(n)+1}^\infty a_i < (P_n S_{n+m(n)})^{-1} P_n f_i, f > \Big|^2 \\
\leq \; & \sum_{i=n+m(n)+1}^\infty |a_i|^2 \cdot sup_{||f||=1} \sum_{i=n+m(n)+1}^\infty | < (P_n S_{n+m(n)})^{-1} P_n f_i, f > |^2 \\
\leq \; & \sum_{i=n+m(n)+1}^\infty |a_i|^2 \cdot sup_{||f||=1} \sum_{i=1}^\infty | < f_i, (P_n S_{n+m(n)})^{-1} P_n f > |^2 \\
\leq \; & B \sum_{i=n+m(n)+1}^\infty |a_i|^2 \cdot sup_{||f||=1} ||(P_n S_{n+m(n)})^{-1} P_n f||^2 \\
\leq \; & \frac{2B}{A} \cdot \sum_{i=n+m(n)+1}^\infty |a_i|^2 \to 0 \text{ for } n \to \infty.
\end{aligned}$$

Q.E.D.

We have the following estimate for the speed of convergence:

$$\begin{aligned} & ||\sum_{i=1}^{\infty} a_i S^{-1} f_i - \sum_{i=1}^{n+m(n)} a_i (P_n S_{n+m(n)})^{-1} P_n f_i|| \\ \leq \; & ||\sum_{i=1}^{\infty} a_i S^{-1} f_i - \sum_{i=1}^{\infty} a_i (P_n S_{n+m(n)})^{-1} P_n f_i|| + ||\sum_{i=n+m(n)+1}^{\infty} a_i (P_n S_{n+m(n)})^{-1} P_n f_i|| \\ \leq \; & ||(S^{-1} - (P_n S_{n+m(n)})^{-1} P_n) \sum_{i=1}^{\infty} a_i f_i|| + \sqrt{\frac{2B}{A} \cdot \sum_{i=n+m(n)+1}^{\infty} |a_i|^2}. \end{aligned}$$

Using the condition from Theorem 3.5 we obtain a result similar to Theorem 4.1 and a corresponding estimate of the speed of convergence. We state the result without proof.

Theorem 4.2:*Let $\epsilon_n, m(n)$ and Q_n be as in Theorem 3.5. Then for all sequences $\{a_i\}_{i=1}^{\infty} \in \ell^2(N)$ we have that*

$$\begin{aligned} & ||\sum_{i=1}^{\infty} a_i S^{-1} f_i - \sum_{i=1}^{n+m(n)} a_i (S_{n+m(n)|\mathcal{H}_n})^{-1} Q_n f_i|| \\ \leq \; & \frac{1}{A}(||(I - Q_n)f|| + \frac{\epsilon_n}{A - \epsilon_n}||Q_n f||) + \sqrt{\frac{B}{A - \epsilon_n} \sum_{i=n+m(n)+1}^{\infty} |a_i|^2}. \end{aligned}$$

5 Examples.

For notational convenience we indexed the frames by the natural numbers in the previous sections. It is clear that the same results can be formulated for any countable index set I. When $\{f_i\}_{i \in I}$ is a frame and $J \subseteq I$ is finite, denote the frame operator for $\{f_i\}_{i \in J}$ by S_J. Let $\mathcal{H}_J := span\{f_i\}_{i \in J}$ and let P_J be the orthogonal projection onto $\mathcal{H}_J$.

Let $\{I_n\}_{n=1}^{\infty}$ be a collection of finite subsets of I such that

$$I_1 \subseteq I_2 \subseteq ...... \subseteq I_n ... \nearrow I.$$

With this notation Theorem 3.5 can be formulated in the following way: Given $n \in N$, chose a finite set J_n containing I_n such that

$$(2) \qquad \sum_{i \notin J_n} |< f, f_i >|^2 \leq \epsilon_n ||f||^2, \; \forall f \in \mathcal{H}_{I_n}.$$

Let Q_{I_n} denote the orthogonal projection of $L^2(R)$ onto $\mathcal{K}_{I_n} := S_{J_n}\mathcal{H}_{I_n}$. Then

$$||S^{-1}f - (S_{J_n|\mathcal{H}_{I_n}})^{-1}Q_{I_n}f|| \leq \frac{1}{A}(||(I-Q_{I_n})f|| + \frac{\epsilon_n}{A-\epsilon_n}||Q_{I_n}f||), \ \forall f \in \mathcal{H}.$$

In the present section we are mainly interested in how to find J_n in concrete situations. The examples we present here involve two types of operators on $L^2(R)$, namely

$$\text{Translation } T_a \text{ with } a \in R: \ (T_a f)(x) = f(x-a), \ f \in L^2(R), \ x \in R$$

and

$$\text{Modulation } E_b, \ b \in R: \ (E_b f)(x) = e^{ibx}f(x), f \in L^2(R), \ x \in R.$$

Theorem 5.1:*Let $f \in L^2(R)$ have compact support and let $\{a_i\}_{i=-\infty}^{\infty}$ be an increasing sequence of real numbers. Suppose that $\{f_i\}_{i=-\infty}^{\infty} := \{T_{a_i}f\}_{i=-\infty}^{\infty}$ is a frame for $\mathcal{H} := \overline{span}\{f_i\}_{i=-\infty}^{\infty}$. Define $I_n := \{-n, -n+1, ..., 0, 1, ..., n\}$. Then there exists a nonnegative integer m independent of n such that*

$$||S^{-1}f - (S_{I_{n+m}|\mathcal{H}_n})^{-1}Q_{I_n}f|| \leq \frac{1}{A}||f - Q_{I_n}f||, \ \forall f \in \mathcal{H}.$$

Proof: By [7], the assumption that $\{f_i\}_{i=-\infty}^{\infty}$ is a frame for $\mathcal{H}$ implies that $\{a_i\}_{i=-\infty}^{\infty}$ is uniformly relatively separated. That is, there exists a finite collection of disjoint index sets $J_k, k = 1, ..., l$, such that $Z = \cup_{k=1}^{l} J_k$ and each set $\{a_i\}_{i \in J_k}$ is δ_k-separated, meaning that

$$\delta_k := \min_{i,j \in J_k, i \neq j}|a_i - a_j| > 0.$$

Now, chose $\delta \in]0, \min\ \delta_j[$. Observe that any interval of length δ contains at most l points $a_i, \ i \in Z$.

By assumption, f has compact support, say, $supp(f) \subseteq [c,d]$. So for $a \in R, supp(T_a f) \subseteq [a+c, a+d]$. Consequently, if $g \in \mathcal{H}_{I_n}$, then $supp(g) \subseteq [a_{-n}+c, a_n+d]$. It follows that $< g, T_a f >= 0$ for all $g \in \mathcal{H}_{I_n}$ if

$$a + c \geq a_n + d \ \text{ or } \ a + d \leq a_{-n} + c,$$

i.e., if

$$a - a_n \geq d - c \ \text{ or } \ a_{-n} - a \geq d - c.$$

Let $[a]$ denote the integer part of the number $a \in R$. An interval of length $d-c$ contains at most $[\frac{d-c}{\delta}]+1$ points from each separated sequence $\{a_i\}_{i\in J_k}$, and thus at most $m := l([\frac{d-c}{\delta}]+1)$ points from $\{a_i\}_{i=-\infty}^{\infty}$. Thus

$$\sum_{i\notin I_{n+m}} | < g, f_i > |^2 = 0, \ \forall g \in \mathcal{H}_{I_n},$$

and now the result follows from the version of Theorem 3.5 that we stated at the beginning of the section.
Q.E.D.

Note that, according to [7], a collection of translates of a single function can not form a frame for $L^2(R)$, so in the theorem above $\mathcal{H}$ is a proper subspace of $L^2(R)$.

Define the Fourier transformation of $f \in L^1(R)$ by

$$\hat{f}(y) = \frac{1}{2\pi}\int f(x)e^{ixy}dx.$$

Note, that i denotes the complex unit number here! As usual we extend the Fourier transformation to an isometry from $L^2(R)$ onto $L^2(R)$.

Our next application of the theorems in section 3 concerns Gabor frames $\{f_{k,l}(x)\}_{k,l\in Z}$, as defined in section 2. Observe that in terms of the translation and modulation operators we have $f_{k,l}(x) = (E_{kb}T_{la}f)(x)$.

Our approach is strongly motivated by Daubechies celebrated paper [9]. For $M \in N$, define two operators Q_M, R_M on $L^2(R)$ by

$$(Q_M g)(x) = \mathbf{1}_{[-M;M]}(x)g(x) \ \text{ and } \ (R_M g)^\wedge(x) = \mathbf{1}_{[-M;M]}(x)\hat{g}(x).$$

On p. 1001 in [9] Daubechies shows that under certain regularity conditions on $f \in L^2(R)$ (see the exact requirements in Theorem 5.2 below), there exists a constant $k(a,b)$ (as the notation indicates, depending only on the values of a, b, which are fixed here) such that for all $M, m \in N$,

$$\sum_{|kb|\geq M+m, l\in Z} | < Q_M g, f_{k,l} > |^2 \leq k(a,b)(1+m^2)^{-2\alpha+1}||g||^2, \ \ \forall g \in L^2(R).$$

The constant $k(a,b)$ is estimated explicitly in [9]. Furthermore, a similar estimate holds with Q_M replaced by R_M and the roles of k,l switched:

$$\sum_{|la|\geq M+m, k\in Z} | < R_M g, f_{k,l} > |^2 \leq k(b,a)(1+m^2)^{-2\alpha+1}||g||^2, \ \ \forall g \in L^2(R).$$

For $n \in N$, we define the finite index set I_n by

$$I_n = \{(k,l) \in Z \times Z|\ |kb| \le n,\ |la| \le n\}.$$

Theorem 5.2: *Let $f \in L^2(R)$ and assume that for constants $C > 0, \alpha > 1/2$ we have*

$$|f(x)| \le C(1+x^2)^{-\alpha},\ \forall x \in R,\ \textit{and}\ |\hat{f}(y)| \le C(1+y^2)^{-\alpha},\ \forall y \in R.$$

Furthermore, assume that $\{f_{k,l}(x)\}_{k,l\in Z}$ is a frame for $L^2(R)$ with bounds A, B and let $\{\epsilon_n\}_{n=1}^{\infty}$ be a decreasing sequence of real numbers converging to 0. Given $n \in N$, chose $M > n$ such that

$$||(I - Q_M)g|| \le \epsilon_n[B(4B+1)]^{-1/2}||g||, \forall g \in \mathcal{H}_{I_n}$$

and

$$||(I - R_M)g|| \le \epsilon_n[B(4B+1)]^{-1/2}||g||,\ \forall g \in \mathcal{H}_{I_n}.$$

Then, chose m such that

$$m > \sqrt{[\frac{{\epsilon_n}^2}{B(4B+1)(k(a,b)+k(b,a))}]^{\frac{1}{-2\alpha+1}} - 1}.$$

Then we have:

i) $\sum_{(k,l)\notin I_{M+m}} |< g, f_{k,l} >|^2 \le \frac{{\epsilon_n}^2}{B}||g||^2,\ \ \forall g \in \mathcal{H}_{I_n}$

and for all $f \in L^2(R)$,

ii) $||S^{-1}f - (S_{I_{M+m}|\mathcal{H}_{I_n}})^{-1}Q_{I_n}f|| \le \frac{1}{A}(||(I - Q_{I_n})f|| + \frac{\epsilon_n}{A-\epsilon_n}||Q_{I_n}f||).$

Proof: Note again that Lemma 3.4 guarante that the choice of M is possible. Now, for $g \in L^2(R)$ and natural numbers M, m, we have

$$\sum_{(k,l)\notin I_{M+m}} |< g, f_{k,l} >|^2$$

$$\le \sum_{|kb|\ge M+m, l\in Z} |< g, f_{k,l} >|^2 + \sum_{|la|\ge M+m, k\in Z} |< g, f_{k,l} >|^2 = (*).$$

In order to estimate the first term in $(*)$, write

$$< g, f_{k,l} >=< (I - Q_M)g, f_{k,l} > + < Q_M g, f_{k,l} > .$$

Then

$$| < g, f_{k,l} > |^2 \leq 2 \cdot | < (I - Q_M)g, f_{k,l} > |^2 + 2 \cdot | < Q_M g, f_{k,l} > |^2.$$

By a similar estimate for the second term in $(*)$, we get

$$\begin{aligned}
(*) \leq\ & 2\sum\nolimits_{|kb| \geq M+m, l \in Z} | < (I - Q_M)g, f_{k,l} > |^2 \\
& +2\sum\nolimits_{|kb| \geq M+m, l \in Z} | < Q_M g, f_{k,l} > |^2 \\
& +2\sum\nolimits_{|la| \geq M+m, k \in Z} | < (I - R_M)g, f_{k,l} > |^2 \\
& +2\sum\nolimits_{|la| \geq M+m, k \in Z} | < R_M g, f_{k,l} > |^2 \\
\leq\ & 2B(||(I - Q_M)g||^2 + ||(I - R_M)g||^2) \\
& +2\sum\nolimits_{|kb| \geq M+m, l \in Z} | < Q_M g, f_{k,l} > |^2 \\
& +2\sum\nolimits_{|la| \geq M+m, k \in Z} | < R_M g, f_{k,l} > |^2 \\
\leq\ & 2B(||(I - Q_M)g||^2 + ||(I - R_M)g||^2) \\
& +[k(a, b) + k(b, a)](1 + m^2)^{-2\alpha+1}||g||^2.
\end{aligned}$$

The choices of M, m in the assumption now implies that

$$\sum\nolimits_{(k,l) \notin I_{M+m}} | < g, f_{k,l} > |^2 \leq \frac{{\epsilon_n}^2}{B} ||g||^2, \quad \forall g \in \mathcal{H}_{I_n}.$$

This proves i). ii) is now a consequence of the version of Theorem 3.5 that we stated at the beginning of the section. **Q.E.D.**

A similar result for wavelets can be proved using the proof of [9], Theorem 3.2. However, a different approach seems more convincing in that case.

Acknowledgements: The authors want to express their gratitude to the two anonymous referees whose remarks greatly improved the presentation of the paper.

References

[1] Casazza, P.G. and Christensen, O.: *Riesz frames and approximation of the frame coefficients.* Appr. Theory and Appl. **14** no.2 (1998).

[2] Casazza, P.G. and Christensen, O.: *Approximation of the frame coefficients using finite-dimensional methods.* J. Elect. Imag. **4** no. 6 (1997), p. 479-483.

[3] Christensen, O.: *Frames and pseudo-inverses.* J. Math. Anal. Appl. **195** (1995) p.401-414.

[4] Christensen, O.: *Frames and the projection method.* Appl. Comp. Harm. Anal. **1** (1993), p.50-53.

[5] Christensen, O.: *Moment problems and stability results for frames with applications to irregular sampling and Gabor frames.* Appl. Comp. Harm. Anal. **3** (1996), p. 82-86.

[6] Christensen, O.: *Frames containing a Riesz basis and approximation of the frame coefficients using finite dimensional methods.* J. Math. Anal. Appl. **199** (1996) p.256-270.

[7] Christensen, O., Baiqiao, D. and Heil, C.: *Density of Gabor frames.* Preprint, 1997.

[8] Daubechies, I.: *Ten lectures on wavelets.* SIAM conf. series in applied math. Boston 1992.

[9] Daubechies, I.: *The wavelet transformation, time-frequency localization and signal analysis.* IEEE Trans. Inform. Theory **36** (1990), p.961- 1005.

[10] Heil, C., Ramanathan, J. and Topiwala, P.: *Linear independence of time-frequency translates.* Proc. Amer. Math. Soc. **124** (1996), p. 2787-2795.

[11] Heil, C. and Walnut, D.: *Continuous and discrete wavelet transforms.* SIAM Review **31** (1989), p.628-666.

[12] Kim, H.O. and Lim, J.K.: *New characterizations of Riesz bases.* Appl. Comp. Harm. Anal. **4** (1997), p. 222-229.

[13] Strohmer, T.: *Numerical algorithms for discrete Gabor expansions.* In "Gabor analysis: theory and applications". Eds. Feichtinger, H.G. and Strohmer, T.. Birkhäuser, 1997.

[14] Strohmer, T.: *Rates of convergence for the approximation of dual shift-invariant systems.* Preprint, 1998.

[15] Young, R.: *An introduction to nonharmonic Fourier series.* Academic Press, New York, 1980.

Peter G. Casazza
Department of Mathematics
University of Missouri
Columbia
Mo 65211
USA
E-mail: pete@casazza.math.missouri.edu

Ole Christensen
Department of Mathematics
University of Missouri
Columbia
Mo 65211
USA
E-mail: ole@newton.math.missouri.edu