THE BOURGAIN-TZAFRIRI CONJECTURE FOR OPERATORS WITH SMALL COEFFICIENTS

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ABSTRACT. It is known that the Kadison-Singer Problem and the Paving Conjecture are equivalent to the Bourgain-Tzafriri Conjecture. In 1989, Bourgain and Tzafriri showed that the class of zero diagonal matrices with small entries (on the order of $\leq 1/log^{1+\epsilon}n$, for an n-dimensional Hilbert space) are *pavable*. It has always been assumed that this result also holds for the BT-Conjecture - although no one formally checked it. We will show that this is not the case. We will show that if the BT-Conjecture is true for vectors with small coefficients (on the order of $\leq C/\sqrt{n}$) then the BT-Conjecture is true and hence KS and PC are true.

1. INTRODUCTION

It is now known that the 1959 Kadison-Singer Problem is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering [5, 6]. In 1979, Anderson [1] showed that the Kadison-Singer Problem is equivalent to the *Paving Conjecture.*

Paving Conjecture (PC). For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a *paving*) $\{A_j\}_{j=1}^r$ of $\{1, \ldots, n\}$, such that

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\| \quad \text{for all } j = 1, 2, \ldots, r,$$

where Q_{A_j} is the natural projection onto the A_j-coordinates of a vector.

Operators satisfying the Paving Conjecture are called **pavable operators**.

In 1989, Bourgain and Tzafriri proved one of the most celebrated theorems in analysis: The *Bourgain-Tzafriri Restricted Invertibility Theorem* [2]. This gave rise to a major open problem in analysis.

Bourgain-Tzafriri Conjecture (BT). There is a universal constant $A > 0$ so that for every $B > 1$ there is a natural number $r = r(B)$ satisfying: For any natural number n, if $T : \ell_2^n \to \ell_2^n$ is a linear operator with $\|T\| \leq B$ and $\|Te_i\| = 1$ for all $i = 1, 2, \ldots, n$, then there is a partition $\{A_j\}_{j=1}^r$ of

Date: July 31, 2009.
The author was supported by NSF DMS 0704216.

$\{1,2,\ldots,n\}$ so that for all $j=1,2,\ldots,r$ and all choices of scalars $\{a_i\}_{i\in A_j}$ we have:

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \geq A \sum_{i\in A_j} |a_i|^2.$$

It was shown in [5] that BT is equivalent to the Paving Conjecture. Bourgain and Tzafriri [2, 3] also showed that the Paving Conjecture has a positive solution for the class of zero diagonal matrix operators $A=(a_{ij})_{i,j=1}^n$ on $\mathbb{H}_n$ with small coefficients. In particular if the coefficients satisfy for some $\epsilon>0$,

$$|a_{ij}| \leq \frac{C}{log^{1+\epsilon} n}.$$

It has always been assumed that the corresponding result holds for BT if

$$|Te_i(j)| \leq \frac{C}{log^{1+\epsilon} n}, \quad \text{for all } i,j=1,2,\ldots,n.$$

We will show that this is not the case. To state our theorem, we need a definition.

Definition 1.1. *A family of vectors* $\{f_i\}_{i=1}^M$ *for an* N*-dimensional Hilbert space* $\mathcal{H}_N$ *is* (δ,r)-**Rieszable** *if there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1,2,\ldots,M\}$ *so that for all* $j=1,2,\ldots,r$ *and all scalars* $\{a_i\}_{i\in A_j}$ *we have*

$$\|\sum_{i\in A_j} a_i f_i\|^2 \geq \delta \sum_{i\in A_j} |a_i|^2.$$

A projection P *on* $\mathcal{H}_N$ *is* (δ,r)-**Rieszable** *if* $\{Pe_i\}_{i=1}^N$ *is* (δ,r)*-Rieszable.*

We are ready for the main theorem of this paper.

Theorem 1.2. *The following are equivalent:*

(1) The Bourgain-Tzafriri Conjecture is true.

(2) There are constants δ *and* $r\in\mathbb{N}$ *so that for every* $C>0$ *there is an* N_0 *so that for every* $N\geq N_0$ *if* $\{f_i\}_{i=1}^{2N}$ *is a unit norm 2-tight frame for* $\mathcal{H}_N$ *satisfying*

$$|f_i(j)| \leq \frac{C}{\sqrt{2N}},$$

then $\{f_i\}_{i=1}^{2N}$ *is* (δ,r)*-Rieszable.*

A **unit norm 2-tight frame** for $\mathcal{H}_N$ can be described as a family of vectors $\{f_i\}_{i=1}^{2N}$ satisfying:

(1) For every $i=1,2,\ldots,2N$, $\|f_i\|=1$,

(2) For every $f\in\mathcal{H}_N$ we have

$$\sum_{i=1}^{2N} |\langle f,f_i\rangle|^2 = 2\|f\|^2.$$

The *analysis oerator* of the frame is the operator $T : \mathcal{H}_N \to \ell_{2N}$ given by

$$T(f) = \{\langle f, f_i\rangle\}_{i=1}^{2N}.$$

For a 2-tight frame, we have $\frac{1}{\sqrt{2}}T$ is an isometry. The adjoint of the analysis operator is the *synthesis operator* $T^* : \ell_{2N} \to \mathcal{H}_N$ given by

$$T(\{a_i\}_{i=1}^{2N} = \sum_{i=1}^{2N} a_i f_i.$$

The **Grammian** of the 2-tight frame is the $2N \times 2N$ matrix given by

$$G = \{\langle f_i, f_j\rangle\}_{i,j=1}^{2N},$$

and $P = \frac{1}{\sqrt{2}}G$ is the orthogonal projection of ℓ_{2N} onto the range of the analysis operator.

The following result shows the relationship between the conjectures above.

Proposition 1.3. *The following are equivalent:*

(1) The Paving Conjecture.
(2) The class of projections with constant diagonal $1/2$ are pavable.
(3) The class of projections with constant diagonal $1/2$ are Rieszable.
(4) The class of unit norm 2-tight frames $\{f_m\}_{m=1}^{2N}$ for $\mathcal{H}_N$ are Rieszable.

Proof. (1) $\Leftrightarrow$ (2): This is a result of Casazza, Edidin, Kalra and Paulsen [4].

(3) $\Leftrightarrow$ (4): This is clear by the Grammian.

(2) $\Leftrightarrow$ (3): Given a projection P with constant diagonal $1/2$, it is (δ, r)-pavable

if and only if

there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, N\}$ such that

$$\begin{aligned}\langle Q_{A_j} P Q_{A_j} x, x\rangle &= \langle P Q_{A_j} x, Q_{A_j} x\rangle \\ &= \langle P Q_{A_j} x, P Q_{A_j} x\rangle \\ &= \|P Q_{A_j}\|^2 \\ &\le (1-\delta)\|x\|^2,\end{aligned}$$

if and only if

$$\|(I-P)Q_{A_j} x\|^2 \ge \delta\|x\|^2,$$

if and only if

$(I - P)$ is (δ, r)-Rieszable. ∎

2. The Proof of Theorem 1.2

Proof. (1) $\Rightarrow$ (2): This is from Proposition 1.3.

(2) $\Rightarrow$ (1): Let P be a projection with constant diagonal $1/2$ on $\mathcal{H}_{2N}$. So $\{\sqrt{2}Pe_i\}_{i=1}^{2N}$ is a unit norm 2-tight frame for $\mathcal{H}_{2N}$. Let A be the $N \times N$ matrix with row vectors $\{\sqrt{2}Pe_i\}_{i=1}^{2N}$. Define recursively,

$$A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} A & A \\ A & -A \end{bmatrix}$$

and

$$A_{K+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} A_K & A_K \\ A_K & -A_K \end{bmatrix}$$

Note: Each A_K (their rows) is a unit norm 2-tight frame for $\mathcal{H}_{2^K N}$. Since the columns of A_K are orthogonal, this implies that the columns of A_{K+1} are orthogonal. Also, clearly the sums of the squares of the row elements are still one and the sums of the squares of the column elements are still one.

Also, the entries $(a_{i,j})$ of A_K satisfy

$$|a_{i,j}| \leq \frac{1}{\sqrt{2^K}} = \frac{\sqrt{N}}{\sqrt{2^K N}}. \tag{1}$$

Letting $C = \sqrt{N}$ in (2) of the theorem, there is some N_0 such that for every $L \geq N_0$, if $\{f_i\}_{i=1}^{2L}$ is a unit norm 2-tight frame for $\mathcal{H}_L$ with

$$|f_{i,j}| \leq \frac{C}{\sqrt{2L}},$$

then $\{f_i\}_{i=1}^{2L}$ is (δ, r)-Rieszable. Hence, for K large enough, Equation 1 has this inequality. So, A_K is (δ, r)-Rieszable. That is, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, 2^K N\}$ so that for every $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\| \sum_{i \in A_j} a_i f_i \|^2 \geq \delta \sum_{i \in A_j} |a_i|^2,$$

where $\{f_i\}$ are the row vectors of A_K. Let

$$B_j = A_j \cap \{1, 2, \ldots, N\}.$$

Then $\{B_j\}_{j=1}^r$ is a partition of $\{1, 2, \ldots, N\}$. Now we compute,

$$\begin{aligned} \delta &\leq \| \sum_{i \in B_j} a_i f_i \|^2 \\ &= \frac{1}{2^K} \sum_{\ell=1}^{2^K} \| \sum_{i \in B_j} a_i \sum_{j=1}^{N} f_{i,\ell+j} \|^2 \\ &= \frac{1}{2^K} \cdot 2^K \| \sum_{i \in B_j} a_i \sqrt{2} P e_i \|^2 \\ &= \| \sum_{i \in B_j} a_i \sqrt{2} P e_i \|^2. \end{aligned}$$

Hence, A is (δ, r)-Rieszable and hence KS holds by Proposition 1.3. $\square$

Remark 2.1. The above points out that there really is a major difference between "paving" and "Rieszing". In the above construction, if G_A is the Grammian of A then the Grammian of A_K is

$$\begin{bmatrix} G_A & 0 & 0 & \cdots \\ 0 & G_A & 0 & \cdots \\ 0 & 0 & G_A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

That is, the coefficients of the Grammian do not get smaller in this construction while the coefficients of the matrix do get smaller.

Remark 2.2. This result also says that passing results on paving from the Grammian back to the matrix and the other way do not hold in general.

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