BRACKET PRODUCTS FOR WEYL-HEISENBERG FRAMES

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ABSTRACT. We provide a detailed development of the L^1 function valued inner product on $L^2(\mathbb{R})$ known as the bracket product. In addition to some of the more basic properties, we show that this inner product has a Bessel's inequality, a Riesz Representation Theorem, and a Gram-Schmidt process. We then apply this to Weyl-Heisenberg frames to show that there exist "compressed" versions of the frame operator, the frame transform and the preframe operator. Finally, we introduce the notion of an a-frame and show that there is an equivalence between the frames of translates for this function valued inner product and Weyl-Heisenberg frames.

1. INTRODUCTION

While working on some deep questions in non-harmonic Fourier series, Duffin and Schaeffer [7] introduced the notion of a frame for Hilbert spaces. Outside of this area, this idea seems to have been lost until Daubechies, Grossman and Meyer [5] brought attention to it in 1986. They show that Duffin and Schaeffer's definition was an abstraction of a concept introduced by Gabor [8] in 1946 for doing signal analysis. Today the frames introduced by Gabor are called **Gabor frames** or **Weyl-Heisenberg frames** and play an important role in signal analysis.

In the study of shift invariant systems and frames several authors, including de Boor, DeVore, Ron and Shen [1, 2, 13, 14], have made extensive use of the so called **bracket product**

$$[f,g](x) = \sum_{\beta \in 2\pi\mathbb{Z}^d} f(x+\beta)\overline{g(x+\beta)}.$$

This turns out to be a special case of an inner product for a Hilbert C^* module used effectively by Rieffel and others to produce results in the field of harmonic analysis on non-commutative groups. For a reference on this, we refer the reader to [12]. In what follows we give a more thorough development of

1991 *Mathematics Subject Classification.* Primary: 42A65, 42C15, 42C30.

Key words and phrases. Weyl-Heisenberg (Gabor) frames, inner products, Hilbert C^* modules .

The first author was supported by NSF DMS 970618.

the bracket product on $L^2(\mathbb{R})$ and its application to univariate principal Weyl-Heisenberg systems. Because we would like to change the shift parameter to arbitrary $a \in \mathbb{R}^+$ we will refer to this bracket product as the **a-inner product**.

Let us outline the organization of the paper. In section 2 we review some of the fundamentals regarding Weyl-Heisenberg systems. In section 3 we give a good reference for the bracket product and some of its basic properties. In section 4 we develop the orthogonality of this function valued inner product including the notions of Bessel's inequality and a Gram-Schmidt process. In section 5 we examine the operators associated with this inner product and prove two Riesz Representation Theorems. Finally, in section 6 we apply these notions to Weyl-Heisenberg systems. Here we show that all the operators associated to a Weyl-Heisenberg system have a compression with regard to this function valued inner product and relate this to the Ron and Shen theory. We go on to introduce the notion of a frame for this function valued inner product and show that frames of translates here coincide with Weyl-Heisenberg frames.
the authors would like to thank the referee's and A.J.E.M. Janssen whose recommendations greatly improved this manuscript.

2. Preliminaries

We use $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ to denote the natural numbers, integers, real numbers and complex numbers, respectively. A **scalar** is an element of $\mathbb{R}$ or $\mathbb{C}$. Integration is always with respect to Lebesgue measure. $L^2(\mathbb{R})$ will denote the complex Hilbert space of square integrable functions mapping $\mathbb{R}$ into $\mathbb{C}$. A bounded unconditional basis for a Hilbert space H is called a **Riesz basis**. That is, (f_n) is a Riesz basis for H if and only if there is an orthonormal basis (e_n) for H and an invertible operator $T : H \to H$ defined by $T(e_n) = f_n$, for all n. We call (f_n) a **Riesz basic sequence** if it is a Riesz basis for its closed linear span. For $E \subset H$, we write span E for the **closed linear span of E**.

In 1952, Duffin and Schaeffer [7] introduced the notion of a frame for a Hilbert space.

Definition 2.1. *A sequence $(f_n)_{n\in\mathbb{Z}}$ of elements of a Hilbert space H is called a* **frame** *if there are constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \text{for all } \ f \in H. \tag{2.1}$$

The numbers A, B are called the **lower** and **upper frame bounds** respectively. The largest number $A > 0$ and smallest number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called the **optimal frame bounds**. The frame is a **tight frame** if $A = B$ and a **normalized tight frame** if

$A = B = 1$. If $f_n \in H$, for all $n \in \mathbb{Z}$, we call $(f_n)_{n\in\mathbb{Z}}$ a **frame sequence** if it is a frame for its closed linear span in H.

We will consider frames from the operator theoretic point of view. To formulate this approach, let (e_n) be an orthonormal basis for an infinite dimensional Hilbert space H and let $f_n \in H$, for all $n \in \mathbb{Z}$. We call the operator $T : H \to H$ given by $Te_n = f_n$ the **preframe operator** associated with (f_n). Now, for each $f \in H$ and $n \in \mathbb{Z}$ we have $\langle T^* f, e_n\rangle = \langle f, Te_n\rangle = \langle f, f_n\rangle$. Thus

$$T^* f = \sum_n \langle f, f_n\rangle\, e_n, \text{ and } \|T^* f\|^2 = \sum_n |\,\langle f, f_n\rangle\,|^2, \text{ for all } f \in H.$$

It follows that the preframe operator is bounded if and only if (f_n) has a finite upper frame bound B. Comparing this to Definition 2.1 we have

Theorem 2.2. *Let H be a Hilbert space with an orthonormal basis (e_n). Also let (f_n) be a sequence of elements of H and let $Te_n = f_n$ be the preframe operator. The following are equivalent:*
(1) (f_n) is a frame for H.
(2) The operator T is bounded, linear and onto.
(3) The operator T^* is an (possibly into) isomorphism called the **frame transform**.
Moreover, (f_n) is a normalized tight frame if and only if the preframe operator is a quotient map (i.e. a co-isometry).

It follows that $S = TT^*$ is an invertible operator on H, called the **frame operator**. Moreover, we have

$$Sf = TT^* f = T(\sum_n \langle f, f_n\rangle\, e_n) = \sum_n \langle f, f_n\rangle\, Te_n = \sum_n \langle f, f_n\rangle\, f_n.$$

A direct calculation now yields

$$\langle Sf, f\rangle = \sum_n |\,\langle f, f_n\rangle\,|^2.$$

Therefore, the **frame operator is a positive, self-adjoint invertible operator** on H. Also, the frame inequalities (2.1) yield that (f_n) is a frame with frame bounds $A, B > 0$ if and only if $A \cdot I \le S \le B \cdot I$. Hence, (f_n) is a normalized tight frame if and only if $S = I$.

We will work here with a particular class of frames called Weyl-Heisenberg frames. To formulate these frames, we first need some notation. For a function f on $\mathbb{R}$ we define the operators:

Translation:	$T_a f(x) = f(x-a),$	$a \in \mathbb{R}$
Modulation:	$E_a f(x) = e^{2\pi i a x} f(x),$	$a \in \mathbb{R}$
Dilation:	$D_a f(x) = \vert a\vert^{-1/2} f(x/a),$	$a \in \mathbb{R} - \{0\}$

We also use the symbol E_a to denote the **exponential function** $E_a(x) = e^{2\pi i a x}$. Each of the operators T_a, E_a, D_a are unitary operators on $L^2(\mathbb{R})$. In 1946 Gabor [8] formulated a fundamental approach to signal decomposition in terms of elementary signals. This method resulted in **Gabor frames** or as they are often called today **Weyl-Heisenberg frames**.

Definition 2.3. *If $a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we call $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ a* **Weyl-Heisenberg system** *(***WH-system** *for short) and denote it by (g, a, b). We denote by (g, a) the family $(T_{na}g)_{n\in\mathbb{Z}}$. We call g the* **window function***.*

If the WH-system (g, a, b) forms a frame for $L^2(\mathbb{R})$, we call this a **Weyl-Heisenberg frame** (**WH-frame** for short). The numbers a, b are the **frame parameters** with a being the **shift parameter** and b being the **modulation parameter.** We will be interested in when there are finite upper frame bounds for a WH-system. That is, we wish to know when the system (g, a, b) is a Bessel system also referred to as **preframe functions**. We denote this class by **PF**. It is easily checked that

Proposition 2.4. *The following are equivalent:*
(1) $g \in$ **PF**.
(2) The operator

$$Sf = \sum_{n,m} \langle f, E_{mb}T_{na}g\rangle\, E_{mb}T_{na}g,$$

is a well defined bounded linear operator on $L^2(\mathbb{R})$.

We will need the WH-frame identity due to Daubechies [4]. To simplify the notation a little we introduce the following auxiliary functions. Define for a $g \in L^2(\mathbb{R})$ and all $k \in \mathbb{Z}$

$$G_k(t) = \sum_{n\in\mathbb{Z}} g(t-na)\overline{g(t-na-k/b)}.$$

In particular,

$$G_0(t) = \sum_{n\in\mathbb{Z}} |g(t-na)|^2.$$

Theorem 2.5. *(***WH-frame Identity.***) If $\sum_n |g(t-na)|^2 \le B$ a.e. and $f \in L^2(\mathbb{R})$ is bounded and compactly supported, then*

$$\sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 = F_1(f) + F_2(f),$$

where

$$F_1(f) = b^{-1}\int_R |f(t)|^2 G_0(t)\ dt,$$

and

$$\begin{aligned} F_2(f) &= b^{-1}\sum_{k\neq 0}\int_R \overline{f(t)}f(t-\frac{k}{b})G_k(t)\ dt \\ &= b^{-1}\sum_{k\geq 1} 2Re\int_R \overline{f(t)}f(t-\frac{k}{b})G_k(t)\ dt. \end{aligned}$$

There are many restrictions on the g, a, b in order that (g, a, b) form a WH-frame. We will make note of a few of them here. The first is a simple application of the WH-frame Identity. That is, if we put functions supported on $[0, \frac{1}{b}]$ into this identity, then $F_2(f) = 0$. Now the WH-frame Identity combined with the frame condition quickly yields,

Theorem 2.6. *If (g, a, b) is a WH-frame with frame bounds A, B then*

$$A \leq \frac{1}{b}G_0(t) \leq B, \quad a.e.$$

A basic result of Ron and Shen [14] yields a similar upper bound condition with a replaced by $\frac{1}{b}$.

Proposition 2.7. *(g, a, b) is a WH-frame with frame bounds A,B, iff the operator $\mathcal{K}(x) := \{g(x - na - j/b)\}_{n,j}$ satisfies*

$$bAI \leq \mathcal{K}\mathcal{K}^*(x) \leq bBI.$$

Hence if (g, a, b) has a finite upper frame bound, then $\sum |g(t-\frac{n}{b})|^2 \leq bB$ *a.e.*

A recent very important result was proved independently by Daubechies, H. Landau and Z. Landau [6], Janssen [10], and Ron and Shen [14].

Theorem 2.8. *For $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, the following are equivalent:*
(1) (g, a, b) is a WH-frame.
(2) The family $(E_{\frac{m}{a}}T_{\frac{n}{b}}g)_{m,n\in\mathbb{Z}}$ is a Riesz basic sequence in $L^2(\mathbb{R})$.

We end these preliminaries with a brief discussion of the Ron and Shen theory of Gramian analysis for shift invariant systems from [13, 14]. At the heart of their technique is the pre-Gramian operator J and its adjoint J^*. Ron and Shen show that J^* is a Fourier transform analogue of the operator $\mathcal{T}^*$. In the Weyl-Heisenberg case with a=1 it takes the form

$$J^*(f) :\to ([f, T_{mb}\hat{g}])_{m\in\mathbb{Z}},$$

where we consider $\mathcal{T}^* : L^2(\mathbb{R}) \to \ell_2(\mathbb{Z})$. Then they calculate the dual Gramian $\tilde{\mathcal{G}} = JJ^*$ which will correspond to a Fourier transform version of what we later refer to as the compression of the frame operator. In order to avoid confusion we point out a subtle difference in our approach. We use the preframe and frame operators which are bounded operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ instead of the operator $\mathcal{T}^*$ above and its dual.

3. Pointwise Inner Products

A number of the results below can be found in the early papers [1, 2, 13, 14]. For the sake of completeness, and to create a good reference for this inner product for use in WH-systems we list them here. To guarantee that our inner product is well defined, we need to first check some convergence properties for elements of $L^2(\mathbb{R})$.

Proposition 3.1. *For $f, g \in L^2(\mathbb{R})$ and $a \in \mathbb{R}^+$ the series*

$$\sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)}$$

converges unconditionally a.e. to a function in $L^1[0,a]$.

Proof. If $f, g \in L^2(\mathbb{R})$ then $f\overline{g} \in L^1(\mathbb{R})$. Hence,

$$\|fg\|_{L^1} = \int_0^a \sum_{n\in\mathbb{Z}} |f(t-na)\overline{g(t-na)}| \, dt < \infty.$$

The Monotone Convergence Theorem yields both the interchange of the integral and the sum and the existence of $\sum f(t-na)\overline{g(t-na)}$ as a function in $L^1[0,a]$. □

A simple application of the Lebesgue Dominated Convergence Theorem combined with Proposition 3.1 yields

Corollary 3.2. *For all $f, g \in L^2(\mathbb{R})$ we have*

$$\langle f, g\rangle = \int_0^a \sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)} \, dt.$$

Now we introduce the pointwise inner product for WH-frames. We can view this as a function valued inner product.

Definition 3.3. *Fix $a \in \mathbb{R}^+$. For all $f, g \in L^2(\mathbb{R})$ we define the* a**-pointwise inner product of f and g** *(called the* a**-inner product** *for short) by*

$$\langle f, g\rangle_a(t) = \sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)}, \quad \text{for all } \ t \in \mathbb{R}.$$

We define the a**-norm of f** *by*

$$\|f\|_a(t) = \sqrt{\langle f, f\rangle_a(t)}.$$

We emphasize here that the a-inner product and the a-norm are *functions* on $\mathbb{R}$ which are clearly a-periodic. To cut down on notation, whenever we have an a-periodic function on $\mathbb{R}$, we will also consider it a function on $[0, a]$. The convergence of these series is guaranteed by our earlier discussion. In fact, the a-inner product $< \cdot, \cdot >_a$ is an onto (but not 1-1) mapping from $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ to the a-periodic functions on $\mathbb{R}$ whose restriction to $[0, a]$ lie in $L^1[0, a]$.

First we show that the a-inner product really is a good generalization of the standard notion of inner products for a Hilbert space. Since all of these proofs follow directly from the definitions we omit them.

Theorem 3.4. *Let $f, g, h \in L^2(\mathbb{R})$, $c, d \in \mathbb{C}$, and $a, b \in \mathbb{R}$. The following properties hold:*

(1) $\langle f, g\rangle_a$ is a periodic function of period a on $\mathbb{R}$ with $\langle f, g\rangle_a \in L^1[0, a]$.

(2) $\|f\|_{L^2(\mathbb{R})} = \Big\| \|f\|_a(t) \Big\|_{L^2[0,a]}$

(3)$\langle f, g\rangle = \int_0^a \langle f, g\rangle_a (t)\ dt$.

(4)$\langle cf + dg, h\rangle_a = c\langle f, h\rangle_a + d\langle g, h\rangle_a$.
(5) $\langle f, cg + dh\rangle_a = \overline{c}\langle f, g\rangle_a + \overline{d}\langle f, h\rangle_a$.
(6) $\langle f, g\rangle_a = \overline{\langle g, f\rangle_a}$.
(7) $\langle fg, h\rangle_a = \langle f, \overline{g}h\rangle_a$, for $fg, gh \in L^2(\mathbb{R})$.
(8) If $\langle f, g\rangle_a = 0$ then $\langle f, g\rangle = 0$.
(9) $\langle T_b f, T_b g\rangle_a = T_b\langle f, g\rangle_a$.
(10) $\|T_b g\|_a^2 = T_b \|g\|_a^2$.
(11) $\langle T_b f, g\rangle_a = T_b\langle f, T_{-b} g\rangle_a$.
(12) $\langle f, g\rangle_a = \frac{1}{\sqrt{ab}} D_{ab} \left\langle D_{\frac{1}{ab}} f, D_{\frac{1}{ab}} g\right\rangle_{\frac{1}{b}}$.

It is not difficult to mimic the standard proofs for the usual inner product on a Hilbert space to obtain the following results for the a-inner product.

Proposition 3.5. *For all $f, g \in L^2(\mathbb{R})$ the following hold a.e.:*

(1) $|\langle f, g\rangle|_a \le \|f\|_a \|g\|_a$. (2) $\|f + g\|_a^2 = \|f\|_a^2 + 2Re\langle f, g\rangle_a + \|g\|_a^2$.
(3) $\|f + g\|_a \le \|f\|_a + \|g\|_a$. (4) $\|f + g\|_a^2 + \|f - g\|_a^2 = 2(\|f\|_a^2 + \|g\|_a^2)$.

Since our a-inner product is an a-periodic function, it enjoys some special properties related to a-periodic functions.

Proposition 3.6. *Let $f,g \in L^2(\mathbb{R})$ and let $h \in L^\infty(\mathbb{R})$ be an a-periodic function. Then*

$$\langle fh, g\rangle_a = h\,\langle f,g\rangle_a \quad \textit{and} \quad \langle f, hg\rangle_a = \overline{h}\,\langle f,g\rangle_a .$$

In particular, if h satisfies $h(t) \neq 0$ a.e., then $\langle f,g\rangle_a = 0$ if and only if $\langle fh, g\rangle_a = \langle f, g\overline{h}\rangle_a = 0$.

Proof. We compute

$$\begin{aligned}
\langle fh, g\rangle_a(t) &= \sum_{n\in\mathbb{Z}} f(t-na)h(t-na)\overline{g(t-na)} \\
&= \sum_{n\in\mathbb{Z}} f(t-na)h(t)\overline{g(t-na)} \\
&= h(t)\sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)} = h(t)\,\langle f,g\rangle_a(t).
\end{aligned}$$

□

Next we normalize our functions in the a-inner product. For $f \in L^2(\mathbb{R})$, we define the **a-pointwise normalization of f** to be

$$N_a(f)(t) = \begin{cases} \frac{f(t)}{\|f\|_a(t)} : & \|f\|_a(t) \neq 0 \\ 0 : & \|f\|_a(t) = 0. \end{cases}$$

We now have

Proposition 3.7. *Let $f,g \in L^2(\mathbb{R})$.*
(1) We have

$$\langle N_a(f), g\rangle_a = \frac{\langle f,g\rangle_a}{\|f\|_a(t)}, \quad \textit{where} \ \ \|f\|_a(t) \neq 0.$$

In particular, $\langle f,g\rangle_a(t) = 0$ if and only if $\langle N_a(f), g\rangle_a(t) = 0$.
(2) For $f \neq 0$ a.e. we have

$$\langle N_a(f), N_a(f)\rangle_a(t) = \sum_{n\in\mathbb{Z}} |N_a(f)(t-na)|^2 = 1, a.e.$$

(3) We have

$$\|N_a(f)\|^2_{L^2(\mathbb{R})} = \lambda(supp\ \|f\|_a|_{[0,a]}(t)) \leq a.$$

where λ denotes Lebesgue measure.
(4) $N_a(N_a(f)) = N_a(f)$.

Proof. (1) We compute

$$\langle N_a(f), g\rangle_a = \sum_{n\in\mathbb{Z}} N_a(f)(t-na)\overline{g(t-na)} = \sum_{n\in\mathbb{Z}} \frac{f(t-na)}{\|f\|_a(t-na)}\overline{g(t-na)}.$$

Since our inner product is a-periodic, this equality becomes,

$$\frac{1}{\|f\|_a(t)}\sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)} = \frac{\langle f,g\rangle_a(t)}{\|f\|_a(t)}, \quad \text{where} \quad \|f\|_a(t)\neq 0.$$

(2)-(4) are straightforward calculations. □

4. a-ORTHOGONALITY

The notion of orthogonality with respect to the a-inner product has been used primarily to describe the orthogonal complement in the usual inner product for shift-invariant spaces. In this section we explore more thoroughly what it means to be a-orthogonal and develop such things as a-orthonormal sequences and a Bessel inequality for the a-inner product. This property gives one of the main applications of the a-inner product in Weyl-Heisenberg frame theory. For as we will see, orthogonality in this form is very strong.

Definition 4.1. *For $f, g \in L^2(\mathbb{R})$, we say that f and g are* a**-orthogonal***, and write $f\perp_a g$, if $\langle f,g\rangle_a(t)=0$ a.e.. We define the* a**-orthogonal complement** *of $E\subset L^2(\mathbb{R})$ by*

$$E^{\perp_a} = \{g : \langle f,g\rangle_a = 0,\ \text{for all } f\in E\}.$$

Similarly, an a**-orthogonal sequence** *is a sequence (f_n) satisfying $f_n\perp_a f_m$, for all $n\neq m$. This is an* a**-orthonormal sequence** *if we also have $\|f_n\|_a = 1$ a.e.*

We now identify an important class of functions for working with the a-inner product.

Definition 4.2. *We say that $g\in L^2(\mathbb{R})$ is* a**-bounded** *if there is a $B>0$ so that*

$$|\langle g,g\rangle_a(t)| \le B, \quad \text{for a.a. } t$$

We let $L^\infty_a(\mathbb{R})$ denote the family of a-bounded functions.

We have that $L^\infty_a(\mathbb{R})$ is a non-closed (in the $L^2(\mathbb{R})$ norm) linear subspace of $L^\infty(\mathbb{R})$. Note also that the Wiener amalgam space is a subspace of $L^\infty_a(\mathbb{R})$.

We have not defined orthonormal bases for the a-inner product yet since, as we will see, this requires a little more care. First we need to develop the basic properties of a-orthogonality.

Proposition 4.3. *If $E\subset L^2(\mathbb{R})$ and $B_a=\{f : f\in L^\infty(\mathbb{R})$ and f is a-periodic$\}$ then*

$$E^{\perp_a} = \cap_{\phi\in B_a}(\phi E)^\perp = (span_{\phi\in B_a}\phi E)^\perp.$$

Proof. Let $f \in E^{\perp_a}$. For any $g \in E$ and any a-periodic function $\phi \in B_a$ we have by Proposition 3.6

$$\langle f, \phi g \rangle_a (t) = \overline{\phi}(t) \langle f, g \rangle_a (t) = 0.$$

Hence, $f \perp_a \phi g$. That is, $f \in (\phi E)^{\perp}$.
Now let $f \in \cap (\phi E)^{\perp}$, the intersection being taken over all bounded a-periodic ϕ. Let $g \in E$ and define for $n \in \mathbb{N}$,

$$\phi_n(t) = \begin{cases} \langle f, g \rangle_a (t) : & |\langle f, g \rangle_a (t)| \leq n \\ 0 : & \text{otherwise.} \end{cases}$$

Note that ϕ_n is a-periodic. Now we compute,

$$\begin{aligned} 0 = \langle f, \phi_n g \rangle &= \int_{\mathbb{R}} f(t)\overline{\phi_n(t) g(t)} dt \\ &= \int_0^a \left(\sum_{n\in\mathbb{Z}} f(t - na)\overline{g(t - na)} \right) \overline{\phi_n(t)}\ dt \\ &= \int_0^a \langle f, g \rangle_a (t) \overline{\phi_n(t)}\ dt = \int_0^a |\phi_n(t)|^2\ dt. \end{aligned}$$

Therefore, $\phi_n = 0$, for all $n \in \mathbb{Z}$. Hence, $\langle f, g \rangle_a (t) = 0$ a.e., and so $f \perp_a g$. That is, $f \perp_a E$. $\square$

By Theorem 3.4 (8), we have that $E^{\perp_a} \subset E^{\perp}$.

Corollary 4.4. *For $E \subset L^2(\mathbb{R})$, $E^{\perp_a}$ is a norm closed linear subspace of $E^{\perp}$.*

The next result which can be found in [1] shows more clearly what orthogonality means in this setting .

Proposition 4.5. *For $f, g \in L^2(\mathbb{R})$, the following are equivalent:*
(1) $f \perp_a g$.
(2) $span_{m\in\mathbb{Z}} E_{\frac{m}{a}} f \perp span_{m\in\mathbb{Z}} E_{\frac{m}{a}} g$.

Proof. Fix $m \in \mathbb{Z}$ and compute

$$\left\langle f, E_{\frac{m}{a}} g \right\rangle = \int_0^a \langle f, g \rangle_a (t) e^{-2\pi i (\frac{m}{a})t}\ dt = \widehat{< f, g >}_a(m).$$

It follows that $\left\langle f, E_{\frac{m}{a}} g \right\rangle = 0$, for all $m \in \mathbb{Z}$ if and only if all the Fourier coefficients of $< f, g >_a (t)$ are zero. A moment's reflection should convince the reader that this is all we need. $\square$

Definition 4.6. *We say that $E \subset L^2(\mathbb{R})$ is an a-**periodic closed set** if for any $f \in E$ and any $\phi \in L_a^{\infty}(\mathbb{R})$ we have that $\phi f \in E$.*

The next result follows immediately from Propositions 4.3 and 4.5.

Corollary 4.7. *For any $E \subset L^2(\mathbb{R})$, $E^{\perp_a}$ is an a-periodic closed set. If E is an a-periodic closed set then $E^{\perp} = E^{\perp_a}$.*

Now we observe what orthogonality means for $(E_{\frac{m}{a}} g)$ in terms of the regular inner product.

Proposition 4.8. *If $g \in L^2(\mathbb{R})$ and $\|g\|_a = 1$ a.e., then $(\frac{1}{\sqrt{a}} E_{\frac{m}{a}} g)_{m\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.*

Proof. For any $n, m \in \mathbb{Z}$ we have

$$\begin{aligned} \left\langle E_{\frac{n}{a}} g, E_{\frac{m}{a}} g \right\rangle &= \int_{\mathbb{R}} |g(t)|^2 e^{2\pi i[(n-m)/a]t}\, dt \\ &= \int_0^a \|g\|_a^2(t) e^{2\pi i[(n-m)/a]t}\, dt \\ &= \int_0^a e^{2\pi i[(n-m)/a]t}\, dt = a\delta_{nm}. \end{aligned}$$

□

Corollary 4.9. *If $(g_n)_{n\in\mathbb{N}}$ is an a-orthonormal sequence in $L^2(\mathbb{R})$, then $(E_{\frac{m}{a}} g_n)_{n,m\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.*

Proof. We need that for all $(n, m) \neq (\ell, k) \in \mathbb{Z} \times \mathbb{Z}$, $E_{\frac{m}{a}} g_n \perp E_{k/a} g_\ell$. But, if $n \neq \ell$, this is Proposition 4.5, and if $n = \ell$, this is Proposition 4.8. □

Corollary 4.9 tells us how to define an a-orthonormal basis.

Definition 4.10. *Let $g_n \in L^2(\mathbb{R})$. We call (g_n) an a-**orthonormal basis** for $L^2(\mathbb{R})$ if it is an a-orthonormal sequence and*

$$span\ (E_{\frac{m}{a}} g_n)_{n,m\in\mathbb{Z}} = L^2(\mathbb{R}).$$

Proposition 4.11. *A sequence (g_n) in $L^2(\mathbb{R})$ is an a-orthonormal basis if and only if $(E_{\frac{m}{a}} g_n)_{n,m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

We would like to capture the important Bessel's Inequality for a-orthonormal sequences but before we do so we need to insure that $\langle f, g\rangle_a g$ remains in $L^2(\mathbb{R})$ for functions $g \in L_a^\infty(\mathbb{R})$.

Proposition 4.12. *If $g, h \in L_a^\infty(\mathbb{R})$ then $\langle f, g\rangle_a h \in L^2(\mathbb{R})$ for all $f \in L^2(\mathbb{R})$.*

Proof. First we need to show $\langle f, g\rangle_a \in L^2([0, a])$. Let $B = esssup_{[0,a)} \|g\|_a^2(t)$ and $C = esssup_{[0,a)} \|h\|_a^2(t)$. This follows from the Cauchy-Schwarz inequality for the a-inner product:

$$\begin{aligned}\| \langle f, g\rangle_a (t)\|^2_{L^2[0,a]} &= \int_0^a |\langle f, g\rangle_a (t)|^2 \, dt \\ &\le \int_0^a \langle f, f\rangle_a (t) \langle g, g\rangle_a (t) \, dt \\ &\le B \int_0^a \langle f, f\rangle_a (t) \, dt = B\|f\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Now we can prove the proposition using the Monotone Convergence Theorem and the result above:

$$\begin{aligned}\| \langle f, g\rangle_a h\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |\langle f, g\rangle_a (t)h(t)|^2 \, dt \\ &= \sum_n \int_0^a |\langle f, g\rangle_a (t)|^2 |h(t-na)|^2 dt \\ &\le \int_0^a |\langle f, g\rangle_a (t)|^2 \langle h, h\rangle_a (t) dt \\ &\le BC\|f\|^2_{L^2(\mathbb{R})}\end{aligned}$$

□

Theorem 4.13. *Let $(g_n)_{n\in\mathbb{N}}$ be an a-orthonormal sequence in $L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$.*
(1) the series of functions $\sum_{n\in\mathbb{N}} \langle f, g_n\rangle_a g_n$ converges in $L^2(\mathbb{R})$.

(2) We have "Bessel's Inequality",

$$\langle f, f\rangle_a \quad \ge \quad \sum_{n=1}^{\infty} |\langle f, g_n\rangle_a|^2.$$

Note that this is an inequality for functions.
Moreover, if $f \in span\ (E_{\frac{m}{a}} g_n)_{m,n\in\mathbb{Z}}$, then

$$\langle f, f\rangle_a = \sum_{n=1}^{\infty} |\langle f, g_n\rangle_a|^2.$$

Proof. First we note that the g_n are in $L^\infty_a(\mathbb{R})$ so each $\langle f, g_n\rangle_a g_n$ is in $L^2(\mathbb{R})$. Fix $1 \le m$ and let

$$h = \sum_{n=1}^{m} \langle f, g_n\rangle_a g_n.$$

Using the fact that the a-inner product of two functions is a-periodic (and hence may be factored out of the a-inner product) we have

$$\begin{aligned} \langle h, h\rangle_a &= \left\langle \sum_{n=1}^{m} \langle f, g_n\rangle_a g_n, \sum_{k=1}^{m} \langle f, g_k\rangle_a g_k \right\rangle_a \\ &= \sum_{n,k=1}^{m} \langle f, g_n\rangle_a \overline{\langle f, g_k\rangle_a} \langle g_n, g_k\rangle_a = \sum_{n=1}^{m} |\langle f, g_n\rangle_a|^2. \end{aligned}$$

Letting $g = f - h$ we have by the same type of calculation as above,

$$\begin{aligned} \langle h, g\rangle_a &= \left\langle \sum_{n=1}^{m} \langle f, g_n\rangle_a g_n, f - \sum_{k=1}^{m} \langle f, g_k\rangle_a g_k \right\rangle_a \\ &= \sum_{n=1}^{m} |\langle f, g_n\rangle_a|^2 - \sum_{k=1}^{m} |\langle f, g_k\rangle_a|^2 = 0. \end{aligned}$$

So we have decomposed f into two a-orthogonal functions h, g. Therefore,

$$\begin{aligned} \langle f, f\rangle_a &= \langle h + g, h + g\rangle_a \\ &= \langle h, h\rangle_a + \langle g, g\rangle_a \\ &= \sum_{n=1}^{m} |\langle f, g_n\rangle_a|^2 + \langle g, g\rangle_a \geq \sum_{n=1}^{m} |\langle f, g_n\rangle_a|^2. \end{aligned}$$

Since m was arbitrary, we have (2) of the Theorem. For (1), we just put together what we know. By (2) and the Monotone Convergence Theorem, we have that the series of functions $\sum_{n\in\mathbb{N}} |\langle f, g_n\rangle_a|^2$ converges in $L^1[0, a]$. But, by our calculations above and the properties of the a-norm,

$$\begin{aligned} \|\sum_{n=k}^{m} \langle f, g_n\rangle_a g_n\|_{L^2(\mathbb{R})}^2 &= \int_0^a \|\sum_{n=k}^{m} \langle f, g_n\rangle_a g_n\|_a^2(t)\ dt \\ &= \int_0^a \left\langle \sum_{n=k}^{m} \langle f, g_n\rangle_a g_n, \sum_{n=k}^{m} \langle f, g_n\rangle_a g_n \right\rangle_a (t)\ dt \\ &= \int_0^a \sum_{n=k}^{m} |\langle f, g_n\rangle_a|^2(t)\ dt. \end{aligned}$$

Now, $\sum_{n\in\mathbb{N}} |\langle f, g_n\rangle_a|^2$ converges in $L^1[0, a]$ implies that the right hand side of our equality goes to zero as $k \to \infty$.

□

We end this section with a Gram-Schmidt process for the a-inner product. First we need a result which shows that this process produces functions which are in the proper spans.

Proposition 4.14. *Let $f, g, h \in L^2(\mathbb{R})$. We have:*
(1) $N_a(g) \in$ span $(E_{\frac{m}{a}}g)_{m\in\mathbb{Z}}$.
(2) If any two of f, g, h are in $L_a^\infty(\mathbb{R})$, then $\langle f, g\rangle_a h \in$ span $(E_{\frac{m}{a}}g)_{m\in\mathbb{Z}}$.

Proof. (1): For each $n \in \mathbb{N}$ let

$$E_n = \{t \in [0, a] : |\langle g, g\rangle_a (t)|^2 \geq n \text{ or } \langle g, g\rangle_a (t) \leq \frac{1}{n}\}.$$

Also, let

$$\tilde{E}_n = \cup_{m\in\mathbb{Z}}(E_n + m).$$

Since $g \in L^2(\mathbb{R})$, we have

$$\|g\|^2 = \int_0^a \langle g, g\rangle_a (t)\; dt < \infty.$$

Hence, $\lim_{n\to\infty} \lambda(E_n) = 0$. Let $F_n = [0, a] - E_n$ and

$$\tilde{F}_n = \cup_{m\in\mathbb{Z}}(F_n + m).$$

Now,

$$\frac{1}{n} \leq \frac{1}{\left\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g\right\rangle_a} \leq n.$$

Hence,

$$\frac{1}{\left\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g\right\rangle_a} \in L_a^\infty(\mathbb{R}).$$

Hence,

$$\frac{\chi_{\tilde{F}_n} g}{\left\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g\right\rangle_a} + \chi_{\tilde{E}_n} g \in \text{span } (E_{\frac{m}{a}}g)_{m\in\mathbb{Z}}.$$

Also,

$$\begin{aligned}
\left\|\frac{\chi_{\tilde{F}_n} g}{\left\langle \chi_{\tilde{F}_n} g, \chi_{\tilde{F}_n} g\right\rangle_a} + \chi_{\tilde{E}_n} g - N_a(g)\right\|_{L^2(\mathbb{R})} &= \left\|\chi_{\tilde{E}_n} g - \frac{\chi_{\tilde{E}_n} g}{\langle g, g\rangle_a}\right\| \\
&\leq \|\chi_{\tilde{E}_n} g\| + \left\|\frac{\chi_{\tilde{E}_n} g}{\langle g, g\rangle_a}\right\| \\
&= \left(\int_{\mathbb{R}} |\chi_{\tilde{E}_n} g|^2\; dt\right)^{1/2} + \|N_a(\chi_{\tilde{E}_n} g)\| \\
&\leq \left(\int_{E_n} \langle g, g\rangle_a (t)\; dt\right)^{1/2} + \lambda(E_n).
\end{aligned}$$

But the right hand side of the above inequality goes to zero as $n \to \infty$.

(2): Assume first that $f, g \in L^\infty_a(\mathbb{R})$. Let $B = esssup_{[0,a)} \|f\|_a^2(t)$ and $C = esssup_{[0,a)} \|g\|_a^2(t)$. Now by Cauchy-Schwarz

$$|\langle f, g\rangle_a (t)| \leq \|f\|_a(t)\|g\|_a(t) \leq \sqrt{B}\sqrt{C}.$$

Therefore, $\langle f, g\rangle_a$ is a bounded a-periodic function on $\mathbb{R}$. This implies that $\langle f, g\rangle_a h \in L^2(\mathbb{R})$. Now suppose that $g, h \in L^\infty_a(\mathbb{R})$. Then $\langle f, g\rangle_a h \in L^2(\mathbb{R})$ by Proposition 4.12. A direct calculation shows

$$\text{span } (E_{\frac{m}{a}} g)_{m\in\mathbb{Z}} = \{\phi g : \phi \text{ is a-periodic and } \phi g \in L^2(\mathbb{R})\}.$$

So by the above, we have that $\langle f, h\rangle_a g \in \text{span } (E_{\frac{m}{a}} g)_{m\in\mathbb{Z}}$. □

Definition 4.15. *Let $g_n \in L^2(\mathbb{R})$, for $1 \leq n \leq k$. We say that $(g_n)_{n=1}^k$ is a-* **linearly independent** *if for each $1 \leq n \leq k$, $g_n \notin$ span $(E_{\frac{m}{a}} g_i)_{m\in\mathbb{Z};1\leq i\neq n\leq k}$. An arbitrary family is a-***linearly independent** *if every finite sub-family is a-linearly independent.*

Now we carry out the Gram-Schmidt process.

Theorem 4.16. *(Gram-Schmidt ortho-normalization procedure) Let $(g_n)_{n\in\mathbb{N}}$ be an a-linearly independent sequence in $L^2(\mathbb{R})$ for $a > 0$. Then there exists an a-orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ satisfying for all $n \in \mathbb{N}$:*

$$\textit{span } (E_{\frac{m}{a}} g_k)_{m\in\mathbb{Z},1\leq k\leq n} = \textit{span } (E_{\frac{m}{a}} e_k)_{m\in\mathbb{Z},1\leq k\leq n}.$$

Proof. We proceed by induction. First let $e_1 = N_a(g_1)$. If $(e_i)_{i=1}^n$ have been defined to satisfy the theorem, let

$$e_{n+1} = N_a(g_{n+1} - \sum_{i=1}^n \langle g_i, e_i\rangle_a e_i) \text{ and } h = g_{n+1} - \sum_{i=1}^n \langle g_{n+1}, e_i\rangle_a e_i.$$

Note that $h \neq 0$ by our a-linearly independent assumption and Proposition 4.14. Now, for $1 \leq k \leq n$ we have

$$\begin{aligned} \langle e_{n+1}, e_k\rangle_a &= \frac{1}{\langle h, h\rangle_a}\left(\langle g_{n+1}, e_k\rangle_a - \sum_{i=1}^n \langle g_{n+1}, e_i\rangle_a \langle e_i, e_k\rangle_a\right) \\ &= \frac{1}{\langle h, h\rangle_a}(\langle g_{n+1}, e_k\rangle_a - \langle g_{n+1}, e_k\rangle_a \langle e_k, e_k\rangle_a) = 0. \end{aligned}$$

The statement about the linear spans follows from Proposition 4.14. □

5. a-Factorable Operators

Now we consider operators on $L^2(\mathbb{R})$ which behave naturally with respect to the a-inner product. We will call these operators a-factorable operators.

Definition 5.1. *Fix $1 \le p \le \infty$. We say that a linear operator $L : L^2(\mathbb{R}) \to L^p(E)$ is an a-**factorable operator** if for any factorization $f = \phi g$ where $f, g \in L^2(\mathbb{R})$ and ϕ is an a-periodic function on $\mathbb{R}$ we have*

$$L(f) = L(\phi g) = \phi L(g).$$

First we show it is enough to consider factorizations over $L^\infty([0,a))$ and see these are exactly the operators that commute with all $E_{\frac{m}{a}}$

Proposition 5.2. *Let L be a bounded operator from $L^2(\mathbb{R})$ to $L^2([0,a))$. Then L is a-factorable if and only if $L(\phi f) = \phi L(f)$ for all $f \in L^2(\mathbb{R})$ and all a-periodic $\phi \in L^\infty(\mathbb{R})$.*

Proof. Assume ϕ is a-periodic, $f, g \in L^2(\mathbb{R})$ and $f = \phi g$. For all $n \in \mathbb{N}$ let

$$F_n = \{t \in [0,a] : |\phi(t)| > n\}.$$

Let $E_n = [0,1] - F_n$ and

$$\tilde{E}_n = \cup_{m\in\mathbb{Z}}(E_n + m) \quad \text{and} \quad \tilde{F}_n = \cup_{m\in\mathbb{Z}}(F_n + m).$$

Now,

$$\begin{aligned} \|\chi_{\tilde{E}_n}\phi g - \phi g\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} |\chi_{\tilde{F}_n}\phi(t)g(t)|^2 \, dt \\ &= \int_0^a |\chi_{F_n}\phi(t)|^2 \langle g, g\rangle_a (t) \, dt. \end{aligned}$$

Since $\phi g \in L^2(\mathbb{R})$ and $\lim_{n\to\infty} \lambda(F_n) = 0$, it follows that $h_n =: \chi_{\tilde{E}_n}\phi g$ converges to ϕg in $L^2(\mathbb{R})$. Since L is a bounded linear operator, it follows that $L(h_n)$ converges to $L(\phi g)$. But, $L(h_n) = \chi_{\tilde{E}_n}\phi L(g)$ by our assumption. Now,

$$\|L(h_n)\| \le \|L\|\|h_n\| \le \|L\|\|\phi g\| = \|L\|\|f\|.$$

Finally, since $|L(h_n)| \uparrow |\phi L(g)|$ it follows from the Lebesgue Dominated Convergence Theorem that $\phi L(g) \in L^2(\mathbb{R})$ and $L(h_n) \to \phi L(g)$. This completes the proof of the Proposition. □

We have immediately

Corollary 5.3. *An operator $L : L^2(\mathbb{R}) \to L^p(E)$ is a-factorable if and only if $L(E_{\frac{m}{a}} g) = E_{\frac{m}{a}} L(g)$, for all $m \in \mathbb{Z}$. That is, L is a-factorable if and only if it commutes with $E_{\frac{m}{a}}$.*

Next we derive our first Riesz Representation Theorem for a-factorable operators. To simplify this proof as well as later arguments we first prove a lemma.

Lemma 5.4. *Let L_1 and L_2 be a-factorable operators from $L^2(\mathbb{R}) \to L^1[0,a]$. Then $L_1 = L_2$ iff*

$$\int_0^a L_1(f)(t)dt = \int_0^a L_2(f)(t)dt.$$

Proof. Fix $f \in L^2(\mathbb{R})$. By our assumption, for all $m \in \mathbb{Z}$ we have

$$\begin{aligned}\int_0^a L_1(E_{\frac{m}{a}}f)(t)dt &= \int_0^a L_1(f)(t)E_{\frac{m}{a}}(t)dt \\ &= \int_0^a L_2(f)(t)E_{\frac{m}{a}}(t)dt = \int_0^a L_2(E_{\frac{m}{a}}f)(t)dt.\end{aligned}$$

Hence, the Fourier coefficients for $L_1(f)$ and $L_2(f)$ are the same for all $f \in L^2(\mathbb{R})$ and therefore $L_1 = L_2$ □

Our original proof of the Riesz Representation Theorem for a-factorable operators was cumbersome. The direct proof below was kindly communicated to us by A.J.E.M. Janssen.

Theorem 5.5. *(Riesz Representation Theorem 1): L is a bounded a-factorable operator from $L^2(\mathbb{R})$ to $L^1[0,a]$ iff there is a $g \in L^2(\mathbb{R})$ so that $L(f) = \langle f,g\rangle_a(t)$ for all $f \in L^2(\mathbb{R})$. Moreover $\|L\| = \|g\|$.*

Proof. $\Leftarrow$ Fix $g \in L^2(\mathbb{R})$ and define L on $L^2(\mathbb{R})$ by $L(f) = \langle f,g\rangle_a(t)$. Now for any $f \in L^2(\mathbb{R})$

$$\begin{aligned}\|Lf\| &= \|\langle f,g\rangle_a(t)\|_{L^1[0,a]} \\ &= \int_0^a |\sum_{n\in\mathbb{Z}} f(t-na)\overline{g(t-na)}|\ dt \\ &\le \int_0^a \sqrt{\sum_{n\in\mathbb{Z}} |f(t-na)|^2}\sqrt{\sum_{n\in\mathbb{Z}} |g(t-na)|^2}\ dt \\ &\le \left(\int_0^a \sum_{n\in\mathbb{Z}} |f(t-na)|^2\right)^{1/2}\left(\int_0^a \sum_{n\in\mathbb{Z}} |g(t-na)|^2\right)^{1/2} \\ &= \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.\end{aligned}$$

Letting $g = f$ we see that $\|L(g)\| = \|g\|$ which, combined with the above, shows that $\|L\| = \|g\|$.

$\Rightarrow$ Assume L is an a-factorable operator from $L^2(\mathbb{R}) \to L^1[0,a]$. Define the linear functional Ψ on $L^2(\mathbb{R})$ by

$$\Psi(f) = \int_0^a L(f)(t)dt.$$

By the standard Riesz Representation Theorem, there is a function $g \in L^2(\mathbb{R})$ so that $\Psi(f) = \langle f, g\rangle$ for all $f \in L^2(\mathbb{R})$. Define the operator $\mathcal{L}_g(f) = \langle f, g\rangle_a(t)$. It follows that

$$\Psi(f) = \langle f, g\rangle = \int_0^a \langle f, g\rangle_a(t)dt = \int_0^a \mathcal{L}_g(f)(t)dt = \int_0^a L(f)(t)dt.$$

Since $\mathcal{L}_g$ and L are a-factorable maps from $L^2(\mathbb{R})$ to $L^1[0, a]$, they are equal by Lemma 5.4. □

Now, let L be any a-factorable linear operator from $L^2(\mathbb{R})$ to $L^p([0, a])$, and let $E = \ker L$. If $f \in E$, and $\phi \in L^\infty_a(\mathbb{R})$, then $L(\phi f) = \phi L(f) = 0$. So $\phi f \in E$. We summarize this below.

Proposition 5.6. *If L is any a-factorable linear operator with kernel E, then E is an a-periodic closed set and so $E^\perp = E^{\perp_a}$.*

One more property of a-factorable operators into $L^2[0, a]$ is that the operator is bounded pointwise with respect to the a-norm.

Proposition 5.7. *Let L be an a-factorable linear operator from $L^2(\mathbb{R})$ to $L^2[0, a]$. Then L is bounded if and only if there is a constant $B > 0$ ($B = \|L\|$) so that for all $f \in L^2(\mathbb{R})$ we have*

$$|L(f)(t)| \le B\|f\|_a(t), \quad \text{for a.e.} \;\; t \in [0, a].$$

Moreover, L is an isomorphism if and only if there are constants $A, B > 0$ ($A = \|L^{-1}\|^{-1}$, $B = \|L\|$) so that for all $f \in L^2(\mathbb{R})$ we have

$$A\|f\|_a(t) \le |L(f)(t)| \le B\|f\|_a(t), \quad \text{for a.e.} \;\; t \in [0, a].$$

Proof. For any bounded a-periodic function ϕ on $\mathbb{R}$, and for every $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned} \int_0^a |\phi(t)|^2 |L(f)(t)|^2 |dt &= \int_0^a |L(\phi f)(t)|^2 \, dt = \|L(\phi f)\|^2_{L^2([0,a]} \\ &\le \|L\|^2 \|\phi f\|^2_{L^2(\mathbb{R})} = \|L\|^2 \int_{\mathbb{R}} |\phi(t)|^2 |f(t)|^2 \, dt \\ &= \|L\|^2 \int_0^a |\phi(t)|^2 \|f\|_a^2(t) \, dt. \end{aligned}$$

It follows immediately that

$$|L(f)(t)|^2 \le \|L\|^2 \|f\|_a^2(t), \quad \text{for a.e.} \quad t \in [0, a].$$

The other implication is similar, as is the "moreover" part of the Proposition. □

This gives us another Riesz Representation Theorem for operators from $L^2(\mathbb{R})$ to $L^2[0, a]$.

Theorem 5.8. *(Riesz Representation 2) L is a bounded a-factorable operator from $L^2(\mathbb{R})$ to $L^2[0,a]$ iff there is a $g \in L_a^\infty(\mathbb{R})$ so that $L(f) = \langle f, g\rangle_a$ for all $f \in L^2(\mathbb{R})$. Moreover $\|L\|^2 = \text{ess sup}_{[0,a]} \langle g, g\rangle_a$.*

Proof. $\Leftarrow$ Let g be in $L_a^\infty(\mathbb{R})$ and define L to be $L(f) = \langle f, g\rangle_a$. The rest follows directly from the first part of the proof of Proposition 4.12 and again, letting $g = f$ above gives the norm of the operator.

$\Rightarrow$ Let L be a bounded a-factorable operator from $L^2(\mathbb{R})$ to $L^2[0,a]$. Since $L^2[0,a] \subset L^1[0,a]$ it is clear from Theorem 5.5 that there exists $g \in L^2(\mathbb{R})$ so that $L(f) = \langle f, g, \rangle_a (t)$. By Proposition 5.7 we get

$$|\langle g, g\rangle_a (t)| = \|g\|_a^2(t) = |L(g)(t)| \leq \|g\|_a(t)\|L\|.$$

Hence $\|g\|_a(t) \leq \|L\|$ a.e. and $g \in L_a^\infty(\mathbb{R})$ □

We have a corresponding a-norm bound for a-factorable operators into $L^2(\mathbb{R})$.

Proposition 5.9. *If $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an a-factorable operator then L is bounded if and only if there is a constant $B > 0$ ($A = \|L\|$) so that for all $f \in L^2(\mathbb{R})$ we have*

$$\|Lf\|_a(t) \leq B\|f\|_a(t), \quad \text{for a.e. } t \in [0,a].$$

Moreover, L is an isomorphism if and only if there are constants $A, B > 0$ ($A = \|L^{-1}\|^{-1},\ B = \|L\|$) so that for all $f \in L^2(\mathbb{R})$ we have

$$A\|f\|_a(t) \leq \|L(f)(t)\| \leq B\|f\|_a(t), \quad \text{for all } t \in \mathbb{R}.$$

Proof. For any $f \in L^2(\mathbb{R})$ and any bounded a-periodic function ϕ we compute

$$\begin{aligned}
\|L(\phi f)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |L(\phi f)(t)|^2 \, dt = \int_{\mathbb{R}} |\phi(t)|^2 |(Lf)(t)|^2 \, dt \\
&= \int_0^a |\phi(t)|^2 \sum_{n\in\mathbb{Z}} |(Lf)(t-na)|^2 \, dt = \int_0^a |\phi(t)|^2 \|Lf\|_a^2(t) \, dt \\
&\leq \|L\|^2 \|\phi f\|_{L^2(\mathbb{R})}^2 = \|L\|^2 \int_{\mathbb{R}} |\phi(t)|^2 |f(t)|^2 \, dt \\
&= \|L\|^2 \int_0^a |\phi(t)|^2 \|f\|_a^2(t) \, dt.
\end{aligned}$$

It follows that

$$\|Lf\|_a^2(t) \leq \|L\|^2 \|f\|_a^2(t), \quad \text{a.e.}$$

The rest of the proposition follows similarly. □

Note that Proposition 5.9 shows that a-factorable operators must map a-bounded functions to a-bounded functions. We end this section by verifying that for a-factorable operators L, the operator L^* behaves as it should relative to the a-inner product.

Proposition 5.10. *If L is an a-factorable operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, then for all $f, g \in L^2(\mathbb{R})$ we have*

$$\langle L(f), g\rangle_a (t) = \langle f, L^*(g)\rangle_a (t).$$

Proof. Consider the operator $\mathcal{L}(f) = \langle L(f), g\rangle_a (t)$ and $\mathcal{L}^*(f) = \langle f, L^*(g)\rangle_a (t)$. Both of these are a-factorable operators from $L^2(\mathbb{R}) \to L^1[0, a]$. Also,

$$\int_0^a \mathcal{L}(f)(t)dt = \langle L(f), g\rangle = \langle f, L^*(g)\rangle = \int_0^a \mathcal{L}^*(f)(t)dt.$$

We are done by Lemma 5.4. □

6. Weyl-Heisenberg frames and the a-inner product

Now we apply our a-inner product theory to Weyl-Heisenberg frames. This will produce **compression representations** of the various operators associated with frames. We call these **compressions** because they no longer have the modulation parameter explicitly represented. That is, we are compressing the modulation parameter into the $1/b$-inner product. We will also relate our results to the Ron-Shen Theory [13, 14]. An excellent accessible account of this theory (and much more) can be found in Janssen's article [11]. This treatment is done for general shift-invariant systems using only basic facts from Fourier analysis and Lebesgue integration. This is then applied to Weyl-Heisenberg systems including representations of the frame operator as well as representations and classifications of the dual systems.

For any WH-frame (g, a, b), it is well known that the frame operator S commutes with E_{mb}, T_{na}. Thus, Corollary 5.3 yields:

Corollary 6.1. *If (g, a, b) is a WH-frame, then the frame operator S is a $\frac{1}{b}$-factorable operator.*

We next show that the WH-frame Identity for (g, a, b) has an interesting representation in both the a and the $\frac{1}{b}$ inner products. The known WH-frame identity requires that the function f be bounded and of compact support. While this remains a condition for the WH-frame Identity derived from the a-inner product we are able to extend this result to all $f \in L^2(\mathbb{R})$ when we use the $\frac{1}{b}$-inner product. For this reason we present the theorems separately.

The proof of both these theorems have their roots in the Heil and Walnut proof of the WH-frame Identity (see [9], Theorem 4.1.5). We refer the reader to Proposition 3.1 and Corollary 3.2 for questions concerning convergence of the series and integrals below.

Theorem 6.2. *Let $g \in L_a^\infty(\mathbb{R})$, and $a, b \in \mathbb{R}^+$. For all $f \in L^2(\mathbb{R})$ which are bounded and compactly supported we have*

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 = b^{-1}\sum_k \int_0^a \left\langle T_{\frac{k}{b}}f, f\right\rangle_a \left\langle g, T_{\frac{k}{b}}g\right\rangle_a dt.$$

Proof. We start with the WH-frame Identity realizing that $\left\langle g, T_{\frac{k}{b}}g\right\rangle_a$ is a-periodic. Hence

$$\begin{aligned}(6.1)\qquad & \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \\ &= b^{-1}\sum_k \int_{\mathbb{R}} \overline{f(t)} f(t-\frac{k}{b}) \sum_n g(t-na)\overline{g(t-na-\frac{k}{b})}dt \\ &= b^{-1}\sum_k \sum_j \int_0^a \overline{f(t-ja)} f(t-\frac{k}{b}-ja) \left\langle g, T_{\frac{k}{b}}g\right\rangle_a dt \\ &= b^{-1}\sum_k \int_0^a \left\langle T_{\frac{k}{b}}f, f\right\rangle_a \left\langle g, T_{\frac{k}{b}}g\right\rangle_a dt\end{aligned}$$

□

For the rest of this section we concentrate on the $\frac{1}{b}$ inner product and its relationship to WH-frames. In a forthcoming paper on the WH-frame identity we show that one may relax the condition on g. That is, the original WH-frame identity holds for all $g \in L^2(\mathbb{R})$ when f is bounded and compactly supported.

Theorem 6.3. *Let $g \in L_a^\infty(\mathbb{R})$, and $a, b \in \mathbb{R}^+$. For all $f \in L^2(\mathbb{R})$ we let $f_n = \langle f, T_{na}g\rangle_{\frac{1}{b}}$ then*

$$\sum_{m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 = \|\langle f, T_{na}g\rangle_{\frac{1}{b}}\|^2_{L^2[0,\frac{1}{b}]} = \|f_n\|^2_{L^2[0,\frac{1}{b}]},$$

and so

$$\sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 = \sum_{n\in\mathbb{Z}} \|\langle f, T_{na}g\rangle_{\frac{1}{b}}\|^2_{L^2[0,\frac{1}{b}]} = \sum_{n\in\mathbb{Z}} \|f_n\|^2_{L^2[0,\frac{1}{b}]}.$$

Proof. Since $g \in L_a^\infty(\mathbb{R})$ we know each $f_n \in L^2[0, \frac{1}{b}]$ and now we just compute

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 &= \sum_{m\in\mathbb{Z}} |\int_{\mathbb{R}} f(t)\overline{g(t-na)}e^{-2\pi imbt}dt|^2 \\
&= b^{-1}\sum_{m\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} \int_0^{\frac{1}{b}} f(t-\frac{k}{b})\overline{g(t-na-\frac{k}{b})}e^{-2\pi imbt}dt|^2 \\
&= b^{-1}\sum_{m\in\mathbb{Z}} |\int_0^{\frac{1}{b}} \langle f, T_{na}g\rangle_{\frac{1}{b}}(t)e^{-2\pi imbt}dt|^2 \\
&= \sum_{m\in\mathbb{Z}} |\widehat{f_n}(m)|^2\ dt = \|f_n\|^2_{L^2[0,\frac{1}{b}]} = \|\langle f, T_{na}g\rangle_{\frac{1}{b}}\|^2_{L^2[0,\frac{1}{b}]}.
\end{aligned}$$

□

Now we want to directly relate our a-inner product to WH-frames. We begin with the compression we referred to above.

Proposition 6.4. *If $g, h \in L^\infty_{\frac{1}{b}}(\mathbb{R})$, then for all $f \in L^2(\mathbb{R})$ we have*

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}g\rangle E_{mb}h = \frac{1}{b}\langle f, g\rangle_{\frac{1}{b}} h,$$

where the series converges unconditionally in $L^2(\mathbb{R})$. Hence,
$\langle f, g\rangle_{\frac{1}{b}} g \in \overline{\text{span}}\ (E_{mb}g)_{m\in\mathbb{Z}}$.

Proof. By our second Riesz Representation Theorem 5.8 we know that $\langle f, g\rangle_{\frac{1}{b}} \in L^2[0, \frac{1}{b}]$. Next, for any $m \in \mathbb{Z}$ we have

$$\langle f, E_{mb}g\rangle = \int_0^{\frac{1}{b}} \langle f, g\rangle_{\frac{1}{b}}(t)e^{-2\pi imbt}\ dt = \widehat{\langle f, g\rangle}_{\frac{1}{b}}(mb).$$

Therefore, if we restrict ourselves to $L^2[0, \frac{1}{b}]$ we have

$$\sum_{m\in\mathbb{Z}} \langle f, E_{mb}g\rangle E_{mb} = \sum_{m\in\mathbb{Z}} \widehat{\langle f, g\rangle}(mb)e^{2\pi imbt} = \frac{1}{b}\langle f, g\rangle_{\frac{1}{b}}.$$

Now by 4.12 we have $\frac{1}{b}\langle f, g\rangle_{\frac{1}{b}} h \in L^2(\mathbb{R})$ □

There are many interesting consequences of this proposition. First we recapture the following result due to de Boor, DeVore, and Ron [1]

Corollary 6.5. *For $g \in L^2(\mathbb{R})$ and $b \in \mathbb{R}$, the orthogonal projection P of $L^2(\mathbb{R})$ onto $\overline{\text{span}}\ (E_{mb}g)_{m\in\mathbb{Z}}$ is*

$$Pf = \frac{1}{\|g\|^2_{\frac{1}{b}}}\langle f, g\rangle_{\frac{1}{b}}\ g,$$

where if $\|g\|_{\frac{1}{b}}(t) = 0$ *then* $g(t) = 0$ *so we interpret* $\frac{g(t)}{\|g\|^2_{\frac{1}{b}}(t)} = 0$.

Proof. By Proposition 4.9, we have that $(\sqrt{b}E_{mb}\frac{g}{\|g\|_{\frac{1}{b}}})_{m\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$. Hence, for all $f \in L^2(\mathbb{R})$ we have by Proposition 6.4

$$\begin{aligned} Pf &= \sum_{m\in\mathbb{Z}} \left\langle f, \sqrt{b}E_{mb}\frac{g}{\|g\|_{\frac{1}{b}}} \right\rangle \sqrt{b}E_{mb}\frac{g}{\|g\|_{\frac{1}{b}}} \\ &= b\sum_{m\in\mathbb{Z}} \left\langle f, E_{mb}\frac{g}{\|g\|_{\frac{1}{b}}} \right\rangle E_{mb}\frac{g}{\|g\|_{\frac{1}{b}}} = \left\langle f, \frac{g}{\|g\|_{\frac{1}{b}}} \right\rangle_{\frac{1}{b}} \frac{g}{\|g\|_{\frac{1}{b}}} = \frac{1}{\|g\|^2_{\frac{1}{b}}} \langle f, g\rangle_{\frac{1}{b}} \; g. \end{aligned}$$

□

Combining Theorem 4.13 and Corollary 6.5 we have:

Proposition 6.6. *If* $(g_n)_{n\in\mathbb{Z}}$ *is a* $\frac{1}{b}$*-orthonormal sequence in* $L^2(\mathbb{R})$*, then*

$$P(f) = \sum_{n\in\mathbb{Z}} \langle f, g_n\rangle_{\frac{1}{b}} g_n,$$

is the orthogonal projection of $L^2(\mathbb{R})$ *onto span* $(E_{mb}g_n)_{n,m\in\mathbb{Z}}$

This allows us to compress the operators associated with a WH-system (g, a, b). In [13] and [14], Ron and Shen make use of the Dual Gramian to analyze a WH-system $(g, 1, b)$. This will correspond to a compression of the frame operator of the system $(\hat{g}, b, 1)$. Here we produce similar results for the frame operator, preframe operator and frame transform in the space domain.

Proposition 6.7. *If* $\|g\|_{\frac{1}{b}} \leq B$ *a.e. then the frame transform and the preframe operator have the following compressions*

$$T(f) = \sqrt{\frac{1}{b}} \sum_n \langle f, e_n\rangle_{\frac{1}{b}} T_{na}g \text{ and } T^*(f) = \sqrt{\frac{1}{b}} \sum_n \langle f, T_{na}g\rangle_{\frac{1}{b}} e_n$$

where $e_n = T_{\frac{n}{b}}\mathbf{1}_{[0,\frac{1}{b})}$*, the standard* $\frac{1}{b}$*-orthonormal basis.*

Proof. If $\|g\|_{\frac{1}{b}} \leq B$, by Proposition 6.4 we have

$$T^*(f) = \sum_{m,n} \langle f, E_{mb}T_{na}g\rangle \sqrt{b}E_{mb}T_{\frac{n}{b}}(\mathbf{1}_{[0,\frac{1}{b}]}) = \sqrt{\frac{1}{b}} \sum_n \langle f, T_{na}g\rangle_{\frac{1}{b}} e_n,$$

where we used the fact that $(\sqrt{b}E_{mb}T_{\frac{n}{b}}(\mathbf{1}_{[0,\frac{1}{b}]}))$ is an orthonormal basis for $L^2(\mathbb{R})$. Hence T^* must be $\frac{1}{b}$ factorable which in turn implies T is. Continuing we get

$$T^*(f) = \sum_n \langle T^*(f), e_n\rangle_{\frac{1}{b}} e_n = \sum_n \langle f, T(e_n)\rangle_{\frac{1}{b}} e_n$$

So $T(e_n) = \sqrt{\frac{1}{b}} T_{na} g$ and the rest follows.

□

Theorem 6.8. *If (g, a, b) is* **PF** *with frame operator S, then S has the form*

$$S(f) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na} g \rangle_{\frac{1}{b}} T_{na} g = \frac{1}{b} \sum_{n \in \mathbb{Z}} P_n f \cdot T_{na} \|g\|^2_{\frac{1}{b}},$$

where P_n is the orthogonal projection of $L^2(\mathbb{R})$ onto span $(E_{mb} T_{na} g)_{m \in \mathbb{Z}}$ and the series converges unconditionally in $L^2(\mathbb{R})$.

Proof. If (g, a, b) is a WH-frame then by Proposition 2.7 we have that $\langle g, g \rangle_{\frac{1}{b}} \leq B$ a.e. Now, by definition of the frame operator S we have

$$\begin{aligned} S(f) &= \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} g \right) \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \langle f, T_{na} g \rangle_{\frac{1}{b}} \; T_{na} g. \end{aligned}$$

An application of Corollary 6.5 and Theorem 3.4 (10) completes the proof. □

We summarize some of the known results about normalized tight WH-frames in the the following Proposition. These results are due to various authors. Direct proofs from the definitions as well as a historical development may be found in [3].

Proposition 6.9. *Let (g, a, b) be a WH-frame. the following are equivalent:*
(1) $(E_{mb} T_{na} g)_{n,m \in \mathbb{Z}}$ is a normalized tight Weyl-Heisenberg frame.
(2) $(\frac{1}{\sqrt{b}} T_{\frac{n}{b}} g)_{n \in \mathbb{Z}}$ is an orthonormal sequence in the a-inner product.
(3) We have that $g \perp_a T_{\frac{k}{b}} g$, for all $k \neq 0$ and $\langle g, g \rangle_a = b$ a.e.

Putting Corollary 6.6 and Proposition 6.9 together we have

Corollary 6.10. *If (g, a, b) is a normalized tight Weyl-Heisenberg frame, then*

$$P(f) = \frac{1}{b} \sum_{n \in \mathbb{Z}} \left\langle f, T_{\frac{n}{b}} g \right\rangle_a T_{\frac{n}{b}} g$$

is the orthogonal projection of $L^2(\mathbb{R})$ onto span $(E_{\frac{m}{a}} T_{\frac{n}{b}} g)_{n,m \in \mathbb{Z}}$.

Given the compressed representation of the frame operator it is now natural to examine the notion of a frame and a Riesz basis with respect to a pointwise inner product.

Definition 6.11. *We say that a sequence $f_n \in L^2(\mathbb{R})$ is a* a**-Riesz basic sequence** *if there is an a-orthonormal basis $(g_n)_{n\in\mathbb{Z}}$ and an a-factorable operator L on $L^2(\mathbb{R})$ with $L(g_n) = f_n$ so that L is invertible on its range. If L is surjective, we call (f_n) a* a**-Riesz basis** *for $L^2(\mathbb{R})$.*

Proposition 6.12. *Let $f_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*
(1) $(f_n)_{n\in\mathbb{Z}}$ is an a-Riesz basic sequence.
(2) $(E_{\frac{m}{a}} f_n)_{n\in\mathbb{Z}}$ is a Riesz basic sequence.

Proof. (1) $\Rightarrow$ (2): By assumption, there is an a-orthonormal basis (g_n) and an a-factorable operator L with $L(g_n) = f_n$ for all $n \in \mathbb{Z}$. By the definition of an a-orthonormal basis we have that $(\frac{1}{\sqrt{a}} E_{\frac{m}{a}} g_n)_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. Since L is an isomorphism, it follows that

$$(L(\frac{1}{\sqrt{a}} E_{\frac{m}{a}} g_n))_{n,m\in\mathbb{Z}} = (\frac{1}{\sqrt{a}} E_{\frac{m}{a}} L(g_n))_{n,m\in\mathbb{Z}} = (\frac{1}{\sqrt{a}} E_{\frac{m}{a}} f_n)_{n,m\in\mathbb{Z}}$$

is a Riesz basic sequence.
(2) $\Rightarrow$ (1): Let $g = \chi_{[0,a)}$ so that $(\frac{1}{\sqrt{a}} E_{\frac{m}{a}} T_{na} g)_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$. Then

$$L(\frac{1}{\sqrt{a}} E_{\frac{m}{a}} T_{na} g) = E_{\frac{m}{a}} f_n$$

is an a-factorable linear operator which is an isomorphism because $(E_{\frac{m}{a}} f_n)$ is a Riesz basic sequence. Hence, (f_n) is an a-Riesz basic sequence. ☐

Corollary 6.13. *Let $f_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*
(1) $(f_n)_{n\in\mathbb{Z}}$ is an a-Riesz basis.
(2) $(E_{\frac{m}{a}} f_n)_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$.

Since the inner product on a Hilbert space is used to define a frame, we can get a corresponding concept for the a-inner product.

Definition 6.14. *If $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$, we call $(g_n)_{n\in\mathbb{Z}}$ an* a**-frame sequence** *if there exist constants $A, B > 0$ so that for all $f \in$ span $(E_{\frac{m}{a}} g_n)_{m,n\in\mathbb{Z}}$ we have*

$$A\|f\|_a^2(t) \le \sum_{n\in\mathbb{Z}} |\langle f, g_n\rangle_a (t)|^2 \le B\|f\|_a^2(t).$$

If the inequality above holds for all $f \in L^2(\mathbb{R})$ then we call $(g_n)_{n\in\mathbb{Z}}$ an a**-frame**.

Now we have the corresponding result to Theorem 2.2.

Theorem 6.15. *Let $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*
(1) $(g_n)_{n\in\mathbb{Z}}$ is an a-frame.
(2) If $(e_n)_{n\in\mathbb{Z}}$ is an a-orthonormal basis for $L^2(\mathbb{R})$, and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $T(e_n) = g_n$ is a-factorable, then T is a bounded, linear surjective operator on $L^2(\mathbb{R})$.

Proof. If $T(e_n) = g_n$, then

$$\langle T^*(f), e_n\rangle_a = \langle f, T(e_n)\rangle_a = \langle f, g_n\rangle_a .$$

Hence, by Theorem 4.13 we have that $T^*(f) = \sum_{n\in\mathbb{Z}} \langle f, g_n\rangle_a e_n$ and

$$\|T^*(f)\|_a^2 = \sum_{n\in\mathbb{Z}} |\langle f, g_n\rangle_a|^2.$$

Hence, (g_n) is an a-frame sequence if and only if

$$A\|f\|_a^2(t) \le \|T^*(f)\|_a^2(t) \le B\|f\|_a^2(t), \quad \text{for all} \quad f \in L^2(\mathbb{R}).$$

But, by Proposition 5.9, this is equivalent to T^* being an isomorphism, which itself is equivalent to T being a bounded, linear onto operator. □

Finally, we can relate this back to our regular frame sequences.

Proposition 6.16. *Let $g_n \in L^2(\mathbb{R})$, for all $n \in \mathbb{Z}$. The following are equivalent:*
(1) $(g_n)_{n\in\mathbb{Z}}$ is an a-frame sequence.
(2) $(E_{\frac{m}{a}} g_n)_{m,n\in\mathbb{Z}}$ is a frame sequence.

Proof. (1) ⇒ (2): If (g_n) is an a-frame sequence, then there is an a-orthonormal basis (e_n) for $L^2(\mathbb{R})$ and an a-factorable onto (closed range) operator $T(e_n) = g_n$. Now, $(E_{\frac{m}{a}} e_n)_{n,m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ and

$$T(E_{\frac{m}{a}} e_n) = E_{\frac{m}{a}} T(e_n) = E_{\frac{m}{a}} g_n.$$

Hence, $(E_{\frac{m}{a}} g_n)_{m,n\in\mathbb{Z}}$ is a frame sequence.
(2) ⇒ (1): Reverse the steps in part I above. □

The following Corollary is immediate from Theorem 6.15 and Proposition 6.16.

Corollary 6.17. *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. The following are equivalent:*
(1) (g, a) is a $\frac{1}{b}$-frame.
(2) (g, a, b) is a Weyl-Heisenberg frame.

We conclude this paper with the following remark about some ongoing research of Michael Frank and the two authors. One can show that the space $L_a^\infty(\mathbb{R})$ may be viewed as the Lebesgue space $L^\infty(\ell_2)$. If $f, g \in L^\infty{}_a(R)$ then $\langle f, g\rangle_a (t) \in L^\infty[0, a]$ instead of $L^1[0, a]$. This dense subspace of $L^2(\mathbb{R})$ equipped with the a-inner product can be viewed as a Hilbert C^*-Module where the range space of the a inner product is now the C^*-Algebra $L^\infty[0, a]$. The corollary above shows that there is a strong connection between "frames" of translates for this Hilbert C^*-Module and WH-frames. This opens the door for passing results back and forth between WH-systems and certain Hilbert C^*-modules.

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