Contemporary Mathematics

Classes of Finite Equal Norm Parseval Frames

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ABSTRACT. Finite equal norm Parseval frames are a fundamental tool in applications of Hilbert space frame theory. We will derive classes of finite equal norm Parseval frames for use in applications as well as reviewing the status of the currently known classes.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [**20**] in 1952 to study some deep problems in nonharmonic Fourier series. Duffin and Schaeffer abstracted the fundamental notion of Gabor [**22**] for signal processing. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmannn, and Meyer [**19**] in 1986. After this ground breaking work the theory of frames began to be widely studied.

Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector [**14**]. It is this redundancy that makes frames useful in applications. Frames have traditionally been used in signal processing because of their resilience to additive noise [**18**], resilience to quantization [**24**], as well as their numerical stability of reconstruction [**18**], and their greater freedom to capture important signal characteristics [**14**]. Today, frames play an important role in many applications in mathematics, science, and engineering. Some of these applications include time-frequency analysis [**25**], internet coding [**23**], speech and music processing [**39**], communication [**35**], medicine [**36**], quantum computing [**30**], and many other areas.

Applications generally use equal norm Parseval frames (see Section 2 for the definitions) because of the rapid reconstruction of vectors in addition to giving somewhat equal weight to each vector in the space. Unfortunately, there are a small number of examples of these frames and, far worse, there is no place to go to find the known classes of equal norm Parseval frames. In this paper we will derive several new classes of equal norm Parseval frames as well as reviewing the current known classes.

1991 *Mathematics Subject Classification.* Primary: 42C15, 46C99.
The first author was supported by NSF DMS 0405376.

2. An introduction to Parseval frames

We will denote by ℓ_2^N an N-dimensional real or complex Hilbert space while $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) will denote an N-dimensional real (respectively, complex) Hilbert space. If the result holds also for infinite dimensional Hilbert spaces, we will denote this $\ell_2(I)$ where I may be a finite or infinite index set.

A family of vectors $\{f_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a **Riesz basic sequence** if there are constants $A, B > 0$ so that for all scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I}a_if_i\|^2 \le B\sum_{i\in I}|a_i|^2. \tag{1}$$

We call A, B the **lower and upper Riesz basis bounds** for $\{f_i\}_{i\in I}$. If the Riesz basic sequence $\{f_i\}_{i\in I}$ spans $\mathbb{H}$ we call it a **Riesz basis** for $\mathbb{H}$. So $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ means there is an orthonormal basis $\{e_i\}_{i\in I}$ so that the operator $T(e_i) = f_i$ is invertible. In particular, each Riesz basis is **bounded**. That is, $0 < \inf_{i\in I}\|f_i\| \le \sup_{i\in I}\|f_i\| < \infty$.

Hilbert space frames were introduced by Duffin and Schaeffer [**20**] to address some very deep problems in nonharmonic Fourier series (see [**40**]). A family $\{f_i\}_{i\in I}$ of elements of a (finite or infinite dimensional) Hilbert space $\mathbb{H}$ is called a **frame** for $\mathbb{H}$ if there are constants $0 < A \le B < \infty$ (called the **lower and upper frame bounds**, respectively) so that for all $f \in \mathbb{H}$

$$A\|f\|^2 \le \sum_{i\in I}|\langle f, f_i\rangle|^2 \le B\|f\|^2. \tag{2}$$

A good introduction to frames and Riesz bases is [**14**]. If we only have the right hand inequality in Equation 2 we call $\{f_i\}_{i\in I}$ a **Bessel sequence with Bessel bound B**. If $A = B$, we call this an A**-tight frame** and if $A = B = 1$, it is called a **Parseval frame**. If all the frame elements have the same norm, this is an **equal norm** frame and if the frame elements are of unit norm, it is a **unit norm frame**. It is immediate that $\|f_i\|^2 \le B$. If also $\inf\|f_i\| > 0$, $\{f_i\}_{i\in I}$ is a **bounded frame**. The numbers $\{\langle f, f_i\rangle\}_{i\in I}$ are the **frame coefficients** of the vector $f \in \mathbb{H}$. If $\{f_i\}_{i\in I}$ is a Bessel sequence, the **synthesis operator** for $\{f_i\}_{i\in I}$ is the bounded linear operator $T : \ell_2(I) \to \mathbb{H}$ given by $T(e_i) = f_i$ for all $i \in I$. The **analysis operator** for $\{f_i\}_{i\in I}$ is T^* and satisfies: $T^*(f) = \sum_{i\in I}\langle f, f_i\rangle e_i$. In particular,

$$\|T^*f\|^2 = \sum_{i\in I}|\langle f, f_i\rangle|^2, \quad \text{for all } f \in \mathbb{H}, \tag{3}$$

and hence the smallest Bessel bound for $\{f_i\}_{i\in I}$ equals $\|T^*\|^2$. Comparing this to Equation 2 we have:

Theorem 2.1. *Let $\mathbb{H}$ be a Hilbert space and $T : \ell_2(I) \to \mathbb{H}$, $Te_i = f_i$ be a bounded linear operator. The following are equivalent:*

(1) *$\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$.*
(2) *The operator T is bounded, linear, and onto.*
(3) *The operator T^* is an (possibly into) isomorphism.*

Moreover, if $\{f_i\}_{i\in I}$ is a Riesz basis, then the Riesz basis bounds are A, B, the frame bounds for $\{f_i\}_{i\in I}$.

It follows that a Bessel sequence is a Riesz basic sequence if and only if T^* is onto. The **frame operator** for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathbb{H} \to \mathbb{H}$. That is,

$$Sf = TT^*f = T\left(\sum_{i\in I}\langle f, f_i\rangle e_i\right) = \sum_{i\in I}\langle f, f_i\rangle Te_i = \sum_{i\in I}\langle f, f_i\rangle f_i. \tag{4}$$

In particular,

$$\langle Sf, f\rangle = \sum_{i\in I}|\langle f, f_i\rangle|^2. \tag{5}$$

It follows that $\{f_i\}_{i\in I}$ is a frame with frame bounds A, B if and only if $A\cdot I \le S \le B\cdot I$. So $\{f_i\}_{i\in I}$ is a Parseval frame if and only if $S = I$. **Reconstruction** of vectors in $\mathbb{H}$ is achieved via the formula:

$$\begin{aligned} f &= SS^{-1}f = \sum_{i\in I}\langle S^{-1}f, f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i \\ &= \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i \end{aligned}$$

$$= \sum_{i\in I}\langle f, S^{-1/2}f_i\rangle S^{-1/2}f_i. \tag{6}$$

It follows that $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame *equivalent* to $\{f_i\}_{i\in I}$. Two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in a Hilbert space are *equivalent* if there is an invertible operator T between their spans with $Tf_i = g_i$ for all $i \in I$.

Remark 2.2. *Any finite set of vectors $\{f_i\}_{i=1}^M$ in a Hilbert space $\mathbb{H}$ has a frame operator $Sf = \sum_{i=1}^M\langle f, f_i\rangle f_i$ associated with it. S is a positive and self-adjoint operator but is not invertible unless $\{f_i\}_{i=1}^M$ spans $\mathbb{H}$.*

Proposition 2.3. *Let $\{f_i\}_{i=1}^M$ be a frame for ℓ_2^N. If $\{g_j\}_{j=1}^N$ is an orthonormal basis of ℓ_2^N consisting of eigenvectors for the frame operator S with respective eigenvalues $\{\lambda_j\}_{j=1}^N$, then for every $1 \le j \le N$, $\sum_{i=1}^M|\langle f_i, g_j\rangle|^2 = \lambda_j$. In particular, $\sum_{i=1}^M\|f_i\|^2 =$ Trace S ($= N$ if $\{f_i\}_{i=1}^M$ is a Parseval frame). Furthermore, if $\{f_i\}_{i\in I}$ is an equal norm Parseval frame for ℓ_2^N then $\|f_i\|^2 = \frac{N}{M}$.*

Another important result is

Theorem 2.4. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds A, B and P is any orthogonal projection on $\mathbb{H}$, then $\{Pf_i\}_{i\in I}$ is a frame for $P\mathbb{H}$ with frame bounds A, B.*

Proof: For any $f \in P(\mathbb{H})$,

$$\sum_{i\in I}|\langle f, Pf_i\rangle|^2 = \sum_{i\in I}|\langle Pf, f_i\rangle|^2 = \sum_{i\in I}|\langle f, f_i\rangle|^2.$$

□

A fundamental result in frame theory was proved independently by Naimark and Han/Larson [**14, 26**]. For completeness we include its simple proof.

THEOREM 2.5. *A family $\{f_i\}_{i\in I}$ is a Parseval frame for a Hilbert space $\mathbb{H}$ if and only if there is a containing Hilbert space $\mathbb{H} \subset \ell_2(I)$ with an orthonormal basis $\{e_i\}_{i\in I}$ so that the orthogonal projection P of $\ell_2(I)$ onto $\mathbb{H}$ satisfies $P(e_i) = f_i$ for all $i \in I$.*

Proof: The "only if" part is Theorem 2.4. For the "if" part, if $\{f_i\}_{i\in I}$ is a Parseval frame, then the synthesis operator $T : \ell_2(I) \to \mathbb{H}$ is a partial isometry. So T^* is an isometry and we can associate $\mathbb{H}$ with $T^*\mathbb{H}$. Now, for all $i \in I$ and all $g = T^*f \in T^*(\mathbb{H})$ we have

$$\langle T^*f, Pe_i\rangle = \langle T^*f, e_i\rangle = \langle f, Te_i\rangle = \langle f, f_i\rangle = \langle T^*f, T^*f_i\rangle.$$

It follows that $Pe_i = T^*f_i$ for all $i \in I$. □

Theorem 2.5 helps explain why so few classes of equal norm Parseval frames are known. Namely, to get an equal norm Parseval frame we need to find orthogonal projections which map an orthonormal basis to equal norm vectors. There is very little known about such projections, consequently there lies the challenge. There is a universal method for obtaining Parseval frames given in the next lemma (See [**11**]).

LEMMA 2.6. *There is a unique method for constructing Parseval frames in ℓ_2^N. Let U be an $M \times M$, $M \geq N$, unitary matrix,*

$$U = \begin{bmatrix} u_{11} & . & . & . & u_{1M} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ u_{M1} & . & . & . & u_{MM} \end{bmatrix}.$$

Define

$$\begin{bmatrix} \varphi_1 \\ . \\ . \\ . \\ \varphi_M \end{bmatrix} = \begin{bmatrix} u_{11} & . & . & . & u_{1N} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ u_{M1} & . & . & . & u_{MN} \end{bmatrix}, \quad N \leq M.$$

The rows $\{\varphi_i\}_{i=1}^M$ form a Parseval frame for ℓ_2^N.

Another important property of frames comes from [**13**].

THEOREM 2.7. *A family of vectors $\{f_i\}_{i\in I}$ is a frame with frame bounds A and B if and only if forming a matrix C with the $f_i's$ as row vectors, the corresponding column vectors of C form a Riesz basic sequence with Riesz basis bounds A, B.*

There is a classification of the sequence of norms of frame vectors which yield a given frame operator due to Casazza and Leon [**12**]. This result can also be derived from the Schur-Horn Theorem [**2**].

THEOREM 2.8. *Let S be a positive self-adjoint operator on an N-dimensional Hilbert space l_2^N. Let $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_N$ be eigenvalues of S. Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq ... \geq a_M \geq 0$. The following are equivalent:*

(1) There is a frame $\{f_i\}_{i=1}^M$ for l_2^N with frame operator S and $\|f_i\| = a_i$ for all $i = 1, 2, \cdots, M$.

(2) For every $1 \le k \le N$ we have

$$\sum_{j=1}^{k} a_j^2 \le \sum_{j=1}^{k} \lambda_j, \tag{7}$$

and

$$\sum_{j=1}^{M} a_j^2 = \sum_{j=1}^{N} \lambda_j. \tag{8}$$

For some reason the following important corollary of Theorem 2.8 has been overlooked until now.

Corollary 2.9. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space l_2^N. For any $M \ge N$ there is an equal norm sequence $\{f_m\}_{m=1}^M$ in l_2^N which has S as its frame operator.*

Proof. Let $\lambda_1 \ge \lambda_2 \ge \ldots \lambda_N \ge 0$ be the eigenvalues of S. Let

$$a^2 = \frac{1}{M} \sum_{i=1}^{N} \lambda_i. \tag{9}$$

Now we check the conditions of Theorem 2.8 to see that there is a sequence $\{f_i\}_{i=1}^M$ in l_2^N with $\|f_i\| = a$ for all $i = 1, 2, \ldots, M$ and the frame operator of $\{f_i\}_{i=1}^M$ is precisely S. We are letting $a_1 = a_2 = \ldots a_M = a$. For the second equality in Theorem 2.8, by Equation 9 we have

$$\sum_{i=1}^{M} \|f_i\|^2 = \sum_{i=1}^{M} a_i^2 = Ma^2 = \sum_{j=1}^{N} \lambda_j. \tag{10}$$

For the first inequality in Theorem 2.8, we note that by Equation 9 we have that

$$a_1^2 = a^2 = \frac{1}{M} \sum_{j=1}^{N} \lambda_j \le \frac{1}{N} \sum_{j=1}^{N} \lambda_j \le \lambda_1.$$

So our inequality holds for $i = 1$. Suppose there is an $1 < i \le N$ for which this inequality fails and i is the first time this fails. So,

$$\sum_{m=1}^{i-1} a_m^2 = (i-1)a^2 \le \sum_{j=1}^{i-1} \lambda_j,$$

while

$$\sum_{m=1}^{i} a_m^2 = ia^2 > \sum_{j=1}^{i} \lambda_j.$$

It follows that

$$a_i^2 = a^2 > \lambda_i \ge \lambda_{i+1} \ge \lambda_N.$$

Hence,

$$
\begin{aligned}
Ma^2 = \sum_{i=1}^{M} a_i^2 \quad &\geq \quad \sum_{m=1}^{i} a_m^2 + \sum_{m=i+1}^{N} a_m^2 \\
&> \quad \sum_{j=1}^{i} \lambda_j + \sum_{m=i+1}^{N} a_m^2 \\
&\geq \quad \sum_{j=1}^{i} \lambda_j + \sum_{j=i+1}^{N} \lambda_j \\
&= \quad \sum_{j=1}^{N} \lambda_j .
\end{aligned}
$$

But this contradicts Equation 9. □

We give two more important consequences of Theorem 2.8.

Corollary 2.10. *For every $m \geq n$ there is an equal norm Parseval frame for ℓ_2^n containing exactly m-elements.*

Corollary 2.11. *Given an N-dimensional Hilbert space l_2^N and a sequence of positive numbers $\{a_i\}_{i=1}^M$ with $a_1 \geq a_2 \geq \ldots \geq a_M$, there exists a tight frame $\{f_i\}_{i=1}^M$ for l_2^N with $\|f_i\| = a_i$ for all $i = 1, 2, \ldots, M$ if and only if*

$$a_1^2 \leq \frac{1}{N} \sum_{i=1}^{M} a_i^2 .$$

3. Constructing Tight Frames from sets of vectors

This section will address four methods used to construct tight frames from finite sets of vectors. We also discuss each method's advantages and disadvantages.

Method I: Let $\{f_i\}_{i=1}^M$ be a set of norm one vectors in ℓ_2^N. For every $j = 1, ..., M$ let $\{f_{ij}\}_{i=1}^N$ be an orthonormal basis for ℓ_2^N with $f_{1j} = f_j$. The family $\{f_{ij}\}_{i \in \{1,\ldots,N\},\, j \in \{1,\ldots,M\}}$ is an A-tight frame with tight frame bound A=M.

Note: Method I shows that every finite set of vectors is part of a tight frame for a Hilbert space. But, this technique has the disadvantage that the tight frame bound is exceptionally large, i.e $A = M$.

Method II: Let $\{f_i\}_{i=1}^M$ be a set of vectors in ℓ_2^N not all of which are zero. We can add N-1 vectors $\{h_j\}_{j=2}^N$ to the family so that $\{f_i\}_{i=1}^M \cup \{h_j\}_{j=2}^N$ is a tight frame.

Proof of Method II: Let $\{g_j\}_{j=1}^N$ be an eigenbasis for the frame operator of $\{f_i\}_{i=1}^M$ with respective eigenvalues $\{\lambda_j\}_{j=1}^N$, some of which may be zero but one of which must be non-zero. Without loss of generality assume $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_N \geq 0$. For $2 \leq j \leq N$, let $h_j = \sqrt{\lambda_1 - \lambda_j}\, g_j$. If S_1 is the frame operator for $\{f_i\}_{i=1}^M \cup \{h_j\}_{j=2}^N$ then for all $f \in \ell_2^N$

$$\begin{aligned} S_1 f &= \sum_{i=1}^{M} \langle f, f_i \rangle f_i + \sum_{j=2}^{N} \langle f, h_j \rangle h_j \\ &= \sum_{j=1}^{N} \lambda_j \langle f, g_j \rangle g_j + \sum_{j=2}^{N} \sqrt{\lambda_1 - \lambda_j} \langle f, g_j \rangle \sqrt{\lambda_1 - \lambda_j} g_j \\ &= \sum_{j=1}^{N} \lambda_j \langle f, g_j \rangle g_j + \sum_{j=2}^{N} (\lambda_1 - \lambda_j) \langle f, g_j \rangle g_j \\ &= \lambda_1 \langle f, g_1 \rangle g_1 + \sum_{j=2}^{N} \lambda_1 \langle f, g_j \rangle g_j \\ &= \lambda_1 \sum_{j=1}^{N} \langle f, g_j \rangle g_j \\ &= \lambda_1 f \end{aligned}$$

Therefore $\{f_i\}_{i=1}^M \cup \{h_j\}_{j=2}^N$ is a λ_1-tight frame.

Remark: The advantage to this method is that the upper frame bound is the same as the upper frame bound of the original set of vectors and we have to add very few vectors to make the frame tight. However, if the original set consists of equal norm vectors, this method does not ensure that the new frame will be an equal norm frame. In general, even if $\{f_i\}_{i=1}^M$ is an equal norm frame for ℓ_2^N we can't make $\{f_i\}_{i=1}^M \cup \{h_j\}_{j=2}^N$ tight be adding N-1 vectors of the same norm as f_i.

The next method for producing equal norm tight frames comes from [**3**].

Method III: Let $\{f_i\}_{i=1}^M$ be a unit norm Bessel sequence in l_2^N with Bessel bound B. There is a unit norm family $\{g_j\}_{j=1}^K$ so that $\{f_i\}_{i=1}^M \cup \{g_j\}_{j=1}^K$ is a unit norm tight frame with tight frame bound $\lambda \leq B + 2$.

Proof of Method III: This theorem and proof are done in the finite dimensional case. The infinite dimensional case also holds by a similar argument using the results of [**10**]. Let $\{f_i\}_{i=1}^M$ be a unit norm Bessel sequence with Bessel bound B in l_2^N. Let S be the frame operator for this family and let $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_N \geq 0$ be the eigenvalues with respective eigenvectors $\{e_i\}_{i=1}^N$ for S. So B=λ_1. If we consider $N(\lambda_1 + 1 + \epsilon) - M$, we see that this equals $N\lambda_1 + N - M$ if $\epsilon = 0$ and it equals $N\lambda_1 + 2N - M$ If $\epsilon = 1$. In particular, there is an $0 \leq \epsilon \leq 1$ so that $N(\lambda_1 + 1 + \epsilon) - M = K \geq N$ where $K \in \mathbb{N}$. Now let S_0 be the positive self-adjoint operator on $\mathbb{H}_N$ given by

$$S_0 \left(\sum_{j=1}^{N} c_j e_j \right) = \sum_{j=1}^{N} \left[(\lambda_1 + 1 + \epsilon) - \lambda_j \right] c_j e_j. \tag{1}$$

So S_0 is a positive self-adjoint operator on $\mathbb{H}_N$ with eigenvectors $\{e_j\}_{j=1}^N$ having respective eigenvalues $\{\lambda_1 + 1 + \epsilon - \lambda_j\}_{j=1}^N$ (which are now in increasing order).

Since each of these eigenvalues is greater than 1, letting $a_i = 1$ for $i = 1, 2, ..., K$ we immediately have the first inequality given in (2) of Theorem 2.8. Also,

$$\sum_{j=1}^{N} [(\lambda_1 + 1 + \epsilon) - \lambda_j] = N (\lambda_1 + 1 + \epsilon) - \sum_{j=1}^{N} \lambda_j = N (\lambda_1 + 1 + \epsilon) - M = K.$$

The last equality above follows from the fact that

$$\sum_{i=1}^{M} \|f_i\|^2 = M = \sum_{j=1}^{N} \lambda_j.$$

Applying Theorerm 2.8, there is a family of unit norm vectors $\{g_j\}_{j=1}^{K}$ in l_2^N having S_0 for its frame operator. If follows that $\{f_i\}_{i=1}^{M} \cup \{g_j\}_{j=1}^{N}$ is a unit norm frame for l_2^N having frame operator $S + S_0$. But $S + S_0$ has eigenvetors $\{e_j\}_{j=1}^{N}$ with respective eigenvalues

$$[(\lambda_1 + 1 + \epsilon) - \lambda_j] + \lambda_j = \lambda + 1 + \epsilon =: \lambda$$

So our unit norm frame is tight with tight frame bound $\lambda \leq \lambda_1 + 2$. □

For our next method, we first need to recall a standard result.

Proposition 3.1. *If S is a positive, self-adjoint bounded operator on $\ell_2(I)$, then $\{S^{1/2}e_i\}_{i \in I}$ is a sequence of vectors with frame operator S for any orthonormal basis $\{e_i\}_{i \in I}$. In particular, if S is also invertible, then we conclude that there is a Riesz basis for $\mathbb{H}$ having frame operator S.*

Proof. For any $f \in \mathbb{H}$ we have:

$$\begin{aligned} \sum_{i \in I} \langle f, S^{1/2}e_i \rangle S^{1/2}e_i &= S^{1/2} \left(\sum_{i \in I} \langle S^{1/2}f, e_i \rangle e_i \right) \\ &= S^{1/2} \left(S^{1/2} f \right) = Sf. \end{aligned}$$

□

Method IV: Given a Bessel sequence of vectors $\{f_i\}_{i=1}^{M}$ in l_2^N with Bessel bound B and frame operator S, let $\{g_j\}_{j \in J}$ be a family of vectors which has $BI - S$ as its frame operator. Then $\{f_i\}_{i=1}^{M} \cup \{g_j\}_{j \in J}$ is a frame for l_2^N with frame operator $S + (BI - S) = BI$. i.e. This is a tight frame.

Massey and Ruiz [**33**] generalized Theorem 2.8 to the case where we want to add vectors of prescribed norms to a given family of vectors and end up with a tight frame. There are also variations of this result in [**33**] including the infinite dimensional case.

Theorem 3.2 (Massey and Ruiz). *Given vectors $\{f_i\}_{i \in I}$ in $\mathbb{H}_N$ with frame operator S having trace α and eigenvalues $\{\lambda_j\}_{j=1}^{N}$ and a non-increasing sequence $\{a_i\}_{i=1}^{M}$ of positive real numbers, there is a sequence of vectors $\{g_i\}_{i=1}^{M}$ in $\mathbb{H}_N$ with $\|g_i\|^2 = a_i$ and $\{f_i\}_{i \in I} \cup \{g_i\}_{i=1}^{M}$ is a tight frame if and only if*

$$\frac{1}{N} \left(\sum_{i=1}^{M} a_i + \alpha \right) \geq \lambda_1,$$

and

$$\frac{1}{N}\left(\sum_{i=1}^{M} a_i + \alpha\right) \geq \frac{1}{k}\sum(a_i + \lambda_{n-i+1}), \quad 1 \leq k \leq min\{N, M\}.$$

4. Harmonic Frames

In this section we will define three types of harmonic frames and show each type is an equal norm Parseval frame. For results on harmonic frames we refer the reader to [**11, 23, 37, 41**].

4.1. Real Harmonic Frames.

THEOREM 4.1. *The family* $\{\varphi_i\}_{i=0}^{M-1}$ *is an orthonormal basis for* $\mathbb{R}^M$ *where for* $M = 2k+1$

$$(1)\qquad \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos 2\pi\frac{1}{M} & \cos 2\pi\frac{2}{M} & \cdots & \cos 2\pi\frac{(M-1)}{M} \\ 0 & \sin 2\pi\frac{1}{M} & \sin 2\pi\frac{2}{M} & \cdots & \sin 2\pi\frac{M-1}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos 2\pi\frac{k}{M} & \cos 2\pi\frac{2k}{M} & \cdots & \cos 2\pi\frac{(k(M-1))}{M} \\ 0 & \sin 2\pi\frac{k}{M} & \sin 2\pi\frac{2k}{M} & \cdots & \sin 2\pi\frac{(k(M-1))}{M} \end{bmatrix}$$

and for $M=2k$

(2)

$$\begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{M-3} \\ \varphi_{M-2} \\ \varphi_{M-1} \end{bmatrix} = \sqrt{\frac{2}{M}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ 1 & \cos 2\pi\frac{1}{M} & \cos 2\pi\frac{2}{M} & \cdots & \cos 2\pi\frac{(M-1)}{M} \\ 0 & \sin 2\pi\frac{1}{M} & \sin 2\pi\frac{2}{M} & \cdots & \sin 2\pi\frac{M-1}{M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos 2\pi\frac{k-1}{M} & \cos 2\pi\frac{2(k-1)}{M} & \cdots & \cos 2\pi\frac{(k-1)(M-1))}{M} \\ 0 & \sin 2\pi\frac{(k-1)}{M} & \sin 2\pi\frac{2(k-1)}{M} & \cdots & \sin 2\pi\frac{((k-1)(M-1))}{M} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

PROOF. Let $\{\varphi_j\}_{j=0}^{M-1}$ be defined above. First, we want to show that each φ_j has norm one.
For this we just need to check the terms in (1) above. For j=1, $\|\varphi_j\|^2 = \frac{2}{M} \times \frac{M}{2} = 1$.
If $1 \leq 2q-1 \leq M-1$,

$$\begin{aligned}
\|\varphi_{2q-1}\|^2 &= \frac{2}{M}\sum_{j=0}^{M-1}\cos^2 2\pi\frac{qj}{M} \\
&= \frac{2}{M}\frac{1}{2}\sum_{j=0}^{M-1}\left(1+\cos 2\pi 2\frac{qj}{M}\right) \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}\sum_{j=0}^{M-1}\cos 2\pi 2\frac{qj}{M}\right) \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}Re\sum_{j=0}^{M-1}\left(\omega^{2q}\right)\right),\ \omega = e^{\frac{2\pi i}{M}} \\
&= \frac{2}{M}\left(\frac{M}{2}+\frac{1}{2}Re\left(\frac{1-\left(\omega^{2q}\right)^j}{1-\omega^{2q}}\right)\right) \\
&= 1.
\end{aligned}$$

Similarly, if $0 < 2q \leq M-1$ is even

$$\begin{aligned}
\|\varphi_{2q}\|^2 &= \frac{2}{M}\sum_{j=0}^{M-1}\sin^2 2\pi\frac{qj}{M} \\
&= \frac{2}{M}\left(\frac{M}{2}-\frac{1}{2}\sum_{j=0}^{M-1}\cos 2\pi 2\frac{qj}{M}\right) \\
&= 1.
\end{aligned}$$

It remains to show that the φ_j's are orthogonal. Again, it suffices to check (1). Let $\omega = e^{\frac{2\pi i}{M}}$. For $0 < 2q$ even

$$\begin{aligned}
\langle\varphi_0, \varphi_{2q-1}\rangle &= \frac{2}{M\sqrt{2}}\sum_{j=0}^{M-1}\cos 2\pi\frac{qj}{M} \\
&= \frac{2}{M\sqrt{2}}Re\sum_{j=0}^{M-1}\left(\omega^q\right)^j \\
&= 0.
\end{aligned}$$

For $2q-1$ odd

$$\begin{aligned}\langle\varphi_0,\varphi_{2q-1}\rangle &= \frac{2}{M\sqrt{2}}\sum_{j=0}^{M-1}\sin 2\pi\frac{qj}{M}\\ &= \frac{2}{M\sqrt{2}}Im\sum_{j=0}^{M-1}(\omega^q)^j\\ &= 0.\end{aligned}$$

Finally,

$$\begin{aligned}\langle\varphi_2 q,\varphi_{2\ell-1}\rangle &= \frac{2}{M}\sum_{j=1}^{M-1}\cos\frac{2\pi j\,(q-\ell)}{M}\sin\frac{2\pi j\,(q-\ell)}{M}\\ &= \frac{1}{M}\sum_{j=1}^{M-1}\sin\frac{2\pi j2\,(q-\ell)}{M}\\ &= \frac{1}{M}Im\sum_{j=1}^{M-1}\left(\omega^{2(q-\ell)}\right)^j\\ &= \frac{1}{M}Im\left[\sum_{j=1}^{M-1}\left(\omega^{2(q-\ell)}\right)^j-1\right]\\ &= \frac{1}{M}Im\,(0-1)\\ &= 0.\quad\Box\end{aligned}$$

$\Box$

By Lemma 2.6 if we take any N-columns, $N<M$, from the matrices given in Theorem 4.1, the corresponding row vectors form a Parseval frame for ℓ_2^N called a **(real) harmonic frame**. Similarly, we could take any N-columns from the transpose of these matrices, then the corresponding row vectors form a Parseval frame for ℓ_2^N.

4.2. Complex Harmonic Frames. In this subsection we look at the complex versions of the harmonic frames.

THEOREM 4.2. *The family* $\{\varphi_i\}_{i=0}^{M-1}$ *in* $\mathbb{C}^M$ *is an orthonormal basis for* $\mathbb{C}^M$ *where for* $\omega=e^{\frac{2\pi i}{M}}$

$$\begin{bmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \vdots\\ \varphi_{M-2}\\ \varphi_{M-1}\end{bmatrix}=\sqrt{\frac{1}{M}}\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & \omega^1 & \omega^2 & \cdots & \omega^{M-1}\\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & \omega^{(M-2)} & \omega^{2(M-2)} & \cdots & \omega^{(M-2)(M-1)}\\ 1 & \omega^{(M-1)} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(M-1)}\end{bmatrix}.$$

PROOF. Let $\{\varphi_j\}_{j=0}^{M-1}$ and ω be defined as above. It is obvious that each φ_j has norm one. Now we want to show that the rows are orthogonal. For $\ell \neq j$ we have

$$\begin{aligned}
\langle \varphi_k, \varphi_\ell \rangle &= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{jk} \overline{\omega^{\ell k}} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{jk} \omega^{-\ell k} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \omega^{k(j-\ell)} \\
&= \frac{1}{M} \left(\frac{1 - \left(\omega^{j-\ell}\right)^M}{1 - \omega^{j-\ell}} \right) \\
&= \frac{1}{M} \left(\frac{1 - \left(\omega^{M}\right)^{j-\ell}}{1 - \omega^{j-\ell}} \right) \\
&= \frac{1}{M} \left(\frac{1 - 1^{j-\ell}}{1 - \omega^{j-\ell}} \right) = 0 \quad \square
\end{aligned}$$

$\square$

4.3. General Harmonic Frames. The material in this section is due to Casazza and Kovačević and comes from [**11**].

DEFINITION 4.3. *Fix $M \geq N$, $|c| = 1$, and $\{b_i\}_{i=1}^N$ with $|b_i| = \frac{1}{\sqrt{M}}$. Let $\{c_i\}_{i=1}^N$ be distinct M^{th} roots of c, and for $0 \leq k \leq M-1$ let $\varphi_k = \left(c_1^k b_1, c_2^k b_2, \cdots, c_N^k b_N\right)$. Then $\{\varphi_i\}_{i=0}^{M-1}$ is a* general harmonic frame *for $\mathbb{C}^N$.*

PROPOSITION 4.4. *Every general harmonic frame for $\mathbb{C}^N$ is unitarily equivalent to a frame of the form $\left\{c^k \psi_k\right\}_{k=0}^{M-1}$, where $|c| = 1$ and $\{\psi_k\}_{k=0}^{M-1}$ is a harmonic frame.*

PROPOSITION 4.5. *Let $\{\psi_k\}_{k=0}^{M-1}$ be a harmonic frame and let $|c| = 1$. Then $\left\{c^k \varphi_k\right\}_{k=0}^{M-1}$ is equivalent to $\{\psi_k\}_{k=0}^{M-1}$ if and only if c is an M^{th} root of unity and there is a permutation σ of $\{1, 2, \cdots, N\}$ so that $\varphi_{kj} = \psi_{k\sigma(j)}$, for all $0 \leq k \leq M-1$ and all $1 \leq j \leq N$. A general harmonic frame is equivalent to a harmonic tight frame if and only if it equals a harmonic tight frame.*

PROPOSITION 4.6. *The family $\left\{c^k \psi_k\right\}_{k=0}^{M-1}$ is a general harmonic frame for $\mathbb{H}_N$ if and only if there is a vector $\varphi_0 \in \mathbb{C}^N$ with $\|\varphi_0\|^2 = \frac{N}{M}$, an orthonormal basis $\{e_i\}_{i=1}^N$ for $\mathbb{C}^N$ and a unitary operator U on $\mathbb{H}_N$ with $Ue_i = c_i e_i$, with $\{c_i\}_{i=1}^N$ distinct M^{th} roots of some $|c| = 1$ so that $\varphi_k = U^k \varphi_0$, for all $0 \leq k \leq M-1$.*

THEOREM 4.7. *Let U be a unitary operator on $\mathbb{C}^N$, $\varphi_0 \in \mathbb{C}^N$ and assume $\left\{U^k \varphi_0\right\}_{k=0}^{M-1}$ is a equal norm Parseval frame for $\mathbb{C}^N$. Then $U^M = cI$ for some $|c| = 1$ and $\left\{U^k \varphi_0\right\}_{k=0}^{M-1}$ is a general harmonic frame. That is, the general harmonic frames are the only equal norm Parseval frames generated by a group of unitary operators with a single generator.*

A listing of all low dimensional harmonic frames and their properties can be found in [**37**].

4.4. Maximal Robustness to Erasures. The results in this section are due to Kovačević and Puschel and can be found in [**34**].

DEFINITION 4.8. *Let* $\{f_i\}_{i=1}^M$ *be a frame for* ℓ_2^N. *Let* F *be the matrix having the vectors* f_i *as its rows. We call* F **maximally robust to erasures** *if every* $n \times n$ *submatrix of* F *is invertible. If* U *is an* $m \times m$ *matrix and* F *is constructed by keeping all columns with indices in the set* $I \subset \{0, 1, \ldots, m\}$, *then we write*

$$F = U[I].$$

We will work with the **discrete Fourier transform** defined by

$$DFT_m = \frac{1}{\sqrt{m}} \left[\omega_m^{k\ell}\right]_{0 \le k, \ell \le m}, \quad \omega_m = e^{2\pi i/m}.$$

We have from [**34**]

THEOREM 4.9. *For* $n \le m$,

$$F = DFT_m[0, 1, \ldots, n],$$

is an equal norm Parseval frame which is maximally robust to erasures.

The above frames are complex. There are also real versions of these frames given in [**34**]. Define for odd $n = 2k + 1$,

$$U_n = \begin{bmatrix} 1 & & \\ & I_k & -iJ_k \\ & J_k & iI_k \end{bmatrix} \tag{3}$$

where J_k is I_k with the columns in reversed order.

THEOREM 4.10. *Let* $0 \le k < \frac{m-1}{2}$ *and* $n = 2k + 1$. *Then*

$$\begin{aligned} F &= DFT_m[0, 1, \ldots, k, m-k, m-k+1, \ldots, m-1]U_n \\ &= \left[[cos\ \frac{2j\ell\pi}{n}]_{0 \le j < m, 0 \le \ell \le k} [-sin\ \frac{2j\ell\pi}{n}]_{0 \le j < m, 1 \le \ell \le k} \right]. \end{aligned}$$

is an equal norm (real) tight frame with maximal robustness to erasures.

For even n, we first define

$$\widetilde{DFT_m} = \left[\omega_m^{(k+\frac{1}{2})\ell}\right]_{0 \le k, \ell \le m},$$

and

$$V_n = \begin{bmatrix} I_k & -iJ_k \\ J_k & iI_k \end{bmatrix} \tag{4}$$

THEOREM 4.11. *Let* $1 \le k \le \frac{m}{2}$ *and let* $n = 2k$. *Then*

$$\begin{aligned} F &= \widetilde{DFT_m}\,[0, 1, \ldots, k-1, m-k, m-k+1, \ldots m-1]\, V_n \\ &= \left[[cos\ \frac{2(j+\frac{1}{2})\ell\pi}{n}]_{0 \le j < m, o \le \ell < k} [-sin\ \frac{2(j+\frac{1}{2})\ell\pi}{n}]_{0 \le j < m, 1 \le \ell \le k} \right]. \end{aligned}$$

is a equal norm (real) tight frame which is maximally robust to erasures.

5. Structured Parseval Frames

5.1. Using Existing Tight Frames to Construct New Tight Frames. Fix $N, K, M \in \mathbb{N}$ with $K \leq N$. Let

$$\mathcal{K} = \{A \subset \{1, 2, \ldots, N\} : |A| = K\}.$$

For every $A \in \mathcal{K}$, let $\{f_i^A\}_{i=1}^M$ be a unit norm tight frame for $\mathbb{H}_K$ (so the tight frame bound is M/K). For every $A \in \mathcal{K}$ with $A = \{i_1 < i_2 < \cdots i_K\}$ define $T_A : \mathbb{H}_N \to \mathbb{H}_K$ by $T_A(f)(j) = f(i_j)$.

Proposition 5.1. *The family*

$$\{T_A^* f_i^A\}_{i=1, A\in\mathcal{K}}^M$$

is a unit norm tight frame for $\mathbb{H}_N$ with tight frame bound

$$\frac{M}{N}\binom{N}{K}.$$

Proof. Each T_A^* is an isometric embedding. Moreover, if $f \in \mathbb{H}_N$ we have

$$\begin{aligned}
\sum_{A\in\mathcal{K}}\sum_{i=1}^M |\langle f, T_A^* f_i^A\rangle|^2 &= \sum_{A\in\mathcal{K}}\sum_{i=1}^M |\langle T_A f, f_i^A\rangle|^2 \\
&= \sum_{A\in\mathcal{K}} \frac{M}{K}\|T_A f\|^2 \\
&= \frac{M}{K}\sum_{A\in\mathcal{K}}\sum_{i\in A} |F(i)|^2 \\
&= \frac{M}{K}\sum_{i=1}^N \sum_{A\in\mathcal{K},\ i\in A} |f(i)|^2 \\
&= \frac{M}{K}\sum_{i=1}^N \binom{N}{K}|f(i)|^2 \\
&= \frac{M}{K}\binom{N}{K}\sum_{i=1}^N |f(i)|^2 \\
&= \frac{M}{N}\binom{N}{K}\|f\|^2. \quad \square
\end{aligned}$$

□

5.2. M-Circle and M-Semicircle Frames. The results in this section are from Bodmann and Paulsen [**5**]. Bodmann and Paulsen introduced the notion of frame paths to construct M-circle and M-semicircle frames and an alternate frame path definition of real harmonic frames.

Definition 5.2. *A continuous map f:$[a, b] \to \mathbb{R}^M$ (respectively, $\mathbb{C}^M$) is called a* **uniform frame path** *iff $\|f(t)\| = 1$ for all t and there are infinitely many choices of N such that $F_N = \{f\left(a + \frac{b-a}{N}\right), f\left(a + \frac{2(b-a)}{N}\right), \cdots, f(b)\}$ is an equal norm $\frac{N}{M}$-tight frame for $\mathbb{R}^M$ (respectively, $\mathbb{C}^M$). We call any such F_N a* **frame obtained by regular sampling of** *f.*

Examples of such frames would include real and complex harmonic frames [**5**].

In this section we discuss M-circle frame paths in $\mathbb{R}^M$, $M > 2$, where the image of the frame path is the union of M circles. Let $\{e_i\}_{i=1}^M$ be the canonical orthonormal basis for $\mathbb{R}^M$. The image of $\{e_i\}_{i=1}^M$ will be the union of unit circles in the $e_1 - e_2$-plane, $e_2 - e_3$-plane,$\cdots$,$e_{M-1} - e_M$ plane. For the cases we will consider let $N = 4M$. To define the continuous path one needs only to see that it is possible to traverse this union of M circles in a continuous manner, passing through each quarter circle exactly once. Since the intersection of these circles occur at the 2M points, $\pm e_1, \cdots, \pm e_M$, to define the path it is enough to make clear the order in which one passes through the above points.

Proposition 5.3. *When $M > 2$ is even, the following path traverses each of the quarter circles exactly once,*

$$\begin{array}{lllllllll}
+e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
-e_1 & \to & -e_2 & \to & -e_2 & \to & \cdots & -e_M & \to \\
+e_1 & \to & -e_2 & \to & +e_3 & \to & \cdots & -e_M & \to \\
-e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & +e_M & \to \\
+e_1 & & & & & & & &
\end{array}$$

where the sign in the third and fourth rows alternates.

Proposition 5.4. *When $M > 1$ is odd, the following path traverses each of the quarter circles exactly once,*

$$\begin{array}{lllllllll}
+e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
-e_1 & \to & -e_2 & \to & -e_2 & \to & \cdots & -e_M & \to \\
-e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & -e_M & \to \\
+e_1 & \to & -e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\
+e_1 & & & & & & & &
\end{array}$$

where the sign in the third and fourth rows alternates.

Definition 5.5. *Now, using the above ordering we will define a piecewise smooth map $f : [0, 4M] \to \mathbb{R}^M$ so that on ith interval $[i-1, i]$ the image of f traces the ith quarter circle given in the above ordering. So in the case that $M \geq 4$ is even this is accomplished be setting*

$$f(t) = \begin{cases} \left(\cos\frac{\pi t}{2}, \sin\frac{\pi t}{2}, 0, \cdots, 0\right) & 0 \le t \le 1 \\ \left(0, \cos\frac{\pi(t-1)}{2}, \sin\frac{\pi(t-1)}{2}, 0, \cdots, 0\right) & 1 \le t \le 2 \\ \left(0, \cdots, 0, \cos\frac{\pi(t-M+1)}{2}, \sin\frac{\pi(t-M+1)}{2}\right) & M-1 \le t \le M \\ \left(-\sin\frac{\pi(t-M+1)}{2}, 0, \cdots, \cos\frac{\pi(t-M+1)}{2}\right) & M \le t \le M+1 \\ \left(\sin\frac{\pi(t-4M+1)}{2}, 0, \cdots, \cos\frac{\pi(t-4M+1)}{2}\right) & 4M-1 \le t \le 4M \end{cases} \tag{1}$$

[**5**].

Theorem 5.6. *(The Circle Frames). If $n \in \mathbb{N}$, $N = 4nM$ with $M \geq 3$ and let $f : [0, 4M] \to \mathbb{R}^M$ denote the M-circle path defined above, then $\{f\left(\frac{4Mj}{N}\right) : j = 1, \cdots, N\}$ is an equal norm $\frac{N}{M}$-tight frame for $\mathbb{R}^M$.*

THEOREM 5.7. *(The Semicircle Frames). If $f : [0,1] \to \mathbb{R}^2$ be defined by $f(t) = (\cos(\pi t), \sin(\pi t))$, then for any $N > 2$, the set $F_N = \{f\left(\frac{j}{N}\right) : 1 \le j \le N\}$ is an equal norm, $\frac{N}{2}$-tight frame for $\mathbb{R}^2$.*

The construction of the **M-semicircles path** will be similar to the construction of the M-circles path in that the construction of the map $f : [0, 2M] \to \mathbb{R}^M$ is identical. The construction of these paths differs because to construct the M-semicircles path we need only choose a path that exhausts a connected semicircle on each of the M-circles.

PROPOSITION 5.8. *When $M > 2$ is even, the following path traverses a connected semicircle on each of the M circles exactly once,*

$$\begin{array}{ccccccccc} +e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\ -e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & +e_M & \to \\ +e_1 & & & & & & & & \end{array}$$

When $M > 1$ is odd, the following path traverses a connected semicircle on each of the M circles exactly once,

$$\begin{array}{ccccccccc} +e_1 & \to & +e_2 & \to & +e_3 & \to & \cdots & +e_M & \to \\ -e_1 & \to & +e_2 & \to & -e_3 & \to & \cdots & -e_M & \to . \\ +e_1 & & & & & & & & \end{array}$$

[**5**]

THEOREM 5.9. *If $n \in \mathbb{N}$, $N = 2nM$ with $M \ge 3$ and $f : [0, 2M] \to \mathbb{R}^M$ denote the M-semicircle path defined above, then F_N is an equal norm $\frac{N}{M}$-tight frame obtained by regular sampling.*

6. Frames of Translates

Translates of a single function play a fundamental role in frame theory, time-frequency analysis, sampling theory and more [**1, 17, 18**].

There is a simple classification of which functions give (tight) frames of translates. For this we need a definition. For $x \in \mathbb{R}$ we define *translation by x* by

$$\tau_x : L^2(\mathbb{R}) \to L^2(\mathbb{R}), (\tau_x f)(y) = f(y - x),\ y \in \mathbb{R}.$$

We first introduce some notation. For a function $\phi \in L^1(\mathbb{R})$ we denote by $\hat{\phi}$ the *Fourier transform of ϕ*

$$\hat{\phi}(\xi) = \int \phi(x) e^{-2\pi i \xi x}(x) dx.$$

As usual the definition of the Fourier transform extends to an isometry $\phi \to \hat{\phi}$ on $L^2(\mathbb{R})$.

Now suppose $\phi \in L^2(\mathbb{R})$ and that $b > 0$. Let us identify the circle $\mathbb{T}$ with the interval $[0, 1)$ via the standard map $\xi \to e^{2\pi i \xi}$. We define the function $\Phi_b : \mathbb{T} \to \mathbb{R}$ by

$$\Phi_b(\xi) = \sum_{n \in \mathbb{Z}} |\hat{\phi}(\frac{\xi + n}{b})|^2.$$

Note that $\Phi_b \in L^1(\mathbb{T})$.

For any $n \in \mathbb{Z}$ we note that

$$\langle \tau_{nb}\phi, \phi\rangle = \langle e^{-2\pi in\xi b}\hat{\phi}, \hat{\phi}\rangle = \frac{1}{b}\int_0^1 \Phi_b(\xi)e^{-2\pi in\xi}d\xi = \frac{1}{b}\hat{\Phi}_b(n).$$

We now have a classification of (tight) frames of translates originally due to Benedetto and Li [**4**]. Casazza, Christensen and Kalton [**9**] and Kim and Lim [**29**] removed an unnecessary assumption from the results in [**4**] as well as generalizing these results.

THEOREM 6.1. *If $\phi \in L^2(\mathbb{R})$, and $b > 0$ then:*
(1) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$\Phi_b(\gamma) = b \quad a.e.$$

(2) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is a Riesz basic sequence with frame bounds $A,\ B$ if and only if

$$bA \le \Phi_b(\gamma) \le bB \quad a.e.$$

(3) $(\tau_{nb}\phi)_{n\in\mathbb{Z}}$ is a frame sequence with frame bounds $A,\ B$ if and only if

$$bA \le \Phi_b(\gamma) \le bB \quad a.e.$$

on $\mathbb{T} \setminus N_b$ where $N_b = \{\xi \in \mathbb{T} : \Phi_b(\xi) = 0\}$.

We also mention a surprising result from [**9**].

THEOREM 6.2. *Let $I \subset \mathbb{Z}$ be bounded below, $a > 0$ and $g \in L^2(\mathbb{R})$. Then $\{T_{na}g\}_{n\in I}$ is a frame sequence if and only if it is a Riesz basic sequence.*

7. Gabor Frames

An excellent reference for time-frequency analysis is Gröchenig's book [**25**]. Although we are only dealing with finite tight frames in this note, since we now discuss discrete Gabor frames we briefly mention the infinite dimensional equivalent for comparison.

Given a function $g \in L^2(\mathbb{R}^d)$, for any $f \in L^2(\mathbb{R}^d)$ we define **translation of f by** $x \in \mathbb{R}^d$ and **modulation of f by** $y \in \mathbb{R}^d$ respectively as

$$T_x(f)(t) = f(t-x) \ \text{ and } \ M_y(f)(t) = e^{2\pi i y\cdot t}f(t).$$

DEFINITION 7.1. *Given a non-zero* **window function** $g \in L^2(\mathbb{R}^d)$ *and lattice parameters $\alpha, \beta > 0$, the set of time-frequency shifts*

$$\mathcal{G}(g, \alpha, \beta) = \{T_{\alpha k}M_{\beta n}g\}_{k,n\in\mathbb{Z}^d}$$

is called a **Gabor system**. *If this family forms a frame for $L^2(\mathbb{R}^d)$, it is called a* **Gabor frame** *or* **Weyl-Heisenberg frame**.

It is a very deep question when g, α, β yields a Gabor frame [**25**]. It can be shown [**25**] that the frame operator for a Gabor frame commutes with translation and modulation which yields:

THEOREM 7.2. *For any Gabor frame $\mathcal{G}(g, \alpha, \beta)$ with frame operator S, the family $\mathcal{G}(S^{-1/2}g, \alpha, \beta)$ is an equal norm Parseval frame for $L^2(\mathbb{R}^d)$.*

The Wexler-Raz biorthogonality relations give an exact calculation for determining when a Gabor frame is Parseval (See [**25**]).

THEOREM 7.3 (Wexler-Raz). *A Gabor frame $\mathcal{G}(g, \alpha, \beta)$ is a Parseval frame if and only if*

$$(\alpha\beta)^{-1}\langle g, M_{\frac{k}{\alpha}} T_{\frac{n}{\beta}}\rangle = \delta_{k0}\delta n0, \quad \text{for all } k, n \in \mathbb{Z}.$$

Now let us look at finite Gabor systems. Let $\omega = e^{2\pi i/n}$. The **translation operator** T is the unitary operator on $\mathbb{C}^n$ given by

$$Tx = T(x_0, x_1, \ldots, x_{n-1}) = (x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and the **modulation operator** M is the unitary operator defined by

$$Mx = M(x_0, x_1, \ldots, x_{n-1}) = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1}).$$

Given a vector $g \in \mathbb{C}^n$, the **finite Gabor system with window function g** is the family

$$\left\{M^\ell T^k g\right\}_{\ell, k \in \mathbb{Z}_n}.$$

In the discrete case, Gabor systems always form tight frames. Although this has been folklore for quite some time, the first formal proof we have seen is in [**32**].

THEOREM 7.4. *For any $0 \neq g \in \mathbb{C}^n$, the collection $\{M^\ell T^k g\}_{\ell, k \in \mathbb{Z}_n}$ is an equal norm tight frame for $\mathbb{C}^n$ with tight frame bound $n^2\|f\|^2$.*

Lawrence, Pfander and Walnut [**32**] examined the linear independence of discrete Gabor systems.

THEOREM 7.5. *If n is prime, there is a dense open set E of full measure (i.e. The Lebesgue measure of $\mathbb{C}^n \setminus E$ is 0) in $\mathbb{C}^n$ such that for every $f \in E$, every subset of the Gabor system $\{M^\ell T^k g\}_{\ell, k \in \mathbb{Z}_n}$ containing n-elements is linearly independent.*

8. Filter Bank Frames

Important results on filter bank frames can be found in [**16, 31**]. The results here are due to Casazza, Chebira, and Kovačević [**8**].

DEFINITION 8.1. *Given $k, N, M \in \mathbb{N}$, a* **Filter bank frame** *for $\ell_2(\mathbb{Z})$ is a frame for $\ell_2(\mathbb{Z})$, say $\{\varphi_m\}_{m \in \mathbb{Z}}$, satisfying the following:*

1. For $0 \leq i \leq kN - 1$ and $j \notin \{0, 1, \ldots, kN - 1\}$ we have that $\varphi_i(j) = 0$.

2. For $j = 0, 1, \ldots, M - 1$ and $i \in \mathbb{Z}$, $\varphi_{iM+j} = T_{iN}\varphi_j$, where T_{iN} is translation by iN.

NOTATION 8.2. *Throughout this section we will let $\{e_i\}_{i \in \mathbb{Z}}$ be the natural orthonormal basis for $\ell_2(\mathbb{Z})$.*

PROPOSITION 8.3. *Let $N < L - 1$ be natural numbers and let $\{\varphi_j\}_{j=0}^{M-1}$ be a frame for the span of $\{e_i\}_{i=0}^{L-1}$ with frame bounds A, B. Let $\varphi_{iM+j} = T_{iN}\varphi_j$ for all $0 \leq j \leq M - 1$ and all $i \in \mathbb{Z}$. Then $\{\varphi_i\}_{i \in \mathbb{Z}}$ is a frame for $\ell_2(\mathbb{Z})$ with frame bounds $A\lfloor \frac{L}{N} \rfloor$, $B\lceil \frac{L}{N} \rceil$.*

PROOF. Let $\varphi \in \ell_2(\mathbb{Z})$ and compute

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle \varphi, \varphi_m\rangle|^2 &= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, \varphi_{iM+j}\rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, T_{iN}\varphi_j\rangle|^2 \\
&\leq \sum_{i\in\mathbb{Z}} B \sum_{n=1}^{iN+L-1} |\varphi(n)|^2 \\
&\leq B \left\lceil \frac{L}{N} \right\rceil \sum_{n\in\mathbb{Z}} |\varphi(n)|^2 \\
&= B \left\lceil \frac{L}{N} \right\rceil \|\varphi\|^2.
\end{aligned}$$

The lower frame bound follows similarly. □

Next we see when we can get a tight frame from a filter bank frame.

PROPOSITION 8.4. *Let $0 < L < N$ be natural numbers and Let $\{\varphi_j\}_{j=0}^{M-1}$ be a frame for span $\{e_i\}_{i=0}^{KN+L}$, with frame operator S having eigenvectors $\{e_i\}_{i=0}^{KN+L}$ and respective eigenvalues $\{\lambda_i\}_{i=0}^{KN+L}$. Let $\varphi_{iM+j} = T_{iN}\varphi_j$ for all $0 \leq j \leq M-1$ and all $i \in \mathbb{Z}$. The following are equivalent:*

1. *The family $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a λ-tight frame for $\ell_2(\mathbb{Z})$.*
2. *We have*

$$\lambda = \begin{cases} \sum_{j=0}^{K} \lambda_{jN+m} : & : \quad 0 \leq m \leq L \\ \sum_{j=0}^{K-1} \lambda_{jN+m} & : \quad L < m \leq N \end{cases}$$

PROOF. Let $\varphi \in \ell_2(\mathbb{Z})$ and compute

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} |\langle \varphi, \varphi_m\rangle|^2 &= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, \varphi_{iM+j}\rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{j=0}^{M-1} |\langle \varphi, T_{iN}\varphi_j\rangle|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{n=0}^{(i+K)N+L} \lambda_{n-iN} |\varphi(n)|^2 \\
&= \sum_{i\in\mathbb{Z}} \sum_{n=0}^{KN+L} \lambda_n |\varphi(n+iN)|^2 \\
&= \sum_{m=0}^{L} \sum_{i\in\mathbb{Z}} \left(\sum_{j=0}^{K} \lambda_{jN+m} \right) |\varphi(iN+m)|^2 \\
&+ \sum_{m=L+1}^{N} \sum_{i\in Z} \left(\sum_{j=0}^{K-1} \lambda_{jN+m} \right) |\varphi(iN+m)|^2.
\end{aligned}$$

The result follows immediately from here. □

Since a filter bank frame is an equal norm frame if and only if we start with an equal norm frame before translation, the next corollary also classifies which filter bank frames are equal norm tight frames.

Corollary 8.5. *Let* $\{\varphi_j\}_{j=0}^{M-1}$ *be an* A*-tight frame for the span of* $\{e_i\}_{i=0}^{KN+L}$ *for* $0 \leq L < N$. *Let* $\varphi_{iM+j} = T_{iN}\varphi_j$, *for all* $0 \leq j \leq M-1$ *and all* $i \in \mathbb{Z}$. *Then* $\{\varphi_i\}_{i\in\mathbb{Z}}$ *is a* λ*-tight frame for* $\ell_2(\mathbb{Z})$ *if and only if* $L = 0$. *Moreover, in this case* $\lambda = (K+1)A$.

Proof. Since $\{\varphi\}_{i=0}^{M-1}$ is an A-tight frame, every orthonormal basis consists of eigenvectors of the frame operator for this frame. By Proposition 8.4, $\{\varphi_i\}_{i\in\mathbb{Z}}$ is a frame with $\{e_i\}_{i\in\mathbb{Z}}$ eigenvectors for the frame operator having eigenvalues

$$\sum_{j=0}^{K} A = (K+1)A, \quad \text{for } e_{iN+m}, \quad 0 \leq m \leq L,$$

and

$$\sum_{j=0}^{K-1} A = KA, \quad \text{for } e_{iN+m}, \quad L+1 \leq m \leq N.$$

It follows that this is a tight frame if and only if $L = 0$. □

THANKS: The referee did a very serious job of greatly improving the presentation in this manuscript.

References

[1] A. Aldroubi, *p frames and shift-invariant subspaces of* L^p, Journal of Fourier Analysis and Applications **7** (2001) 1–21.

[2] J. Antezana, P. Massey, M. Ruiz and D. Stojanoff, *The Schur-Horn Theorem for operators and frames with prescribed norms and frame operator*, Illinois Journal of Mathematics, To appear.

[3] Balan, R., Casazza, P.G., Edidin, D., and Kutyniok, G., *Decomposition of Frames and a New Frame Identity*, Proceedings of SPIE, **5914** (2005), 1-10.

[4] Benedetto, J. and Li, S., *The Theory of Multiresolution Analysis Frames and Applications to Filter Banks*, ACHA, **4**, No. 4 (1998) 389-427.

[5] Bodmann, B.G and Paulsen, V.I, *Frame Paths and Error Bounds for Sigma-Delta Quantization*, Applied and Computational Harmonic Analysis, to appear (2006).

[6] Bodmann, B.G, Paulsen, V.I, and Abdulbaki, Soha A., *Smooth Frame-Path Termination for Higher Order Sigma-Delta Quantizations*, to appear (2006).

[7] M. Bownik and D. Speegle, *The Feichtinger conjecture for wavelet frames, Gabor frames and frames of translates*, Preprint.

[8] Casazza, P.G., Chebira, A. and Kovačević, J., *Filterbank frames*, In progress.

[9] Casazza, P.G., Christensen, O., and Kalton, N.J. *Frames of translates*, Collect. Math. **52** (2001) 35-54.

[10] Casazza, P.G., Fickus, M., Leon, M., and Tremain, J.C., *Constructing infinite tight frames*, Preprint.

[11] Casazza, P.G. and Kovačević, J., *Uniform Tight Frames with Erasures*, Adv. in Comp. Math. **18** No. 2-4 (2003), 387-430.

[12] Casazza, P.G. and Leon, M., *Constructing Frames with a given Frame Operator*, preprint.

[13] Casazza, P.G., Kutyniok, G., and Lammers, M.C., *Duality Principles in Frame Theory*, J. of Fourier Anal. and App. **10** (2004), 383-408.

[14] Christensen, O., *An introduction to frames and Riesz bases*, Birkhauser, Boston, 2003.

[15] Christensen, O., Deng,B and Heil, C., *Density of Gabor frames*, Appl. and Com. Har. Anal. **7** (1999) 292-304.

[16] Cvetković, Z. and Vetterli, M., *Oversampled filter banks*, IEEE Trans. Signal Process. **46** No. 5 (1998)1245–1255.

[17] M. Bownik and D. Speegle, *The Feichtinger conjecture for wavelet frames, Gabor frames and frames of translates*, Preprint.
[18] Daubechies, I., *Ten Lectures on Wavelets*, SAIM, Philadelphia, PA, 1992.
[19] Daubechies, I., Grossmannn, A., and Meyer, Y., *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271-1283.
[20] Duffin, R.J and Schaeffer, A.C., *A class of nonharmonic Fourier series*, Trans. Amer. Soc. **72** (1952), 341-366.
[21] Feng, D.J., Wang, L. and Wang, Y., *Generation of finite tight frames by Householder transformations*, Advances in Computational Mathematics **24** (2006) 297-309.
[22] Gabor, D., *Theory of communication* J. Inst. Elec. Engrg. **93**(1946),429-457.
[23] Goyal, V.K., Kovačević, J., and Kelner, J.A., *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), 203-233.
[24] Goyal, V.K, Vetterli, M., and Thoa, N.T., *Quantized overcomplete expansions in* $\mathbb{R}^N$*: Analysis, synthesis, and algorithms*, IEEE Trans. Inform. Th. **44(1)** (January 1998), 16-31.
[25] Gröchening, K., *Foundations of Time Frequency Analysis*, Birkhauser, Boston, 2001.
[26] Han, D. and Larson, D.R., *Frames, bases and group representations*, Memoirs AMS **697** (2000).
[27] Hernández, E. and Weiss, G., *A First Course on Wavelets*, CRC, Inc., 1996.
[28] Hernández, E., Labate, D., and Weiss, G., *A Unified Characterization of Reproducing Systems Generated by a Finite Family, II*, 2002.
[29] H.O. Kim and J.K. Lim, *Frame multiresolution analysis*, Comm. Korean Math. Soc. Vol. **15**, No. 2 (2000) 285-308.
[30] Klapperenecker, A. and Rötteler, M., *Mutually unbiased bases, spherical designs and frames* in "Wavelets XI" (San Diego, CA 2005), Proc. SPIE **5914** , M. Papadakis et al., eds., Bellingham, WA (2005) 5914 P1-13.
[31] Kovačević, J., Dragotti, P.L., Goyal, V.K., *Finter bank frame expansions with erasures*, IEEE Trans. Inform. Theory, **48** No. 6 (2002) 1439–1450.
[32] Lawrence, G., Pfander, E. and Walnut, D., *Linear independence of Gabor systems in finite dimensional vector spaces*, preprint.
[33] P. Massey and R. Ruiz, *Tight frame conpletions with prescribed norms*, Preprint.
[34] Puschel, M. and Kovačević, J., *Real, tight frames with maximal robustness to erasures*, Preprint.
[35] Strohmer, T. and Heath, Jr., R.W., *Grassmannian frames with applications to coding and communications*, Appl. Comput. Harmon. Anal., **13** (2003), 257-275.
[36] Unser, M., Aldroubi, A., and Laine, A., eds., em Special Issue on Wavelets in Medical Imaging, IEEE Trans. Medical Imaging, **22** (2003).
[37] Waldron, S. and Hay, N, *On computing all harmonic frames of n vectors in* $\mathbb{C}^d$,Applied and Computational Harmonic Analysis, to appear (2006).
[38] Walnut, D.F., *An introduction to wavelet analysis*, Birkhäuser, Boston (2002).
[39] Wolfe, P.J., Godsill, S.J., and Ng, W.J., *Bayesain variable selection and regularization for time-frequency surface estimation*, J. R. Stat. Soc. Ser. B Stat. Methodol., **66** (2004), 575-589.
[40] Young, R., *An Introduction to Nonharmonic Fourier Series*, Revised First Edition, Academic Press, San Diego, 2001.
[41] Zimmermann, G., *Normalized Tight Frames in Finite Dimensions*,in "Recent Progress in Multivariate Approximation, ISNM 137" (W. Haussmann and K. Jetter Eds.), 249-252, Birkhäuser, Basel, 2001.

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