Classifying Characteristic functions giving Weyl-Heisenberg Frames

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ABSTRACT

We examine the question of which characteristic functions yield Weyl-Heisenberg frames for various values of the parameters. We also give numerous applications of frames of characteristic functions to the general case (g, a, b).

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1. INTRODUCTION

In 1952 Duffin And Schaeffer[7] introduced the notion of a frame for a Hilbert space.

DEFINITION 1.1. *A sequence (f_n) in a Hilbert space H is a* **frame** *for H if there are constants $0 < A, B$ satisfying*

$$A\|f\|^2 \leq \sum_n |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad \textit{for all} \;\; f \in H.$$

The numbers A, B are called **lower** (resp. **upper**) **frame bounds for the frame**. If $A = B$ we call this a **tight frame** and if $A = B = 1$ we call it a **normalized tight frame**. If (f_n) does not span H but is a frame for its closed linear span, we call it a **frame sequence**. It is clear from the definition that an orthogonal projection takes a frame to a frame sequence with the same frame bounds.

If (f_n) is a sequence of elements of an infinite dimensional Hilbert space H and (e_n) is an orthonormal basis for H, we define the **preframe operator** $T : H \to H$ by: $Te_n = f_n$. It follows that for any $f \in H$, $T^*f = \sum_n \langle f, f_n\rangle e_n$. Hence, (f_n) is a frame if and only if T^* is an isomorphism (called the **frame transform**) and in this case $S = TT^*$ is an invertible operator on H called the **frame operator**. The frame operator is a positive, self-adjoint invertible operator on H satisfying:

$$Sf = \sum_n \langle f, f_n\rangle f_n.$$

A bounded unconditional basis for H is called a **Riesz basis** (or a **Riesz basic sequence** if it is a Riesz basis for its closed linear span in H).

The frames commonly used in signal processing are the Weyl-Heisenberg (or Gabor) frames. If $g \in L^2(\mathbb{R})$ and $0 < a, b \in \mathbb{R}$ we define

$$\text{Translation by a} \quad T_a(g)(t) = g(t-a) \qquad \text{Modulation by b} \quad E_b g(t) = e^{2\pi imbt}g(t).$$

We say that (g, a, b) generates (or **is**) a **Weyl-Heisenberg frame** (**WH-frame** for short) for $L^2(\mathbb{R})$ if $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. If this family has a finite upper frame bound we call g a **preframe function**. The family of preframe functions is denoted **PF**.

There are several known restrictions on the g, a, b in order that (g, a, b) form a WH-frame which we summarize below. These results are due to various authors and may be found in Heil and Walnut[8] or Casazza.[1] To simplify the notation, for all $k \in \mathbb{Z}$ we let

$$G_k(t) = \sum_{n\in\mathbb{Z}} g(t-na)\overline{g(t-na-\frac{k}{b})}.$$

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PROPOSITION 1.2. *Let $g \in$ **PF** and $0 < a, b$.*

(1) If (g, a, b) generates a WH-frame then $ab \leq 1$.

(2) (g, a, b) generates a WH-frame if and only if $(g, \frac{1}{b}, \frac{1}{a})$ generates a Riesz basic sequence. Hence, if $ab = 1$, then (g, a, b) is a WH-frame if and only if it is a Riesz basis of $L^2(\mathbb{R})$.

(3) If (g, a, b) is a WH-frame with frame bounds A, B then $bA \leq G_0(t) \leq bB$ a.e.

(4) If $ab \leq 1$ and supp $g \subset [0, \frac{1}{b}]$, then (g, a, b) is a WH-frame with frame bounds A, B if and only if $bA \leq G_0(t) \leq bB$ a.e.

(5) If $a = b = 1$, and for a.e. $0 \leq t \leq 1$ we have that $(g(t-n))_{n \in \mathbb{Z}}$ has at most one non-zero term, then $(g, 1, 1)$ is a WH-frame with frame bounds A, B if and only if $A \leq G_0(t) \leq B$ a.e.

(6) If (g, a, b) is in PF with upper frame bound B then $\sum |g(x - \frac{k}{b})|^2 \leq B$.

There is a necessary condition for having a WH-frame due to Casazza and Christensen.[2]

THEOREM 1.3 **(CC-Condition).** *Let $g \in L^2(\mathbb{R})$, $a, b > 0$ and assume that*
(1) We have

$$A =: \inf_{t \in [0,a]} \left[G_0(t) - \sum_{k \neq 0} |G_k(t)| \right] > 0,$$

(2) We have,

$$B =: \sup_{t \in [0,a]} \sum_{k \in \mathbb{Z}} |G_k(t)| < \infty.$$

Then $(E_{mb}T_{na}g)_{n,m \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{b}, \frac{B}{b}$.

Note that if g is compactly supported and bounded, we get (2) above automatically. Casazza, Christensen and Janssen[4] have shown that the conditions in Proposition 1.3 are both necessary and sufficient if g is a real-valued positive function on $\mathbb{R}$.

A major question in WH-frame theory is:

PROBLEM 1.4. *Classify all the g, a, b so that (g, a, b) is a WH-frame.*

A special case of this problem which is already extremely difficult (as we will see in this manuscript) is:

PROBLEM 1.5. *Classify all measurable sets $E \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ so that (χ_E, a, b) is a WH-frame.*

An even further special case which is still very difficult is still open.

PROBLEM 1.6. **[a,b,c-Problem]** *Classify all $a, b, c \in \mathbb{R}$ so that $(\chi_{[0,c]}, a, b)$ is a WH-frame.*

In these notes we will examine what is known about the latter two problems and add some new results to the list.

2. COMPACTLY SUPPORTED FUNCTIONS

In this section we will look at the more general problem of compactly supported functions and WH-frames. We will not review what is known in this case, but instead develop some specific results for use later in examining characteristic functions which give WH-frames.

Now we give the corresponding result of Proposition 1.2 (5) for two non-zero elements.

PROPOSITION 2.1. *Let $g \in L^2(\mathbb{R})$ and assume that for all $0 \leq t \leq 1$ at most two elements of $(g(t-n))_{n \in \mathbb{Z}}$ are non-zero. The following are equivalent:*

(1) $(g, 1, 1)$ is a WH-frame.

(2) The CC-condition holds.

(3) We have $G_0 \leq B < \infty$ a.e. and for $H_n(t) = |g(t)| - |g(t-n)|$ there is a $0 < A$ such that

$$A \leq \inf\{|H_n(t)| : 0 \neq n \in \mathbb{Z}, g(t) \neq 0\},$$

where (3) holds for almost every t with $g(t) \neq 0$.

Proof. (1) $\Rightarrow$ (3): By Proposition 1.2 (3), we get the first part of (3). We will assume the H_n condition fails and show that (g,a,b) fails to have a non-zero lower frame bound. In this case, for a fixed $\epsilon > 0$, after a translation, it is easily seen that there is an $m \in \mathbb{Z}$ and a set $E \subset [0,1]$ with $|E| > 0$ so that $g(t) \neq 0$ for all $t \in E$ and $g(t+m) \neq 0$ for all $t \in E$ and

$$||g(t)| - |g(t+m)|| < \epsilon, \text{ for all } t \in E.$$

We construct a function $f \in L^2(\mathbb{R})$ by:

$$f = f_0\chi_E - f_1\chi_{E+m} + f_2\chi_{E+2m} - f_3\chi_{E+3m} + \cdots - f_{2n-1}\chi_{E+(2n-1)m},$$

where (f_0, f_1) are given by

$$f_0 g = |g| \quad \text{on } E, \quad f_1 g = |g| \quad \text{on } E+m$$

and $(f_i)_{i=2}^{(2n-2)m}$ are chosen iteratively so that there is a 1-periodic function h_i with $|h_i(t)| = 1$ and

$$f \cdot T_{-im}g = h_i T_{-im}[|\chi_E g| - |\chi_{E+m}g|].$$

It follows that for all $i \neq 0, 2n-1$ (since the $h_i(t)$ are 1-periodic)

$$\sum_k |< f, E_k T_{-im}g >|^2 = \sum_k |\int_E h_i(t)[|g(t+m)| - |g(t)|]E_k(t)\, dt|^2$$

$$= \|\chi_E h_i[T_m g - g]\|^2_{L^2(E)} \leq \epsilon|E|.$$

For the other two values of i we will get

$$\sum_k |< f, E_k T_i g >|^2 \leq \int_E |g(t)|\, dt \leq |E|B.$$

So

$$\sum_{k,n\in\mathbb{Z}} |< f, E_k T_n g >|^2 \leq ((n-2)\epsilon + B)|E|.$$

Since $\|f\|^2 = n|E|$, it is easily seen that it is now impossible for $(g,1,1)$ to have a non-zero lower frame bound.

(3) $\Rightarrow$ (2): We need to check that if $(g,1,1)$ is a frame then the conditions of Theorem 1.3 are satisfied. By Proposition 1.2, we know that for some $A > 0$ we have (when $g(t) \neq 0$),

$$||g(t)| - |g(t+n)|| \geq A,$$

for all $t \in \mathbb{R}$ and all $0 \neq n \in \mathbb{Z}$. Hence,

$$||g(t)| - |g(t+n)||^2 \geq A^2.$$

That is,

$$|g(t)|^2 + |g(t+n)|^2 - 2|g(t)||g(t+n)| \geq A.$$

But, it is easily checked that this is precisely the first condition of Theorem 1.3 for our case. Now, if $(g,1,1)$ is a frame then g is bounded and by our assumptions, it is easily seen that the second condition must also be satisfied. □

We mention another necessary condition for compactly supported functions to give WH-frames.

PROPOSITION 2.2. *If $g \in L^2(\mathbb{R})$ has compact support and for every $\epsilon > 0$ there is a set $E \subset [0,1]$ with $0 < |E|$ and for a.e. $t \in E$ we have*

$$|\sum_{n\in\mathbb{Z}} g(t+n)| < \epsilon,$$

then $(g,1,1)$ is not a WH-frame.

Proof. Without loss of generality we may assume that supp $g \subset [0,N]$. Fix $n \geq N$ and choose $E \subset [0,1]$ with $|E| > 0$ and

$$|\sum_{k\in\mathbb{Z}} g(t+k)| \leq \frac{1}{n}.$$

Let

$$f = \sum_{k=0}^{n} \chi_{k+E}.$$

Then

$$||f||^2 = n|E|$$

Also, by our hypotheses, (and mimicking the proof of Proposition 2.1) for $n - N$ values of m we have

$$\sum_k | < f, E_k T_m g > |^2 = | \int_E \sum_{k \in \mathbb{Z}} g(t+k) E_k \, dt|^2 \leq |E|^2 \frac{1}{n^2}.$$

For the other terms we get at most:

$$2(1 + 2 + \cdots (N-1))|E|^2 = N(N-1)|E|^2.$$

Hence,

$$\sum_{k,m \in \mathbb{Z}} | < f, E_k T_m g > |^2 \leq |E|^2 (N(N-1) + \frac{n-N}{n^2}) \leq |E|^2 N^2.$$

So to have a frame, we need an $A > 0$ so that

$$An|E| \leq |E|^2 N^2.$$

That is,

$$An \leq |E| N^2 \leq N^2,$$

which is a contradiction for large n. □

3. THE A, B, C-PROBLEM

The operator $Lf(t) = f(t/b)$ is an invertible operator on $L^2(\mathbb{R})$ and satisfies:

$$L(T_{na}\chi_{[0,c]})(t)) = T_{nab}\chi_{[o,bc]}(t)$$

while $L(\chi_{[0,c]}(t) = \chi_{[o,bc]}(t)$ and $L(E_{mb}g) = E_m L(g)$. It follows that we may as well assume that $b = 1$ in the a, b, c Problem. Some of the results in this section were previously announced in Casazza.[1]

We will use an immediate consequence of Proposition 1.2 (2).

REMARK 3.1. *If $(g, a, 1)$ is a WH-frame then $(T_m g)_{m \in \mathbb{Z}}$ is a Riesz basic sequence.*

We start with the case $a = b = 1$.

PROPOSITION 3.2. *Consider $(\chi_{[0,c]}, 1, 1)$. If $c = 1$ this is an orthonormal basis for $L^2(\mathbb{R})$. If $c < 1$ it is a normalized tight frame sequence in $L^2(\mathbb{R})$ which is not a frame. If $1 < c$ it is not a frame sequence in $L^2(\mathbb{R})$.*

Proof. The case $c = 1$ is obvious. If $c < 1$, this is the image of the orthonormal basis $(\chi_{[0,1]}, 1, 1)$ under the orthogonal projection $Pf = \chi_E f$, where $E = \cup_{n \in \mathbb{Z}}([0, c] + n)$. For $1 < c$ we look at two cases:

Case I: $2k - 1 < c \leq 2k$. Let $d = c - (2k - 1) > 0$. For any natural number $n > k$ let

$$f = \sum_{i-0}^{2n} (-1)^i \chi_{[i,i+d)}.$$

Now, if $\ell \leq -(2k)$ or $\ell \geq 2n+2k$ then $f, \chi_{[0,c]}$ have disjoint supports so their inner product is 0. If $\ell = 0, 1, \cdots, 2n-2k$ then $< f, \chi_{[0,c]} >= 0$. Otherwise, $| < f, \chi_{[0,c]} > | = 0$ or d. Hence,

$$\sum_j | < f, T_j \chi_{[0,c]} > |^2 \leq 3(2k)d,$$

while

$$\|f\|^2 = 2nd.$$

Since k is fixed and n is arbitrary, we have that $(T_j\chi_{[0,c]})$ is not a Riesz basic sequence.

Case II: $2k < c \le 2k+1$. This time let

$$f = \chi_{[0,d]} - \frac{1}{2}\chi_{[1,1+d]} - \frac{1}{2}\chi_{[2,2+d]} + \chi_{[3,3+d]} - \frac{1}{2}\chi_{[4,4+d]} - \frac{1}{2}\chi_{[5,5+d]} + \chi_{[6,6+d]} - \cdots,$$

where there are $3n$ terms in the sum above. Now proceed similarly to Case I. □

We can go further.

REMARK 3.3. *For the a, b, c-Problem, we may assume that $a < b = 1 < c$.*

Proof. As we saw, we may assume that $b = 1$ (and the case $a = 1$ is done). Hence, by Proposition 1.2 (1) we have that $a \le 1$. Also, by Proposition 1.2 (4), if $a \le c < 1$ then we have a frame and if $c < a < 1$ we do not have a frame. □

Janssen[10] has shown (using the Walnut representation of the frame operator) just how delicate the a, b, c Problem is.

PROPOSITION 3.4. *Assume $a < 1 < c$.*

(1) If a is not rational and $1 < c < 2$ then $(\chi_{[0,c]}, a, 1)$ is a frame.

(2) If $a = p/q$ is rational, $gcd(p,q) = 1$, and $2 - (1/q) < c < 2$, then $(\chi_{[0,c]}, a, 1)$ is not a frame.

(3) If $a > 3/4$, $c = L - 1 + L(1-a)$ with integer $L \ge 3$, then $(\chi_{[0,c]}, a, 1)$ is not a frame.

(4) If d is the greatest integer $\le c$ and $|c - d - 1/2| < (1/2) - a$, then $(\chi_{[0,c]}, a, 1)$ is a frame.

Janssen[9] also has an interesting chart of certain values where we have a frame or don't have a frame for several cases of the a, b, c-Problem.

We can add to this list the following.

PROPOSITION 3.5. *If $2 \le c \in \mathbb{N}$, then for all $a > 0$, $(\chi_{[0,c]}, a, 1)$ is not a frame.*

Proof. Let $c = n \in \mathbb{N}$ and $g = \chi_{[0,n]}$. Then we have

$$\|\sum_{j=0}^{k-1}(T_{jn}g - T_{jn+1}g)\|^2 = 2.$$

Hence $(T_n g)$ is not a Riesz basic sequence and so $(g, a, 1)$ cannot be a WH-frame by Remark 3.1. □

4. CHARACTERISTIC FUNCTIONS GIVING WH-FRAMES

Casazza and Kalton[5] have shown that the problem of classifying all characteristic functions which give WH-frames for the case $a = b = 1$ is already exceptionally difficult since it is equivalent to a classical unsolved problem in complex function theory.

PROBLEM 4.1. *Classify all integer sets $n_1 < n_2 < \cdots n_k$ so that*

$$\sum_{j=1}^{k} z^{n_j}$$

does not have any zeroes on the unit circle.

For notation, we call a measurable subset F of $\mathbb{R}$ **complete** if $|\mathbb{R} - \cup_{n\in\mathbb{Z}}(F+n)| = 0$, and we say that two measurable sets E, F in $\mathbb{R}$ are **completely disjoint** if $|(E+n) \cap (F+m)| = 0$, for all $m, n \in \mathbb{Z}$. We call E a **WH-frame set for a,b** if (χ_E, a, b) is a frame for $L^2(\mathbb{R})$. The first main result of Casazza and Kalton[5] is:

THEOREM 4.2. *Fix integers $n_1 < n_2 < \cdots < n_k$. The following are equivalent:*

(1) The set $F = \cup_{j=1}^{k}([0,1) + n_j)$ is a Weyl-Heisenberg frame set with frame bounds A, B.

(2) We have $A \leq |\sum_{j=1}^{k} z^{n_j}| \leq B$, for all $|z| = 1$,

(3) For every measurable set $E \subset [0,1]$ of positive measure, and $F_0 = \cup_{j=1}^{k}(E + n_j)$, $(E_m T_n \chi_{F_0})_{m,n\in\mathbb{Z}}$ is a frame for $L^2(F_0)$ with frame bounds A, B.

Casazza and Kalton[5] also gave an equivalent formulation for general characteristic functions to give WH-frames for $a = b = 1$. We call a measurable set $F \subset \mathbb{R}$ an **elementary A-Weyl-Heisenberg sub-frame set of length k** if $F = \cup_{j=1}^{k}(E + n_j)$ for some (n_j) and some measurable subset E in $[0,1)$ and

$$A \leq \inf_{|z|=1} |\sum_{j=1}^{k} z^{n_j}|.$$

Casazza and Kalton[5] also classified all WH-frame sets for $a = b = 1$.

THEOREM 4.3. *Let F be a complete subset of $\mathbb{R}$. The following are equivalent:*

(1) F is a Weyl-Heisenberg frame set.

(2) There are constants $k, A > 0$ so that F can be written as a union (finite or infinite) of pairwise completely disjoint elementary A-Weyl-Heisenberg sub-frame sets of length $\leq k$.

We have the following examples to illustrate the technicalities which arise just for the case $a = b = 1$.

EXAMPLE 4.4. *If $g = \chi_{[0,2]}$ or $g = \chi_{[0,1]} - \chi_{[1,2]}$, then $(g, 1, 1)$ is not a frame.*

Proof. This is immediate by Proposition 2.1. □

EXAMPLE 4.5. *If $g = \chi_{[0,3]}$, then $(g, 1, 1)$ is not a frame.*

Proof. If we consider the function

$$f = \chi_{[0,1]} - \frac{1}{2}\chi_{[1,2]} - \frac{1}{2}\chi_{[2,3]} + \chi_{[3,4]} - \frac{1}{2}\chi_{[4,5]} - \frac{1}{2}\chi_{[5,6]} + \chi_{[6,7]} \cdots,$$

where the sum has $3n$ terms. Then $< f, g >= 0$ for all terms except 4 of them at the ends and these terms yield: $\frac{1}{2}$ twice and 1 twice. So

$$\sum_{k\in\mathbb{Z}} |< f, g >|^2 = \frac{5}{2}.$$

Since $||f|| = (n + \frac{n}{2})^{1/2}$, we see that $(g, 1, 1)$ is not a frame. □

EXAMPLE 4.6. *If $g = \frac{1}{2}\chi_{[0,2]} - \chi_{[2,3]}$, then $(g, 1, 1)$ is not a frame.*

Proof. This is immediate by Proposition 2.2. □

EXAMPLE 4.7. *If $g = \chi_{[0,2]} - \chi_{[2,3]}$, then $(g, 1, 1)$ does give a WH-frame.*

Proof. We just note that $(g, 1, 1)$ satisfies the hypotheses of Theorem 1.3. In particular,

$$\sum_{n\in\mathbb{Z}} |g(t-n)|^2 = 3,$$

while

$$G_k = 0, \quad \text{for} \quad |k| \geq 3, \quad \text{and} \quad k = 1, -1.$$

Finally,

$$G_2 = G_{-2} = -1.$$

So $A = 3 - 2 = 1$ and $B = 3 + 2 = 5$ in Theorem 1.3. □

5. FUNDAMENTAL FRAMES

In this section we give some explicit characteristic functions that yield WH-frames. Since they come from the very natural characteristic functions $\chi_{[0,a)}$ and $\chi_{[0,\frac{1}{b})}$ we refer to these as the **fundamental frames for the system determined by** a **and** b. One can use these fundamental frames to decompose the frame operator of any PF WH-system (g,a,b). The main tool of this section is the $\frac{1}{b}$-inner product and its norm

$$\langle f,g\rangle_{\frac{1}{b}}(t)=\sum_{k\in\mathbb{Z}} f(t-\frac{k}{b})\overline{g(t-\frac{k}{b})}. \qquad \|f\|_{\frac{1}{b}}(t)=\sum_{k\in\mathbb{Z}}|f(t-\frac{k}{b})|^2$$

Here we state a few of the necessary properties of this $\frac{1}{b}$-inner product. First we introduce the standard $\frac{1}{b}$-orthonormal basis. Let $e_k=T_{\frac{k}{b}}\chi_{[0,\frac{1}{b})}$. .

PROPOSITION 5.1. *For all $f,g,h\in L^2(\mathbb{R})$ and $j\in\mathbb{Z}$:*

(1)	$\langle f,g\rangle_{\frac{1}{b}}(t)$ *is* $\frac{1}{b}$ *− periodic*	*(2)*	$f=\sum_{k\in\mathbb{Z}}\langle f,e_k\rangle_{\frac{1}{b}}(t)(t)e_k$
(3)	$\langle e_0,f\rangle_{\frac{1}{b}}(t)e_0=\overline{f}e_0$	*(4)*	$\langle fg,h\rangle_{\frac{1}{b}}(t)=\langle f\overline{h},\overline{g}\rangle_{\frac{1}{b}}(t)$
(5)	$\left\langle T_{\frac{j}{b}}f,g\right\rangle_{\frac{1}{b}}(t)=\left\langle f,T_{-\frac{j}{b}}g\right\rangle_{\frac{1}{b}}(t)$	*(6)*	$T_{\frac{j}{b}}\langle f,g\rangle_{\frac{1}{b}}(t)=\langle f,g\rangle_{\frac{1}{b}}(t)$

Ron and Shen[11,12] and Casazza and Lammers[6] have both used inner products of this type to give what the later called compressions of operators for WH-systems. Associated with this $\frac{1}{b}$-inner product are a special class of operators.

DEFINITION 5.2. *We say that a linear operator $L: L^2(\mathbb{R})\to L^2(\mathbb{R})$ is a* $\frac{1}{b}$**-factorable operator** *if for any factorization $f=\phi g$ where $f,g\in L^2(\mathbb{R})$ and ϕ is an $\frac{1}{b}$-periodic function on $\mathbb{R}$ we have*

$$L(f)=L(\phi g)=\phi L(g).$$

We summarize some of the known results about $\frac{1}{b}$-factorable operators and compressions in the theorem below.

THEOREM 5.3. *Let $\|g\|_{\frac{1}{b}}(t)\le B$ a.e. and $\|h\|_{\frac{1}{b}}(t)\le C$ a.e.*
(1) Let $L(f)=\sum_{m\in\mathbb{Z}}\langle f,E_{mb}g\rangle E_{mb}(t)h$. Then L is a $\frac{1}{b}$ factorable operator and has the following **compression**

$$\mathcal{L}(f)=\langle f,g\rangle_{\frac{1}{b}}(t)h$$

(2) The frame operator, frame transform and preframe operator for the system (g,a,b) are $\frac{1}{b}-$ factorable and may be compressed as follows.

$$S_g(f)=\frac{1}{b}\sum_{n\in\mathbb{Z}}\langle f,T_{na}g\rangle_{\frac{1}{b}}(t)T_{na}g,\ T(f)=\sqrt{\frac{1}{b}}\sum_k\langle f,e_k\rangle_{\frac{1}{b}}(t)T_{ka}g \text{ and } T^*(f)=\sqrt{\frac{1}{b}}\sum_k\langle f,T_{ka}g\rangle_{\frac{1}{b}}(t)e_k.$$

(3) For any bounded $\frac{1}{b}$-factorable operator L from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$

$$\|L(f)\|_{\frac{1}{b}}(t)\le\|L\|\|g\|_{\frac{1}{b}}(t) \text{ a.e. }.$$

In our first application of a frame generated by a characteristic function we use the "frame" $(\sqrt{b}\chi_{[0,\frac{1}{b})},\frac{1}{b},b)$ to produce a pointwise necessary condition for (g,a,b) to be PF. A moments reflection shows this yields the standard orthonormal basis for $L^2(\mathbb{R})$ associated with $e^{2\pi imbt}$. However, our concern will be the connection with the e_k's from above.

The functions G_k have an elementary representation in the a- inner product. Namely $G_k(t)=\left\langle g,T_{\frac{k}{b}}g\right\rangle_a(t)$. It is well known that if the system (g,a,b) is PF then $\sum|G_k|^2\le B<\infty$. We now interchange the roles of a and b.

THEOREM 5.4. *If (g,a,b) is PF with upper frame bound B then*

$$\sum |\langle g, T_{na}g\rangle_{\frac{1}{b}}(t)|^2 \leq bB\|g\|_{\frac{1}{b}}(t) a.e.$$

Proof. If (g,a,b) is PF we know by Theorem 1.2 that $\|g\|_{\frac{1}{b}}(t) \leq B$ a.e. and T is a bounded operator. Hence, because $\|T\| = \|T^*\|$ we have $\|T^*(g)\|_{\frac{1}{b}}(t) \leq \|T\|\|g\|_{\frac{1}{b}}(t)$ and

$$\langle T^*(g), T^*(g)\rangle_{\frac{1}{b}}(t) = \frac{1}{b}\left\langle \sum_n \langle g, T_{na}g\rangle_{\frac{1}{b}} e_n, \sum_k \langle g, T_{ka}g\rangle_{\frac{1}{b}} e_k \right\rangle_{\frac{1}{b}} = \frac{1}{b}\sum |\langle g, T_{na}g\rangle_{\frac{1}{b}}|^2(t).$$

The last inequality follows from the $\frac{1}{b}$-orthnormality of the e_k. □

If we consider the case $a = b = 1$ then this is the same condition as $\sum |G_k|^2 \leq B < \infty$. This condition is not sufficient for having a WH-frame.

EXAMPLE 5.5. *There exist a system $(g, 1, 1)$ such that $\sum |G_k|^2 \leq B < \infty$ yet $(g, 1, 1)$ is not PF.*

Proof. Consider the function $g = \sum_{n>1} \frac{e_n}{n}$. By Casazza, Christensen, and Janssen[3] (Corollary 3.7), if $a = b = 1$ then a positive real valued function is PF iff $\sum |G_k| \leq B < \infty$. However a direct computation shows that $G_k = (1/k)\sum_{n=1}^{k} \frac{1}{n}$ for the above g. These G_k are square summable but not summable. Hence $(g, 1, 1)$ is not PF. □

One can present a large number of necessary conditions for the system (g,a,b) to be PF with these techniques by switching between the frame operator, preframe operator and the frame transform. We do not know if any of them are also sufficient. We present one more representation which is stronger than the one above, at least in the case $a = b = 1$.

PROPOSITION 5.6. *Assume the system (g,a,b) is PF. Then*

$$\lim_{m\to\infty} \sum_j \frac{|\sum_{k=-m}^{m} T_{ka}g(t - j/b)|^2}{2mb} \leq B^2.$$

Proof. Let $f_m = \sum_{k=-m}^{m} e_k$ where e_k is the standard $\frac{1}{b}$-orthonormal basis. Since T is a $\frac{1}{b}$-factorable linear operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ we may apply Theorem 5.2 (3) to get:

$$\|T(f_m)\|_{\frac{1}{b}}^2 \leq B^2 2m \quad and \quad \langle T(f_m), T(f_m)\rangle_{\frac{1}{b}} = \frac{1}{b}\sum_j |\sum_{k=-m}^{m} T_{ka}g(t - j/b)|^2.$$

Since m was arbitrary we have

$$\lim_{m\to\infty} \sum_j \frac{|\sum_{k=-m}^{m} T_{ka}g(t - j/b)|^2}{2mb} \leq B^2.$$

□

By "stronger" we mean that the example presented above ($g = \sum_{n>1} \frac{e_n}{n}$) does not satisfy

$$\lim_{m\to\infty} \sum_j \frac{|\sum_{k=-m}^{m} T_{ka}g(t - j/b)|^2}{2mb} \leq B < \infty.$$

5.1. Frame decompositions

Now we present a representation of the frame operator for the system (g,a,b) that mixes the a-inner product and the $\frac{1}{b}$-inner product. Note that in this theorem we are mixing our a-inner product with the $\frac{1}{b}$-orthonormal basis. The end result is that each $\left\langle\left\langle T_{-\frac{j}{b}}g, T_{-\frac{k}{b}}g\right\rangle_a, e_0\right\rangle_{\frac{1}{b}}(t)$ is a $\frac{1}{b}$ section of an a-periodic function which is then extended $\frac{1}{b}$ periodically.

THEOREM 5.7. *Let (g,a,b) be a PF WH-system. Then the frame operator has the representation*

$$S(f) = \sum_k \langle f, e_k\rangle_{\frac{1}{b}}(t)h_k = \sum_k \langle f, h_k\rangle_{\frac{1}{b}}(t)e_k = \sum_k \langle f, e_k\rangle_{\frac{1}{b}}(t)\sum_j \left\langle\left\langle T_{-\frac{j}{b}}g, T_{-\frac{k}{b}}g\right\rangle_a, e_0\right\rangle_{\frac{1}{b}}(t)e_j$$

where e_k is the standard $\frac{1}{b}$-orthonormal basis and $h_k = S(e_k)$.

Proof. Since the system is PF we know that S is a continuous operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Recall $e_k = T_{\frac{k}{b}}\chi_{[0,\frac{1}{b})}$ and $f = \sum_k \langle f, e_k\rangle_{\frac{1}{b}} e_k$.

Now because S is linear, $\frac{1}{b}$-factorable, continuous and self adjoint we get

$$S(f) = \sum \langle f, e_k\rangle_{\frac{1}{b}} S(e_k) = \sum \langle f, S(e_k)\rangle_{\frac{1}{b}} e_k$$

So it is enough to look at $S(e_k)$. First let us note that the computations below rely heavily on the results from Proposition 5.1. Now we compute:

$$\begin{aligned}
S(e_k) = \sum_j \langle S(e_k), e_j\rangle_{\frac{1}{b}} e_j &= \sum_j \left\langle \sum_n \langle e_k, T_{na}g\rangle_{\frac{1}{b}} T_{na}g, e_j\right\rangle_{\frac{1}{b}} e_j \\
&= \sum_j \left\langle \sum_n \langle e_k, T_{na}g\rangle_{\frac{1}{b}} e_j, \overline{T_{na}g}\right\rangle_{\frac{1}{b}} e_j \\
&= \sum_j \left\langle \sum_n \left\langle e_o, T_{na}T_{-\frac{k}{b}}g\right\rangle_{\frac{1}{b}} e_0, T_{na}T_{-\frac{j}{b}}\overline{g}\right\rangle_{\frac{1}{b}} e_j \\
&= \sum_j \left\langle \sum_n T_{na}T_{-\frac{k}{b}}\overline{g}e_0, T_{na}T_{-\frac{j}{b}}\overline{g}\right\rangle_{\frac{1}{b}} e_j \\
&= \sum_j \left\langle \sum_n T_{na}T_{-\frac{k}{b}}\overline{g}T_{na}T_{-\frac{j}{b}}g, e_0\right\rangle_{\frac{1}{b}} e_j \\
&= \sum_j \left\langle\left\langle T_{-\frac{j}{b}}g, T_{-\frac{k}{b}}g\right\rangle_a, e_0\right\rangle_{\frac{1}{b}} e_j
\end{aligned}$$

This gives the result. □

Now we decompose this frame operator with respect to a natural frame arising from the parameters a, b. Let $\alpha_k = T_{ka}\chi_{[0,a]}$. Then by Theorem 1.2 (4)and the representation of the frame operator given in equation (5) it is clear that $(\sqrt{b}\alpha_0, a, b)$ is a normalized tight frame and for all $f \in L^2(\mathbb{R})$ we have

$$f = \sum_{k\in\mathbb{Z}} \langle f, \alpha_k\rangle_{\frac{1}{b}}(t)\alpha_k.$$

Given the relation ship between (g,a,b) and $(g,1/b,1/a)$ in Theorem 1.2 (2) it is also natural to consider the sequence of characteristic functions $\phi_k = T_{\frac{k}{b}}\chi_{[0,a)}$ Putting this together with the decomposition above yields the following.

PROPOSITION 5.8. *Let (g,a,b) be a PF WH-system. The frame operator has the representation*

$$S(f) = \sum_{k\in\mathbb{Z}} \langle f, \alpha_k\rangle_{\frac{1}{b}}(t)T_{ka}\sum_j G_{-j}(t)\phi_j.$$

Proof. We use the fact that S is $\frac{1}{b}$-factorable and then apply S to $f = \sum_k \langle f, \alpha_k \rangle_{\frac{1}{b}} (t)\alpha_k$. So we get

$$S(f) = \sum_{k\in\mathbb{Z}} \langle f, \alpha_k \rangle_{\frac{1}{b}} (t) S(\alpha_k)$$

It is also well known that S commutes with T_{ka} for all $k \in \mathbb{Z}$ so it is enough to find $S(\alpha_0)$. Let us use the representation above. Since all the $\frac{1}{b}$-inner products are 0 except one we get:

$$S(\alpha_0) = \langle \alpha_0, e_0 \rangle_{\frac{1}{b}} (t) \sum_j \left\langle \left\langle g, T_{-\frac{j}{b}} g \right\rangle_a, e_0 \right\rangle_{\frac{1}{b}} e_j.$$

Now since $\langle \alpha_0, e_0 \rangle_{\frac{1}{b}} (t) e_j = \phi_j$ we get the result. Note that since the ϕ_j are only supported on an interval of length a we no longer need the $\frac{1}{b}$-periodic extension. □

5.2. Equivalent frames and $S^{\frac{-1}{2}}$

In this last section we will give one more fundamental example of a WH-frame obtained from a characteristic function and another normalized tight frame. We go on to show that the two frames are equivalent

First, the sake of simplicity, we do the case where $\frac{1}{2} \le ab \le 1$. Let $\beta_k = T_{ka}\chi_{[0,\frac{1}{b})}$. Then again by Theorem 1.2 (4) we know that $(\sqrt{b}\beta_0, a, b)$ forms a frame with upper frame bound 2 and lower frame bound 1 and frame operator

$$S^b(f) = \sum_{k\in\mathbb{Z}} \langle f, \beta_k \rangle_{\frac{1}{b}} \beta_k.$$

Now let $\delta = \frac{1}{b} - a$ and $\gamma_0 = \frac{1}{\sqrt{2}} \left(\chi_{[0,\delta)} + \chi_{[a,\frac{1}{b})} \right) + \chi_{[\delta,a)}$ and $\gamma_k = T_{ka}\gamma_0$. Then $(\sqrt{b}\gamma_0, a, b)$ forms a normalized tight frame which yields the following decomposition of the identity operator:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \gamma_k \rangle_{\frac{1}{b}} \gamma_k.$$

Finally we let $\psi_0 = \frac{1}{\sqrt{2\sqrt{2}}} \left(\chi_{[0,\delta)} + \chi_{[a,\frac{1}{b})} \right) + \chi_{[\delta,a)}$, $\psi_k = T_{ka}\psi_0$ and consider the frame $(\sqrt{b}\psi, a, b)$ with frame operator

$$S^\psi(f) = \sum_k \langle f, \psi_k \rangle_{\frac{1}{b}} \psi_k.$$

Now we compute $S^\psi(\beta_0)$

$$S^\psi(\beta_0) = \langle \beta_0, \psi_{-1} \rangle_{\frac{1}{b}} \psi_{-1} + \langle \beta_0, \psi_0 \rangle_{\frac{1}{b}} \psi_0 + \langle \beta_0, \psi_1 \rangle_{\frac{1}{b}} \psi_1 = \gamma_0$$

Again since frame operators commute with T_{ka} we get that $S^\psi(\beta_k) = \gamma_k$ and hence the two frames are equivalent. Furthermore since $(\sqrt{b}\gamma_0, a, b)$ is a normalized tight frame we have that

$$S^\psi S^b S^\psi = S^\psi \left(\sum_{k\in\mathbb{Z}} \langle S^\psi(f), \beta_k \rangle_{\frac{1}{b}} \beta_k \right) = S^\psi \left(\sum_{k\in\mathbb{Z}} \langle f, \gamma_k \rangle_{\frac{1}{b}} \beta_k \right) = \sum_{k\in\mathbb{Z}} \langle f, \gamma_k \rangle_{\frac{1}{b}} \gamma_k = f$$

and therefore $S^\psi S^b S^\psi = I$ and hence $(S^b)^{\frac{-1}{2}} = S^\psi$

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