# Classifying Irregular Gabor Frames

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## 1 Introduction

A sequence $\{f_i\}$ is a Hilbert space $H$ is a **frame** for $H$ if there are constants $A, B > 0$ satisfying:

$$A\|f\|^2 \leq \sum_i |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for all} \quad f \in H.$$

$A$ (respectively $B$) is called the **lower** (respectively **upper**) **frame bound** of the frame. For $0 < a, b \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ we define **translation by a** and **modulation by b** as:

$$T_a g(t) = g(t-a) \quad \text{and} \quad E_b g(t) = e^{2\pi i b t} g(t).$$

If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call it a **Gabor frame** (or a **Weyl-Heisenberg frame**). If $(x_m, y_m) \in \mathbb{R}\times\mathbb{R}$ and $\{E_{x_m}T_{y_m}g\}_{m\in\mathbb{N}}$ is a frame for $L^2(\mathbb{R})$, we call this an **irregular Gabor frame**. We refer the reader to [2] and [14] for an overview of Gabor frames.

The motivation for this research was a fundamental result of Feichtinger and Gröchenig [12]:

**Theorem 1.1 (Feichtinger and Gröchenig).** *Let $0 \neq g \in L^2(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle E_x T_y g, g\rangle| dx\ dy < \infty.$$

*Then there is a box $Q = [0,a]\times[0,b]$ so that if $\{Q_m\}_{m\in\mathbb{Z}}$ is a tiling of $\mathbb{R}^2$ by $Q$, then for every $(x_m, y_m) \in Q_m$, the family $\{E_{x_m}T_{y_m}g\}_{m,n\in\mathbb{Z}}$ is an irregular Gabor frame for $L^2(\mathbb{R})$.*

We refer the reader to [9] for a unified treatment of the Feichtinger/Gröchenig theory and a proof of Theorem 1.1. Unfortunately, Theorem 1.1 does not give any information about how to choose the box $Q$. Also, the condition as stated is extremely difficult to check - even for characteristic functions. The motivation for our research, then, was to find an easily verifyable condition which will still guarantee the conclusion of Theorem 1.1 and to classify the functions satisfying the theorem. Although this particular result will appear in a later paper, in this paper [4] we give complete classifications and simply verifiable conditions for the lattice forms of Theorem 1.1.

We say that $(g, a, b)$ generates a Gabor frame for $L^2(\mathbb{R})$ if $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. In general, even in very special cases it is difficult to identify those functions $g$ and $a, b > 0$ so that $(g, a, b)$ generates a Gabor frame. It is known that $(E_{mb}T_{na}g)$ is not complete for any $g \in L^2(\mathbb{R})$ if $ab > 1$. If $ab = 1$, then $(g, a, b)$ generates a frame if and only if it generates a Riesz basis for $L^2(\mathbb{R})$. And if $ab < 1$, then if $(g, a, b)$ generates a Gabor frame, then this frame has infinite excess. It is still an open problem to find all $a, b, c > 0$ so that $(\chi_{[0,c]}, a, b)$ generates a Gabor frame (see [8],[13]). For $a = b = 1$, there are classifications of those functions $g$ for which $(g, 1, 1)$ generates a tight Gabor frame (see [5]). Also, $(\chi_{[0,c]}, 1, 1)$ generates an orthonormal basis for $L^2(\mathbb{R})$ if $c = 1$; it is a normalized tight frame sequence which is not a frame if $c < 1$; and it is not a frame sequence if $1 < c$ (see [8],[13]). The general case of when $(\chi_F, 1, 1)$ generates a Gabor frame for a measurable subset $F$ of $\mathbb{R}$ is a deep question. Casazza and Kalton [6] have shown that this question is equivalent to a classical problem of Littlewood concerning certain complex polynomials and when they have roots on the unit circle.

In light of the above, one can not expect to find good classification theorems in general for functions generating Gabor frames. The reason we are able to get some exact classifications in [4] is that we are requiring $(g, a, b)$ to generate a Gabor frame for a whole range of values of $a, b$. It turns out that this is a strong assumption and puts easily identifiable restrictions on the functions $g$. The underlying tool in this paper is a construction of Daubechies [11] and

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more exactly, a slightly more general form of this construction due to Walnut (see [14],[15]).

**Theorem 1.2.** *Let $g \in L^2(\mathbb{R})$ and $a > 0$ be such that:*

*(1) There exist constants $A, B$ such that*

$$A \leq \sum_{n\in\mathbb{Z}} |g(t-na)|^2 \leq B < \infty \text{ a.e.}$$

*(2) We have*

$$\lim_{b\to 0} \sum_{k\neq 0} \beta(k/b) = 0,$$

*where*

$$\beta(s) = ess\ sup_{x\in\mathbb{R}} |\sum_{n\in\mathbb{Z}} g(x-na)\overline{g(x-s-na)}|$$

$$= \|\sum_{n\in\mathbb{Z}} T_{na}g \cdot T_{na+s}\overline{g}\|_\infty.$$

*Then there exists $b_0 > 0$ such that $(g, a, b)$ generates a Gabor frame for $L^2(\mathbb{R})$ for each $0 < b < b_0$.*

To put this result into practice, Walnut [15],[14] switches to working with $g \in W(L^\infty, \ell^1)$. This allows us to easily get non-compactly supported functions $g$ generating Gabor frames for a whole range of values of $b$.

**Theorem 1.3.** *Let $g \in W(L^\infty, \ell^1)$ be such that $g$ satisfies condition (1) of Theorem 1.2 for some $a$. Then there is a $b_0 > 0$ such that $(g, a, b)$ generates a Gabor frame for $L^2(\mathbb{R})$ for all $0 < b < b_0$.*

It is these constructions which provide the tools necessary for our work. But to put them into practice, we rely on, what is essentially the Poisson summation formula, to transfer the calculations to $L^2[0, 1/b]$ and then use perturbation theory for exponential frames to allow $a$ and $b$ to vary.

## 2 Results

One quickly realizes that there are several intermediate versions of Theorem 1.1 which are also important. In particular, when can we find a box $Q = [0, c]\times[0, d]$ so that whenever $b \in (0, c]$ and $y_n \in [nd, (n+1)d]$ then $\{E_{mb}T_{y_n}g\}$ is a frame for $L^2(\mathbb{R})$? We call these **half-irregular** Gabor frames. Or, when is there a box $Q = [0, c]\times[0, d]$ so that whenever $x_m \in [mc, (m+1)c]$ and $y_n \in [nd, (n+1)d]$ then $\{E_{x_m}T_{y_ng}\}$ is a frame for $L^2(\mathbb{R})$? It is these questions we answer in this paper.

We were surprised to discover during this investigation that the **WH-Frame Identity** of Daubechies holds for half-irregular Gabor frames. Also, the **CC-Condition** [3] holds for this case.

**Theorem 2.1.** *Given $g \in L^2(\mathbb{R})$, $b > 0$ and a sequence $\{y_n\}_{n\in\mathbb{Z}}$, assume that*

$$\sup \sum_{k} \sum_{m\in\mathbb{Z}} |g(t-y_n)| \cdot |g(t-y_n-k/b)| < \infty.$$

*Then the following holds:*

*(i) For all bounded and compactly supported functions $f$ we have:*

$$\sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{y_n}g\rangle|^2 =$$

$$\frac{1}{b}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)}f(t-k/b) \sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-k/b)}\ dt$$

$$= b^{-1}\int_{\mathbb{R}} |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2\ dt +$$

$$b^{-1}\sum_{k\neq 0} \int_{\mathbb{R}} \overline{f(t)}f(t-k/b) \sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-k/b)}\ dt$$

*(ii) Assume that*

$$A = \inf_{t\in\mathbb{R}} \left[\sum_{m\in\mathbb{Z}} |g(t-y_n)|^2 - \sum_{k\neq 0} |\sum_{m\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-\frac{k}{b})}|\right] > 0$$

*and let*

$$B := \sup_{t\in\mathbb{R}} \sum_{k\neq 0} |\sum_{m\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-\frac{k}{b})}|.$$

*Then $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{b}, \frac{B}{b}$.*

Casazza and Lammers [7] showed (in a form disguised by the notation) that there is a pointwise form of the WH-frame Identity which characterizes when a family $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. In this paper we show that this result holds for half-irregular Gabor frames.

**Theorem 2.2.** *Let $g \in L^2(\mathbb{R})$, $b > 0$ and $(y_n)_{n\in\mathbb{Z}}$ be a relatively separated sequence of real numbers. The following are equivalent:*

*(1) $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with frame bounds $A, B$.*

*(2) For all bounded, compactly supported $f \in L^2(\mathbb{R})$ we have:*

$$A\sum_{j\in\mathbb{Z}} |f(t-j/b)|^2 \leq$$

$$\frac{1}{b}\sum_{n\in\mathbb{Z}}|\sum_{k\in\mathbb{Z}}f(t-k/b)\overline{g(t-k/b-y_n)}|^2$$

$$=\frac{1}{b}\sum_{j\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}\overline{f(t-\ell/b)}f(t-\ell/b-j/b)\cdot$$

$$\sum_{n\in\mathbb{Z}}g(t-y_n-\ell/b)\overline{g(t-y_n-(\ell+j)/b)}$$

$$\leq B\sum_{j\in\mathbb{Z}}|f(t-j/b)|^2 \quad a.e.$$

A fundamental result for Gabor frames $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ with frame bounds $A, B$ is that

$$A\leq\sum_{n\in\mathbb{Z}}|g(t-na)|^2\leq B, \quad \text{a.e.}$$

It was shown by Casazza and Christensen [3] that this condition is not a necessary condition for a Gabor family to form a frame for its closed linear span. We show in this paper that this condition is necessary for irregular Gabor frames.

**Proposition 2.3.** *Assume that $\{E_{x_n}T_{y_n}g\}_{n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A, B$. There is an $a>0$ so that $\{E_{x_n}\}_{n\in\mathbb{Z}}$ is a frame for $L^2[0,a]$ with frame bounds say $A_1, B_1$. Furthermore*

$$A/B_1\leq\sum_{m\in\mathbb{Z}}|g(t-y_n)|^2\leq B/A_1 \text{ a.e.} \tag{2.1}$$

In the results that follow we will assume that our $g$ is in the Wiener space $W(L^\infty,\ell^1)$ where for $1\leq p<\infty$ we define

$$W(L^\infty,\ell^p)=\{f:\sum_n\|\chi_{[n,n+1]}g\|_\infty^p<\infty\}.$$

We do not know if this assumption is necessary but we suspect that it is. In the paper we show that $g\in W(L^\infty,\ell^2)$ is necessary.

**Proposition 2.4.** *Let $g\in L^2(\mathbb{R})$ and assume that there exist $a,b>0$ such that for all $y_n\in[na,(n+1)a]$, $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds $A, B$. Then*

$$bA\leq\sum_{m\in\mathbb{Z}} ess\ inf_{t\in[an,a(n+1)]}\ |\chi_{[an,a(n+1)]}g(t)|^2$$

$$\leq\sum_{m\in\mathbb{Z}}||\chi_{[an,a(n+1)]}g||_\infty^2\leq bB.$$

*In particular, $g\in W(L^\infty,\ell^2)$.*

The first main classification for Gabor frames in the paper concerns the question of which functions $g$ have the property that for a whole range of $a, b$ the family $(E_{mb}T_{na}g)_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$.

**Theorem 2.5.** *Let $g\in W(L^\infty,\ell^1)$. Then the following are equivalent:*

*(a) There exists a box $Q:=[a_1,b_1]\times[a_2,b_2]\subseteq\mathbb{R}^2$ and an $A>0$ such that*

$$P(x,y):=\sum_{m\in\mathbb{Z}}|g(x-ny)|^2\geq A \text{ for a.e. } (x,y)\in Q.$$

*(b) There exists $a_0>0$ so that for all $0<c_0<a_0$, there are $b_0, A>0$ such that for all $a\in[c_0,a_0], b\in(0,b_0]$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound $A$.*

In some sense it is a surprise that we can give such exact conditions on a function $g$ for it to generate Gabor frames since the general question of classifying functions $g\in L^2(\mathbb{R})$ giving Gabor frames is open. What is different here is that we are requiring that we get a frame for this fixed $g$ for a whole range of choices of $a, b$ and not just one choice. It turns out that in this case, the function $g$ has to satisfy quite exact conditions. Note that we get the conclusion of Theorem 2.5 if $g$ is continuous and non-zero at one point or if $|g|$ is bounded below an an interval in $\mathbb{R}$. An important part of this work is to give exact estimates for the bounds on $(x_m),(y_n)$ so that $(E_{x_m}T_{y_n}g)_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$. For example, the proof of Theorem 2.5 gives the bounds,

**Corollary 2.6.** *Let $g\in W(L^\infty,\ell^1)$ and assume there is a box $Q=[a_1,b_1]\times[a_2,b_2]$ so that*

$$A\leq\sum_{m\in\mathbb{Z}}|g(x-ny)|^2 \quad a.e. \quad (x,y)\in Q.$$

*Let $a_0=min(b_1-a_1,b_2-a_2)$ and $0<c_0<a_0$. Choose $\epsilon>0$ such that $8\epsilon\|g\|_{W,c_0}+4\epsilon^2\leq\frac{A}{2}$, and choose a natural number $N$ so that $\sum_{|n|\geq N}\|g\cdot\chi_{[c_0n,c_0(n+1)]}\|_\infty\leq\epsilon$. Choose $b_0$ so that $1/b_0\geq 2a_0N$. Then for all $0<c_0\leq a\leq a_0$ and all $0<b\leq b_0$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with frame bounds $A/2, B=A/2+\|g\|_{W,c_0}$.*

The next main result in the paper concerns half-irregular Gabor frames and is also a complete classification.

**Theorem 2.7.** *Let $g\in W(L^\infty,\ell^1)$. The following are equivalent:*

*(a) $g$ is bounded below on an interval in $\mathbb{R}$.*

*(b) There are numbers $a, b_0, A>0$ so that for all $0<b\leq b_0$ and all $y_n\in[an,a(n+1)]$, $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with lower frame bound $A$.*

Note that "$g$ bounded below on an interval" means that there is an interval $I$ in $\mathbb{R}$ so that

$$\text{ess inf}_{t\in I}|g(t)| \geq A > 0.$$

Again, it is important to see exactly how to pick the parameters which work in Theorem 2.7.

**Corollary 2.8.** *Let $|g|^2$ be bounded below by $A > 0$ on some interval $I$. Let $a = |I|/2$ and choose $\epsilon > 0$ so that*

$$8\|g\|_{W,a}\epsilon + 4\epsilon^2 \leq \frac{A}{2}.$$

*Next, choose a natural number $N$ so that*

$$\sum_{|n|\geq N} \|g \cdot \chi_{[an,a(n+1)]}\|_\infty < \epsilon.$$

*Finally, let $b_0 = (2aN)^{-1}$. Then this $a, b_0 A$ work in Theorem 2.7.*

The final main theorem in the paper is the classification theorem for irregular lattices. The proof of this theorem introduces a new approach for computing Gabor frame bounds which should have broader applications in the field.

**Theorem 2.9.** *Let $g \in W(L^\infty, \ell^1)$. Then the following are equivalent:*

*(1) $g$ is bounded below on an interval.*

*(2) There exist $a, b_0, A > 0$ such that for all $b \in ]0, b_0]$, and all $x_m \in [mb,(m+1)b]$, and $y_n \in [na,(n+1)a]$, $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound $A$.*

Again, it is important to know how to pick the parameters in Theorem 2.9.

**Corollary 2.10.** *Assume ess $\sup_{t\in\mathbb{R}}|g(t)| \leq 1$ and that $|g|$ is bounded below by $C > 0$ on an interval $I$. Let $a = |I|/2$ and*

$$A = \frac{(40)(32)}{a}\|g\|^2_{W,a}.$$

*Choose $\epsilon > 0$ so that*

$$4\epsilon(1 + 2\|g\|_{W,a}) \leq \frac{C^2}{4a}.$$

*Choose a natural number $L$ so that for all $\frac{1}{2} \leq b \leq 1$ we have*

$$\sum_{2|n|\geq L} \|\chi_{[n/b,(n+1)/b]}g\|_\infty \leq \epsilon.$$

*Finally, choose $M \in \mathbb{N}$ so that*

$$\frac{MC^2}{4a} \geq \frac{(80)(32)}{a}\|g\|^2_{W,a},$$

*and*

$$\sum_{2|n|\geq M} \|\chi_{[n/b,(n+1)/b]}g\|_\infty \leq \frac{\epsilon a}{2L}.$$

*Now let $b_0 = M^{-1}$. Then this $a, b_0, A$ satisfy Theorem 2.9.*

The proof of Theorem 2.9 relies on an exact calculation of certain constants which arise in the Balan/Christensen generalization of the Kadec 1/4-Theorem (see [1] and [10]). Recall that for a Bessel sequence $\{f_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $H$ the **pre-frame operator** associated with $\{f_n\}$ is the operator $L : \ell^2(\mathbb{Z}) \to H$ given by

$$L\{a_n\}_{n\in\mathbb{Z}} = \sum_{n\in\mathbb{Z}} a_n f_n.$$

It is well-known that the adjoint $L^* : H \to \ell^2(\mathbb{Z})$ is given by

$$L^*f = \{\langle f, f_n\rangle\}_{n\in\mathbb{Z}}.$$

**Theorem 2.11.** *Let $0 < b < 1/4$, $K \in \mathbb{N}$ with $K \geq 4\pi$, and let $x_m \in [\frac{mb}{K}, \frac{(m+1)b}{K}]$. For all $k \in \mathbb{Z}$, $\{\sqrt{b}T_{k/b}E_{x_m}\}_{m\in\mathbb{Z}}$ and $\{\sqrt{b}T_{k/b}E_{\frac{mb}{K}}\}_{m\in\mathbb{Z}}$ are frames for $L^2[0, 1/b]$. Denote the pre-frame operators by $R_k$ and $L_k$ respectively. We have*

*(1) $L_kL^*_\ell = KI$ if $k - \ell = nK$ for some $n \in \mathbb{Z}$, and $L_kL^*_\ell = 0$, otherwise.*

*(2) $\|KI - R_kR^*_\ell\| \leq 20$, if $k - \ell = nK$, for some $n \in \mathbb{Z}$.*

*(3) For $k - \ell \notin K\mathbb{Z}$, $\|R_kR^*_\ell\| \leq 20$.*

*Finally, a set of frame bounds for $\{\sqrt{b}T_{k/b}E_{x_m}\}_{m\in\mathbb{Z}}$ is $A = \frac{K}{4}$ and $B = 4K$.*

We also give a general approach to doing frame bound computations with irregular Gabor frames. Since these two results are relatively straightforward and should have applications outside of this paper, we will state them with proofs below.

**Theorem 2.12.** *Let $g \in L^2(\mathbb{R})$, $b > 0$ and $x_m, y_n \in \mathbb{R}$ with $\{x_n\}$ relatively separated. For each $k \in \mathbb{Z}$, let $R_k : \ell^2(\mathbb{Z}) \to L^2[0, \frac{1}{b}]$ be the pre-frame operator for $\{\sqrt{b}\chi_{[0,1/b]}T_{k/b}E_{x_m}\}_{m\in\mathbb{Z}}$. Then for all $f \in L^2(\mathbb{R})$ we have*

$$\sum_{n,m\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 =$$

$$\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\langle R_k^*\left(\chi_{[0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g}\right),$$
$$R_\ell^*\left(\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g}\right)\rangle_{\ell^2}.$$

*Proof*: We compute:

$$\sum_{m,n\in\mathbb{Z}}|\langle f,E_{x_m}T_{y_n}g\rangle|^2=$$
$$\sum_{m,n\in\mathbb{Z}}|\int_{\mathbb{R}}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt|$$
$$=\sum_{m,n\in\mathbb{Z}}|\sum_{k\in\mathbb{Z}}\int_{k/b}^{(k+1)/b}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt|^2$$
$$=\sum_{m,n\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\int_{k/b}^{(k+1)/b}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt\right)$$
$$\times\overline{\left(\sum_{\ell\in\mathbb{Z}}\int_{\ell/b}^{(\ell+1)/b}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt\right)}$$
$$=\sum_{n,m,k,\ell\in\mathbb{Z}}\left(\int_{k/b}^{(k+1)/b}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt\right)$$
$$\times\overline{\left(\int_{\ell/b}^{(\ell+1)/b}f(t)\overline{T_{y_n}g(t)}e^{-2\pi ix_mt}\ dt\right)}$$
$$=\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\left[\sum_{m\in\mathbb{Z}}\left(\int_0^{1/b}(T_{k/b}f(t))\right.\right.$$
$$\left.\times(\overline{T_{y_n+k/b}g(t)}\sqrt{b}T_{k/b}\overline{E_{x_m}(t)}\ dt\right)$$
$$\left.\times\left(\int_0^{1/b}(\overline{T_{\ell/b}f(t)})(T_{y_n+\ell/b}g(t))(\sqrt{b}T_{\ell/b}\overline{E_{x_m}(t)}\ dt\right)\right]$$
$$=\sum_{n,k,\ell\in\mathbb{Z}}\left[\langle\left(\left(\langle\chi_{[0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g},\sqrt{b}T_{k/b}E_{x_m}\rangle\right)_{m\in\mathbb{Z}},\right.\right.$$
$$\left.\left(\langle\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g},\sqrt{b}T_{\ell/b}E_{x_m}\rangle_{\ell\in\mathbb{Z}}\right)\rangle_{\ell^2}\right]$$
$$=\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\langle R_k^*\left(\chi_{0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g}\right),$$
$$R_\ell^*\left(\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g}\right)\rangle_{\ell^2}.$$

□

Theorem 2.12 leads to an alternative proof of the half-irregular WH-Frame Identity. It is this new proof which forms the basis for our approach to the general irregular lattice cases.

*Alternative Proof of Theorem 2.1 (i)*

We will use Theorems 2.10 and 2.11. To this end, for each $k\in\mathbb{Z}$, let $L_k$ be the preframe operator for $\{\sqrt{b}T_{k/b}E_{mb}\}_{m\in\mathbb{Z}}$. So $K=1$ in Theorem 2.8 and hence $L_kL_\ell^*=I$ for all $k,\ell\in\mathbb{Z}$. Now we just compute starting with Theorem 2.11.

$$\sum_{m,n\in\mathbb{Z}}|\langle f,E_{mb}T_{y_n}g\rangle|^2=$$
$$\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\langle L_k^*\left(\chi_{[0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g}\right),$$
$$L_\ell^*\left(\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g}\right)\rangle_{\ell^2}$$
$$=\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\langle\chi_{[0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g},$$
$$L_kL_\ell^*\left(\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g}\right)\rangle_{L^2[0,1/b]}$$
$$=\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\langle\chi_{[0,1/b]}T_{k/b}f\overline{T_{y_n+k/b}g},$$
$$\chi_{[0,1/b]}T_{\ell/b}f\overline{T_{y_n+\ell/b}g}\rangle_{L^2[0,1/b]}$$
$$=\frac{1}{b}\sum_{n,k,\ell\in\mathbb{Z}}\int_0^1T_{k/b}f(t)\overline{T_{y_n+k/b}g(t)}$$
$$\times T_{y_n+\ell/b}g(t)\overline{T_{\ell/b}f(t)}\ dt$$
$$=b^{-1}\sum_{n,\ell\in\mathbb{Z}}\int_0^1\overline{T_{\ell/b}f(t)}T_{y_n+\ell/b}g(t)$$
$$\times\sum_{k\in\mathbb{Z}}f(t-k/b)\overline{g(t-y_n-k/b)}\ dt$$
$$=b^{-1}\sum_{n,\ell\in\mathbb{Z}}\int_{\ell/b}^{(\ell+1)/b}\overline{f(t)}g(t-y_n)$$
$$\times\sum_{k\in\mathbb{Z}}f(t-k/b)\overline{g(t-y_n-k/b)}\ dt$$
$$=b^{-1}\int_{\mathbb{R}}\overline{f(t)}g(t-y_n)$$
$$\times\sum_{k\in\mathbb{Z}}f(t-k/b)\overline{g(t-y_n-k/b)}\ dt$$
$$=b^{-1}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\overline{f(t)}f(t-k/b)$$
$$\times\sum_{n\in\mathbb{Z}}g(t-y_n)\overline{g(t-y_n-k/b)}\ dt.$$

This completes the proof of the Gabor frame identity for irregular frames. □

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