

# CONCRETE CONSTRUCTIONS OF NON-PAVABLE PROJECTIONS

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ABSTRACT. It is known that the paving conjecture fails for 2-paving projections with constant diagonal  $1/2$ . But the proofs of this fact are existence proofs. We will give concrete examples of these projections and projections with constant diagonal  $1/r$  which are not  $r$ -pavable in a very strong sense.

## 1. INTRODUCTION

It is now known that the 1959 Kadison-Singer Problem is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering [6, 7]. In 1979, Anderson [1] showed that the Kadison-Singer Problem is equivalent to the *Paving Conjecture*.

**Paving Conjecture (PC).** *For  $\epsilon > 0$ , there is a natural number  $r$  so that for every natural number  $n$  and every linear operator  $T$  on  $l_2^n$  whose matrix has zero diagonal, we can find a partition (i.e. a paving)  $\{A_j\}_{j=1}^r$  of  $\{1, \dots, n\}$ , such that*

$$\|Q_{A_j} T Q_{A_j}\| \leq \epsilon \|T\| \quad \text{for all } j = 1, 2, \dots, r,$$

where  $Q_{A_j}$  is the natural projection onto the  $A_j$  coordinates of a vector.

Operators satisfying the Paving Conjecture are called **pavable operators**. A projection  $P$  on  $\mathcal{H}_n$  is  $(\epsilon, r)$ -**pavable** if there is a partition  $\{A_j\}_{j=1}^r$  of  $\{1, 2, \dots, n\}$  satisfying

$$\|Q_{A_j} P Q_{A_j}\| \leq \epsilon, \quad \text{for all } j = 1, 2, \dots, r.$$

It was shown in [5] that projections with constant diagonal  $1/r$  are not  $(r, \epsilon)$ -pavable for any  $\epsilon > 0$ . But the argument in [5] is an existence proof and the actual matrices failing paving were not known. In this note we will construct concrete examples of these projections. As a consequence, we will obtain a stronger result than that of [5].

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## 2. PRELIMINARIES

We will actually work with an equivalent form of the Paving Conjecture for projections with constant diagonal. In 1989, Bourgain and Tzafriri proved one of the most celebrated theorems in analysis: The *Bourgain-Tzafriri Restricted Invertibility Theorem* [2]. This gave rise to a major open problem in analysis.

**Bourgain-Tzafriri Conjecture (BT).** *There is a universal constant  $A > 0$  so that for every  $B > 1$  there is a natural number  $r = r(B)$  satisfying: For any natural number  $n$ , if  $T : \ell_2^n \rightarrow \ell_2^n$  is a linear operator with  $\|T\| \leq B$  and  $\|Te_i\| = 1$  for all  $i = 1, 2, \dots, n$ , then there is a partition  $\{A_j\}_{j=1}^r$  of  $\{1, 2, \dots, n\}$  so that for all  $j = 1, 2, \dots, r$  and all choices of scalars  $\{a_i\}_{i \in A_j}$  we have:*

$$\left\| \sum_{i \in A_j} a_i T e_i \right\|^2 \geq A \sum_{i \in A_j} |a_i|^2.$$

It was shown in [6] that BT is equivalent to the Paving Conjecture.

**Definition 2.1.** *A family of vectors  $\{f_i\}_{i=1}^M$  for an  $N$ -dimensional Hilbert space  $\mathcal{H}_N$  is  $(\delta, r)$ -Rieszable if there is a partition  $\{A_j\}_{j=1}^r$  of  $\{1, 2, \dots, M\}$  so that for all  $j = 1, 2, \dots, r$  and all scalars  $\{a_i\}_{i \in A_j}$  we have*

$$\left\| \sum_{i \in A_j} a_i f_i \right\|^2 \geq \delta \sum_{i \in A_j} |a_i|^2.$$

*A projection  $P$  on  $\mathcal{H}_N$  is  $(\delta, r)$ -Rieszable if  $\{P e_i\}_{i=1}^N$  is  $(\delta, r)$ -Rieszable.*

Recall that a family of vectors  $\{f_i\}_{i \in I}$  is a **frame** for a Hilbert space  $\mathcal{H}$  if there are constants  $0 < A, B < \infty$ , called the **lower (upper) frame bounds** respectively satisfying for all  $f \in \mathcal{H}$ :

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2.$$

If  $\|f_i\| = \|f_j\|$  for all  $i, j$ , we call this an **equal norm** frame and if  $\|f_i\| = 1$  for all  $i$ , it is a **unit norm frame**. If  $A = B$  this is an  **$A$ -tight frame** and if  $A = B = 1$ , it is a **Parseval frame**. It is known [8] that  $\{f_i\}_{i \in I}$  is an  $A$ -tight frame if and only if the matrix with the  $f'_i$ 's as rows has orthogonal columns and the square sums of the column coefficients equal  $A$ . It is also known [4, 8] that  $\{f_i\}_{i=1}^M$  is a Parseval frame for  $\mathcal{H}_n$  if and only if there is an orthogonal projection  $P : \ell_2^M \rightarrow \mathcal{H}_n$  with

$$P e_i = f_i, \quad \text{for all } i = 1, 2, \dots, M,$$

where  $\{e_i\}_{i=1}^M$  is the unit vector basis of  $\ell_2^M$ . It is known [4, 8] that  $\{f_i\}_{i=1}^M$  is an  $A$ -tight frame for  $\mathcal{H}_n$  if and only if the matrix with row vectors  $f_i$  has orthogonal columns and the square sum of the column coefficients all equal  $A$ .

The following result can be found in [5, 9].

**Proposition 2.2.** *Fix a natural number  $r \in \mathbb{N}$ . The following are equivalent:*

- (1) *The Paving Conjecture.*
- (2) *All projections with constant diagonal  $1/r$  are pivable.*
- (3) *All projections with constant diagonal  $1/r$  are Rieszable.*
- (4) *All unit norm  $r$ -tight frames  $\{f_m\}_{m=1}^{nr}$  for  $\mathcal{H}_n$  are Rieszable.*

We will construct concrete counterexamples for (4) of 2.2. These will give concrete counterexamples to 1-4 in the proposition by the following result which can be found in [5]. The point here is that the proof of this proposition gives an explicit representation of each of the equivalences in the proposition in terms of all the others.

**Proposition 2.3.** *Let  $P$  be an orthogonal projection on  $\mathcal{H}_n$  with matrix  $B = (a_{ij})_{i,j=1}^n$ . The following are equivalent:*

- (1) *The family  $\{Pe_i\}_{i=1}^n$  is  $(\delta, r)$ -Rieszable.*
- (2) *There is a partition  $\{A_j\}_{j=1}^r$  of  $\{1, 2, \dots, n\}$  so that for all  $j = 1, 2, \dots, r$  and all scalars  $\{a_i\}_{i \in A_j}$  we have*

$$\left\| \sum_{i \in A_j} a_i (I - P)e_i \right\|^2 \leq (1 - \delta) \sum_{i \in I} |a_i|^2.$$

- (3) *The matrix of  $I - P$  is  $(\delta, r)$ -pivable.*

As a fundamental tool in our work, we will work with the (non-normalized)  $n \times n$  discrete Fourier transform matrices which we will just call DFT matrices. For the  $n \times n$  DFT, we choose a primitive  $n^{\text{th}}$ -root of unity  $\omega$  and define

$$DFT_{n \times n} = (\omega^{ij})_{i,j=1}^n.$$

The main point of  $DFT_{n \times n}$  matrices is that they are an  $\sqrt{n}$ -isometry for which the modulus of all of the entries of the matrix are equal to 1. We will rely on the following simple observation.

**Proposition 2.4.** *If  $A = (a_{ij})_{i,j=1}^n$  is a matrix with  $|a_{ij}|$  constant and orthogonal columns and we multiply the  $j^{\text{th}}$ -column of  $A$  by a constant  $C_j$  to get a new matrix  $B$ , then*

- (1) *The columns of  $B$  are orthogonal.*
- (2) *The square sums of the coefficients of any row of  $B$  all equal*

$$\sum_{j=1}^n C_j^2.$$

- (3) *The square sum of the coefficients of the  $j^{\text{th}}$  column of  $B$  equal  $C_j^2$ .*

### 3. THE CASE $r=2$

Let us first outline our construction. For any natural number  $n$ , we will alter two  $2n \times 2n$  DFT matrices along the lines of Proposition 2.4 and then

stack them on top of one another to get a  $4n \times 2n$  matrix with the following properties:

(1) Each altered DFT has the square sums of the coefficients of any row equal to 1.

(2) The top altered DFT will have the square sums of the coefficients of each column  $j$  with  $1 \leq j \leq n - 1$  equal to 2, and the square sums of the coefficients of the remaining columns will all equal  $2/(n + 1)$ .

(3) The combined matrix will have the square sums of the coefficients of each column equal to 2.

It follows that this is the matrix of a unit norm 2-tight frame and hence multiplying the matrix by  $1/\sqrt{2}$  will turn it into an equal norm Parseval frame, creating the matrix of a rank  $2n$  projection on  $\mathcal{C}^{4n}$  with constant diagonal  $1/2$ . We will then show that the rows of this matrix are not 2-Rieszable to complete the example.

So we start with a  $2n \times 2n$  DFT and multiply the first  $n - 1$  columns by  $\frac{\sqrt{2}}{\sqrt{2n}}$  and the remaining columns by  $\frac{1}{\sqrt{n(n+1)}}$  to get a new matrix  $B_1$ . Now, we take the second  $2n \times 2n$  DFT matrix and multiply the first  $n - 1$  columns by 0 and the remaining columns by  $\frac{1}{\sqrt{2n(n+1)}}$  to get a matrix  $B_2$ . We form the matrix  $B$  by stacking the matrices  $B_1$  and  $B_2$  on top of one another to get the matrix  $B$  given in Figure 1 below.

**Figure 1:**

	$2n \times (n - 1)$ boxes	$2n \times (n + 1)$ boxes
$B_1$	$\sqrt{2}$	$\sqrt{\frac{2}{n+1}}$
$B_2$	0	$\sqrt{\frac{2n}{n+1}}$

Now we can prove:

**Proposition 3.1.** *The matrix  $B$  satisfies:*

(1) *The columns are orthogonal and the square sum of the coefficients of every column equals 2.*

(2) *The square sum of the coefficients of every row equals 1.*

*The row vectors of the matrix  $B$  are not 2-Rieszable.*

*Proof.* Parts (1) and (2) are straightforward calculations.

We will now show that the row vectors of  $B$  are not two Rieszable. So let  $\{A_1, A_2\}$  be a partition of  $\{1, 2, \dots, 4n\}$ . Without loss of generality, we may assume that  $|A_1 \cap \{1, 2, \dots, 2n\}| \geq n$ . Let the row vectors of the matrix  $B$  be  $\{f_i\}_{i=1}^{4n}$  as elements of  $\mathcal{C}^{2n}$ . Let  $P_{n-1}$  be the orthogonal projection of  $\mathcal{C}^{2n}$  onto the first  $n - 1$  coordinates. Since  $|A_1| \geq n$ , there are scalars  $\{a_i\}_{i \in A_1}$

so that  $\sum_{i \in A_1} |a_i|^2 = 1$  and

$$P_{n-1} \left( \sum_{i \in A_1} a_i f_i \right) = 0.$$

Also, let  $\{g_j\}_{j=1}^{2n}$  be the original rows of the DFT. We now have:

$$\begin{aligned} \left\| \sum_{i \in A_1} a_i f_i \right\|^2 &= \left\| (I - P_{n-1}) \left( \sum_{i \in A_1} a_i f_i \right) \right\|^2 \\ &= \frac{2}{n+1} \left\| (I - P_{n-1}) \left( \sum_{i \in A_1} a_i g_i \right) \right\|^2 \\ &\leq \frac{2}{n+1} \left\| \sum_{i \in A_1} a_i g_i \right\|^2 \\ &= \frac{2}{n+1} \sum_{i \in A_1} |a_i|^2 \\ &= \frac{2}{n+1}. \end{aligned}$$

Letting  $n \rightarrow \infty$  we have that this class of matrices is not  $(\delta, 2)$ -pavable for any  $\delta > 0$ .  $\square$

#### 4. THE CASE OF GENERAL $r$

In this section we will extend our construction to projections with constant diagonal  $1/r$  and actually prove a stronger result.

**Proposition 4.1.** *For every natural number  $r \geq 2$ , there is a  $r^2 n \times rn$  projection matrix with constant diagonal  $1/r$  so that whenever we partition  $\{1, 2, \dots, r^2 n\}$  into sets  $\{A_j\}_{j=1}^r$ , and for all  $k = 1, 2, \dots, r$ , if  $D_k = \{(k-1)rn + 1, (k-1)rn + 2, \dots, krn\}$ , then for every  $k = 1, 2, \dots, r-1$ , there is a  $j$  so that the vectors  $\{f_i\}_{i \in A_j \cap D_k}$  are not 2-Rieszable.*

This time, we will take  $r$  DFT matrices of size  $rn \times rn$  and alter their columns by certain amounts so that when we stack them on top of one another to get a matrix  $B$  of size  $r^2 n \times rn$  we will have:

1. The columns of  $B$  are orthogonal and the sums of the squares of the coefficients of each row of  $B$  equals 1.
2. The sums of the squares of the coefficients of each column of  $B$  equals  $r$ .
3.  $B$  satisfies the requirements of the proposition.

For the first matrix  $B_1$  we take the  $rn \times rn$  DFT and multiply the first  $n-1$  columns by  $\sqrt{r}$  and the remaining columns by  $\delta_1$  (to be chosen later). For  $B_2$  we take the  $rn \times rn$  DFT and multiply the first  $n-1$  columns by 0, multiply the columns  $n-1+j$ ,  $j = 1, 2, \dots, n-1$  by  $\sqrt{r - \delta_1}$ , and multiply

the remaining columns by  $\delta_2$  (to be chosen later). And for  $k = 3, \dots, r - 1$  we construct the matrix  $B_k$  by taking the  $re \times rn$  DFT and multiply the first  $(k - 1)(n - 1)$  columns by 0, multiply the columns  $(k - 1)(n - 1) + j$  for  $j = 1, 2, \dots, n - 1$  by

$$\sqrt{r - \sum_{i=1}^{k-1} \delta_{i-1}},$$

and multiply the remaining columns by  $\delta_k$  (to be chosen later). Finally, for  $B_r$  we take the  $rn \times rn$  DFT and multiply the first  $(r - 1)(n - 1)$  columns by 0 and the remaining columns by  $\sqrt{\delta_r}$  (to be chosen later).

We then stack these  $r$ ,  $rn \times rn$  matrices  $\{B_k\}_{k=1}^r$  on top of each other to produce the matrix  $B$  given in figure 2 below.

**Figure 2:**

	$rn \times (n - 1)$ boxes	$rn \times (n - 1)$ boxes	$rn \times (n - 1)$ boxes	$\dots$	$rn \times r$ boxes
$B_1$	$\sqrt{\frac{r}{rn}}$	$\sqrt{\frac{\delta_1}{rn}}$	$\sqrt{\frac{\delta_1}{rn}}$	$\dots$	$\sqrt{\frac{\delta_1}{rn}}$
$B_2$	0	$\sqrt{\frac{r}{rn}}$	$\sqrt{\frac{\delta_2}{rn}}$	$\dots$	$\sqrt{\frac{\delta_2}{rn}}$
$B_3$	0	0	$\sqrt{\frac{r}{rn}}$	$\dots$	$\sqrt{\frac{\delta_3}{rn}}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$B_r$	0	0	0	$\dots$	$\sqrt{\frac{\delta_r}{rn}}$

Now we must show that the matrix  $B$  has all of the properties of Proposition 4.1. It is clear that the columns of  $B$  are orthogonal. To show that the square of the row coefficients of the matrix  $B$  are all equal to 1, we need a lemma.

**Lemma 4.2.** *To get the rows of the matrix  $B$  to square sum to 1, we need*

$$\delta_k = \frac{r^2 n}{[(r - k + 1)n + k - 1][(r - k)n + k]}.$$

*Proof.* We will proceed by induction on  $k$ . For  $k = 1$ , we observe that the coefficients of the first  $n - 1$  columns of  $B_1$  have modulus equal to  $1/n$ , while the coefficients of the remaining  $rn - (n - 1)$  columns of  $B_1$  have modulus  $\sqrt{\frac{\delta_1}{rn}}$ . So the sum of the squares of the coefficients of any row of  $B_1$  equals

$$\frac{1}{rn} [r(n - 1) + \delta_1(rn - (n - 1))] = 1.$$

Hence,

$$\delta_1 = \frac{r}{r(n - 1) + 1}.$$

For  $k = 2$ , our matrix  $B_2$  has coefficients of the first  $n - 1$  columns equal to 0, coefficients of the columns  $(n - 1) + j$ ,  $j = 1, 2, \dots, n - 1$  have modulus equal to

$$\sqrt{\frac{r - \delta_1}{rn}}.$$

and the remaining  $rn - 2(n - 1)$  columns have modulus equal to  $\sqrt{\frac{\delta_2}{rn}}$ . So the square sums of the coefficients of any row of  $B_2$  equals

$$\frac{1}{rn} [(n - 1)(r - \delta_1) + (rn - 2(n - 1))\delta_2] = 1.$$

Since

$$r - \delta_1 = r - \frac{r}{(r - 1)(n + 1)} = \frac{r(r - 1)n}{(r - 1)n + 1},$$

we can solve the equation to get

$$\delta_2 = \frac{r^2 n}{[(r - 1)n + 1][(r - 2)n + 2]}.$$

Now assume our formula holds for any  $k \leq r - 1$  and we check it for  $k + 1$ . The matrix  $B_{k=1}$  has coefficients of the first  $k(n - 1)$  columns equal to 0, coefficients of the columns  $k(n - 1) + j$ ,  $j = 1, 2, \dots, n - 1$  of modulus

$$\left( \frac{r - \sum_{j=1}^k \delta_j}{rn} \right)^{1/2},$$

and the coefficients of the remaining columns have modulus  $\frac{\delta_{k+1}}{rn}$ . It follows that the square sums of the row coefficients of the matrix  $B_{k+1}$  must satisfy

$$(1) \quad \left( r - \sum_{j=1}^k \delta_j \right) (n - 1) + \delta_{k+1} [rn - (k + 1)(n - 1)] = rn.$$

Hence, letting  $a = rn/(n-1)$  we have

$$\begin{aligned}
\sum_{j=1}^k \delta_j &= r^2 n \sum_{j=1}^k \frac{1}{[r-j+1)n+j-1][(r-j)n+j]} \\
&= \frac{r^2 n}{(n-1)^2} \sum_{j=1}^k \frac{1}{(a+1-j)(a-j)} \\
&= \frac{r^2 n}{(n-1)^2} \sum_{j=1}^k \left( \frac{1}{a-j} - \frac{1}{a-(j-1)} \right) \\
&= \frac{r^2 n}{(n-1)^2} \frac{1}{a-k} - \frac{1}{a-0} \\
&= \frac{r^2 n}{(n-1)^2} \frac{k}{a(a-k)} \\
&= \frac{r^2 k n}{rn(rn - k(n-1))} \\
&= \frac{rk}{(r-k)n+k}.
\end{aligned}$$

Combining this with Equation 1 we have

$$\begin{aligned}
\delta_{k+1} &= \frac{r + (n-1) \sum_{j=1}^k \delta_j}{rn - (k+1)(n-1)} \\
&= \frac{r + (n-1) \left( \frac{rk}{(r-k)n+k} \right)}{(r-k+1)n+k-1} \\
&= \frac{r[(r-k)n+k] + (n-1)rk}{[(r-k+1)n+k-1][(r-k)n+k]} \\
&= \frac{r^2 n - rkn + rk + rnk - kr}{[(r-k+1)n+k-1][(r-k)n+k]} \\
&= \frac{r^2 n}{[(r-k+1)n+k-1][(r-k)n+k]}.
\end{aligned}$$

□

By Lemma 4.2, we know that the rows of the matrix  $B$  square sum to 1. Now we need to check the column sums. Most of this is true by our definitions. For a column  $\ell = k(n-1) + j$ , the column coefficients for  $1 \leq j \leq k-1$  and  $i = jrn + m$ ,  $m = 1, 2, \dots, rn$ , have modulus  $\sqrt{\frac{\delta_j}{rn}}$ , and for  $i = krn + m$ ,  $m = 1, 2, \dots, rm$  the modulus of the coefficients are  $\sqrt{\frac{r - \sum_{j=1}^{k-1} \delta_j}{rn}}$ , and all other coefficients are 0. Hence, the square sum of the

column coefficients is

$$rn \sum_{j=1}^{k-1} \frac{\delta_j}{rn} + rn \left( \frac{r - \sum_{j=1}^{k-1} \delta_j}{rn} \right) = r.$$

Finally, we need to show that our matrix  $B$  is not pavable in the strong sense given in the proposition. This follows similarly to the 2 case. Let  $\{f_i\}_{i=1}^{r^2n}$  be the rows of the matrix  $B$  and let  $\{g_i\}_{i=1}^{rn}$  be the rows of the DFT matrix. Also, let  $P_k$  be the orthogonal projection of  $\mathcal{C}_2^{rn}$  onto the first  $k(n-1)$  coordinates. Now let  $\{A_j\}_{j=1}^r$  be a partition of  $\{1, 2, \dots, r^2n\}$  and fix  $1 \leq k \leq r-1$ . Then there is a  $j$  so that  $|A_j \cap D_k| \geq n$ . Since the vectors  $\{f_i\}_{i \in A_j \cap D_k}$  have zero coordinates for all  $j = 1, 2, \dots, (k-1)(n-1)$ , there are scalars  $\{a_i\}_{i \in A_j \cap D_k}$  satisfying

1.  $\sum_{i \in A_j \cap D_k} |a_i|^2 = 1$ .
2. We have

$$P_k \left( \sum_{i \in A_j \cap D_k} a_i f_i \right) = 0.$$

It follows from our construction that

$$\begin{aligned} \left\| \sum_{i \in A_j \cap D_k} a_i f_i \right\|^2 &= \left\| (I - P_k) \left( \sum_{i \in A_j \cap D_k} a_i f_i \right) \right\|^2 \\ &= \delta_k \left\| (I - P_k) \left( \sum_{i \in A_j \cap D_k} a_i g_i \right) \right\|^2 \\ &\leq \delta_k \left\| \sum_{i \in A_j \cap D_k} a_i g_i \right\|^2 \\ &= \delta_k \sum_{i \in A_j \cap D_k} |a_i|^2 \\ &= \delta_k \end{aligned}$$

The main question is whether it is possible to take the concrete constructions in this paper and generalize them to give a complete counterexample to the Paving Conjecture.

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