CONSTRUCTING A LARGE FAMILY OF EQUIANGULAR TIGHT FRAMES

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ABSTRACT

We provide a new method for constructing equiangular tight frames (ETFs). This method is valid in both the real and complex settings, and shows that many of the few previously-known examples of ETFs are but the first representatives of infinite families of such frames. The construction is extremely simple: a tensor-like combination of a Steiner system and a regular simplex. It provides great freedom in terms of the frame's size and redundancy, and it explicitly constructs the frame vectors in their native domain, in which the frame vectors are very sparse.

Keywords— equiangular, tight, frames, Steiner systems

1. INTRODUCTION

Let $F = \{f_n\}_{n=1}^N$ be a finite sequence of vectors in a real or complex M-dimensional Hilbert space $\mathbb{H}_M$. The corresponding *frame operator* is $FF^* = \sum_{n=1}^N f_n f_n^*$, where f_n^* denotes the linear functional that maps a given $f \in \mathbb{H}_M$ to the scalar $\langle f, f_n\rangle$. The sequence F is said to be a *tight frame* if there exists $A > 0$ such that $FF^* = A\mathrm{I}$. Meanwhile, F is *equiangular* if $\|f_n\| = 1$ for all n and if there exists $\alpha \geq 0$ such that $|\langle f_n, f_{n'}\rangle| = \alpha$ for all $n \neq n'$. This paper concerns *equiangular tight frames* (ETFs); writing F as an $M \times N$ matrix, we need the rows of F to be orthogonal and have constant norm, the columns of F to be unit norm, and the inner products of distinct columns of F to have constant modulus. Such frames are useful in applications [6], but up to this point, they have proven notoriously difficult to construct.

This paper highlights the main results of [4], which provides a new method for constructing ETFs. The construction is valid in both the real and complex settings, and shows that many of the few previously-known examples of ETFs are but the first representatives of infinite families of such frames. This method also permits great freedom in selecting M and N, just shy of letting one choose the exact size and redundancy of their liking. Moreover, the frame vectors can be chosen to be very sparse.

In the next section, we show how certain combinatorial block designs—Steiner systems—may be combined with regular simplices to produce ETFs. In the third section, we present the known infinite families of such Steiner systems and the corresponding infinite families of ETFs they generate. We further provide some necessary and asymptotically sufficient conditions to aid in the quest for discovering other ETFs that lie outside of the known infinite families.

2. STEINER EQUIANGULAR TIGHT FRAMES

In this section, we provide new constructions of infinite families of ETFs, namely $M \times N$ matrices $F = [f_1 \dots f_N]$ which have orthogonal rows of constant squared-norm A and unit norm columns whose inner products have constant modulus α. For a fixed M and N, there is no ambiguity [6] as to the values of A and α; we necessarily have $A = \frac{N}{M}$ and $\alpha^2 = \frac{N-M}{M(N-1)}$. This paper constructs ETFs by leveraging the inherent symmetries in combinatorial block designs.

Block designs have been studied for over a century; the background facts presented here on these topics are taken from [1, 3]. In short, a (v, b, r, k, λ)-*block design* is a v-element set V along with a collection of b k-element subsets of V, dubbed *blocks*, that have the property that any element of V lies in exactly r blocks and that any 2-element subset of V is contained in exactly λ blocks. The corresponding *incidence matrix* is a $v \times b$ matrix A that has a one in a given entry if that block contains the corresponding point, and is otherwise zero; in this paper, it is more convenient for us to work with the $b \times v$ transpose A^{T} of this incidence matrix. Our particular construction of ETFs involves a special class of block designs known as $(2, k, v)$-*Steiner systems*, for which any 2-element subset of V is contained in exactly one block, that is, $\lambda = 1$. For example, the transposed incidence matrix of a $(2, 2, 4)$-Steiner system is:

$$A^{\mathrm{T}} = \begin{bmatrix} + & + & & \\ + & & + & \\ + & & & + \\ & + & + & \\ & + & & + \\ & & + & + \end{bmatrix}. \tag{1}$$

We can easily verify that A^{T} corresponds to a Steiner system: there is a set V of $v = 4$ elements, each corresponding to a column of A^{T}; there is also a collection of $b = 6$ subsets of V, each corresponding to a row of A^{T}; every row contains $k = 2$ elements; every column contains $r = 3$ elements; and any given pair of elements is contained in exactly one row, that is, $\lambda = 1$.

This in hand, we present our main result, which is proven in [4]; here, the *density* of a matrix is the ratio of the number of nonzero entries of that matrix to the entire number of its entries:

Theorem 1. *Every $(2,k,v)$-Steiner system generates equiangular tight frames consisting of $N = v(1+\frac{v-1}{k-1})$ vectors in a space of dimension $M = \frac{v(v-1)}{k(k-1)}$. Such frames have redundancy $\frac{N}{M} = k(1+\frac{k-1}{v-1})$ as well as density $\frac{k}{v} = (\frac{N-1}{M(N-M)})^{\frac{1}{2}}$.*

Moreover, if there exists a real Hadamard matrix of size $1+\frac{v-1}{k-1}$, then such frames are real.

Specifically, an ETF matrix F can be constructed as follows:

1. *Let A^{T} be the $\frac{v(v-1)}{k(k-1)} \times v$ transpose of the adjacency matrix of a $(2,k,v)$-Steiner system.*
2. *For each $j = 1,\dots,v$, let H_j be a $(1+\frac{v-1}{k-1}) \times (1+\frac{v-1}{k-1})$ matrix that has orthogonal rows and unimodular entries, such as a possibly complex Hadamard matrix.*
3. *For each $j = 1,\dots,v$, let F_j be a $\frac{v(v-1)}{k(k-1)} \times (1+\frac{v-1}{k-1})$ matrix obtained by replacing each of the one-valued entries of the jth column of A^{T} with a distinct row of H_j, and every zero-valued entry with a row of zeros.*
4. *Concatenate and rescale the F_j's to yield a $\frac{v(v-1)}{k(k-1)} \times v(1+\frac{v-1}{k-1})$ matrix $F = (\frac{k-1}{v-1})^{\frac{1}{2}}[F_1 \cdots F_v]$.*

We refer to the ETFs produced in the manner of Theorem 1 as $(2,k,v)$-*Steiner ETFs*. The idea behind this construction is best understood by considering a simple example, such as the ETF that arises from a $(2,2,4)$-Steiner system, whose transposed incidence matrix is given in (1). To form an ETF, for each of the four columns of A^{T} we must choose a 4×4 matrix H with unimodular entries and orthogonal rows; the size of H is always one more than the number r of ones in a given column of A^{T}. Though in principle one may choose a different H for each column, we choose them all to be the same, namely the Hadamard matrix:

$$H = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}.$$

To form the ETF, for each column of A^{T} we replace each of its 1-valued entries with a distinct row of H. Again, though in principle one may choose a different sequence of rows of H for each column, we decide to use the second, third and fourth rows, in that order. Then normalizing gives a real ETF F of $N = 16$ elements of dimension $M = 6$, namely, $\frac{1}{\sqrt{3}}$ times

$$\left[\begin{array}{cccccccccccccccc} + & - & + & - & + & - & + & - & & & & & & & & \\ + & + & - & - & & & & & + & - & + & - & & & & \\ + & - & - & + & & & & & & & & & + & - & + & - \\ & & & & + & + & - & - & + & + & - & - & & & & \\ & & & & + & - & - & + & & & & & + & + & - & - \\ & & & & & & & & + & - & - & + & + & - & - & + \end{array}\right].$$

One can immediately verify that the rows of F are orthogonal and have constant norm, implying F is indeed a tight frame. One can also easily see that the inner products of two columns from the same block are $-\frac{1}{3}$, while the inner products of columns from distinct blocks are $\pm\frac{1}{3}$. Theorem 1 states that this behavior holds for any appropriate choice of A^{T} and H.

In the next section, we apply Theorem 1 to produce several infinite families of Steiner ETFs. Before doing so, however, we pause to remark on the redundancy and sparsity of such frames. In particular, note that since the parameters k and v of the requisite Steiner system always satisfy $2 \leq k \leq v$, then the redundancy $k(1+\frac{k-1}{v-1})$ of Steiner ETFs is always between k and $2k$; the redundancy is therefore on the order of k, and is always strictly greater than 2. If a low-redundancy ETF is desired, one can always take the Naimark complement [2] of an ETF of N elements in M-dimensional space to produce a new ETF of N elements in $(N-M)$-dimensional space; though the complement process does not preserve sparsity, it nevertheless transforms any Steiner ETF into a new ETF whose redundancy is strictly less than 2. However, such a loss of sparsity should not to be taken lightly. Indeed, the low density of Steiner ETFs gives them a large computational advantage over their non-sparse brethren.

To clarify, the most common operation in frame-theoretic applications is the evaluation of the *analysis* operator F^* on a given $f \in \mathbb{H}_M$. For a non-sparse F, this act of computing F^*f requires $\mathrm{O}(MN)$ operations; for a frame F of density D, this cost is reduced to $\mathrm{O}(DMN)$. Indeed, using the explicit value of $D = (\frac{N-1}{M(N-M)})^{\frac{1}{2}}$ given in Theorem 1 as well as the aforementioned fact that the redundancy of such frames necessarily satisfies $\frac{N}{M} > 2$, we see that the cost of evaluating F^*f when F is a Steiner ETF is on the order of $(\frac{M(N-1)}{N-M})^{\frac{1}{2}}N < (2M)^{\frac{1}{2}}N$ operations, a dramatic cost savings when M is large. Further efficiency is gained when F is real, as its nonzero elements are but a fixed scaling factor times the entries of a real Hadamard matrix, implying F^*f can be evaluated using only additions and subtractions. The fact that every entry of F is either 0 or ± 1 further makes real Steiner ETFs potentially useful for applications that require binary measurements, such as design of experiments.

3. CONSTRUCTING EQUIANGULAR TIGHT FRAMES

In this section, we apply Theorem 1 to produce several infinite families of Steiner ETFs. When designing frames for real-world applications, three considerations reign supreme: size, redundancy and sparsity. As noted above, every Steiner ETF is very sparse, a serious computational advantage in high-dimensional signal processing. Moreover, some of these infinite families, such as those arising from finite affine and projective geometries, provide one great flexibility in choosing the ETF's size and redundancy. Indeed, these constructions provide the first known guarantee that for a given application, one is always able to find ETFs whose frame elements lie in a space whose dimension matches, up to an order of magnitude, that of one's desired class of signals, while simultaneously permitting one to have an almost arbitrary fixed level of redundancy, a handy weapon in the fight against noise. To be clear, recall that the redundancy of a Steiner ETF is always strictly greater than 2. Moreover, as general bounds on the maximal number of equiangular lines [5] require that any ETF satisfy $N \leq \frac{M(M+1)}{2}$ in real spaces and $N \leq M^2$ in complex ones, the redundancy of an ETF is never truly arbitrary. Nevertheless, if one does prescribe a given desired level of redundancy in advance, the Steiner method can produce arbitrarily large ETFs whose redundancy is approximately the prime power nearest to the sought-after level.

3.1. Infinite families of Steiner equiangular tight frames

Table 1 details eight infinite families of ETFs, each generated by applying Theorem 1 to one of the eight completely understood infinite families of $(2,k,v)$-Steiner systems [1, 3] In this subsection, we focus on the first infinite family of Steiner systems, which is so simple that it is usually not discussed in the design-theory literature; details on the remaining seven families are given in [4].

For any $v \geq 2$, let V be a v-element set, and let $\mathcal{B}$ be the collection of all 2-element subsets of V. This gives $b = \frac{v(v-1)}{2}$ blocks, each of which contains $k = 2$ elements; each point is contained in $r = v - 1$ blocks, and each pair of points is indeed contained in but a single block, that is, $\lambda = 1$.

By Theorem 1, the ETFs arising from these $(2,2,v)$-Steiner systems consist of $N = v(1 + \frac{v-1}{k-1}) = v^2$ vectors in dimension $M = \frac{v(v-1)}{k(k-1)} = \frac{v(v-1)}{2}$. Though these frames can become arbitrarily large, they do not provide any freedom with respect to redundancy: $\frac{N}{M} = 2\frac{v}{v-1}$ is essentially 2. These frames have density $\frac{k}{v} = \frac{2}{v}$. Moreover, these ETFs can be real-valued if there exists a real Hadamard matrix of size $1 + \frac{v-1}{k-1} = v$. In particular, it suffices to have v be a power of 2; should the Hadamard conjecture prove true, it would suffice to have v divisible by 4.

One example of such an ETF with $v = 4$ was given in the previous section. For another, consider $v = 3$. The $b \times v$ transposed incidence matrix A^{T} is 3×3, with each row corresponding to a given 2-element subset of $\{0,1,2\}$:

$$A^{\mathrm{T}} = \begin{bmatrix} + & + & \\ + & & + \\ & + & + \end{bmatrix}.$$

To form the corresponding 3×9 ETF F, we need a 3×3 unimodular matrix with orthogonal rows, such as a DFT; letting $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$, we can take

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}.$$

To form F, in each column of A^{T}, we replace each 1-valued entry with a distinct row of H. Always choosing the second and third rows yields an ETF of 9 elements in $\mathbb{C}^3$:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & & & \\ 1 & \omega & \omega^2 & & & & 1 & \omega^2 & \omega \\ & & & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \end{bmatrix}.$$

This is the only known instance of when the Steiner-based construction of Theorem 1 produces a *maximal* ETF, namely one that has $N = M^2$.

3.2. Necessary and sufficient conditions on the existence of Steiner ETFs.

$(2,k,v)$-Steiner systems have been actively studied for over a century, with many celebrated results. Nevertheless, much about these systems is still unknown. In this subsection, we discuss some known partial characterizations of the Steiner systems which lie outside of the eight families we have already noted, as well as what these results tell us about the existence of certain ETFs. We begin with the well-known fact that if a $(2,k,v)$-Steiner system exists, then the number r of blocks that contain a given point is necessarily $\frac{v-1}{k-1}$, while the total number of blocks b is $\frac{v(v-1)}{k(k-1)}$. As such, in order for a $(2,k,v)$-Steiner system to exist, it is necessary for (k,v) to be *admissible*, that is, to have the property that $\frac{v-1}{k-1}$ and $\frac{v(v-1)}{k(k-1)}$ are integers.

However, this property is not sufficient for existence: it is known that a $(2,6,16)$-Steiner system does not exist [1] despite the fact that $\frac{v-1}{k-1} = 3$ and $\frac{v(v-1)}{k(k-1)} = 8$. In fact, letting v be either 16, 21, 36, or 46 results in an admissible pair with $k = 6$, despite the fact that none of the corresponding Steiner systems exist; there are twenty-nine additional values of v which form an admissible pair with $k = 6$ and for which the existence of a corresponding Steiner system remains an open problem [1]. Similar nastiness arises with $k \geq 7$. The good news is that admissibility, though not sufficient for existence, is, in fact, asymptotically sufficient: for any fixed k, there exists a corresponding admissible index $v_0(k)$ for which for all $v > v_0(k)$ such that $\frac{v-1}{k-1}$ and $\frac{v(v-1)}{k(k-1)}$ are integers, a $(2,k,v)$-Steiner system indeed exists [1]. Moreover, explicit values of $v_0(k)$ are known for small k: $v_0(6) = 801$, $v_0(7) = 2605$, $v_0(8) = 3753$, $v_0(9) = 16497$. We now detail the ramifications of these design-theoretic results on frame theory:

Theorem 2. *If an N-element Steiner equiangular tight frame exists for an M-dimensional space, then the corresponding block design has parameters:*

$$v = \frac{N\alpha}{1+\alpha}, \qquad b = M, \qquad r = \frac{1}{\alpha}, \qquad k = \frac{N}{M(1+\alpha)},$$

where $\alpha = (\frac{N-M}{M(N-1)})^{\frac{1}{2}}$. In particular, if such a frame exists, then these expressions for v, k and r are necessarily integers.

Conversely, for any fixed $k \geq 2$, there exists an index $v_0(k)$ for which for all $v > v_0(k)$ such that $\frac{v-1}{k-1}$ and $\frac{v(v-1)}{k(k-1)}$ are integers, there exists a Steiner equiangular tight frame of $v(1 + \frac{v-1}{k-1})$ vectors for a space of dimension $\frac{v(v-1)}{k(k-1)}$.

In particular, for any fixed $k \geq 2$, letting $v = jk(k-1) + 1$ or $v = jk(k-1) + k$ for increasingly large values of j yields a sequence of Steiner equiangular tight frames whose redundancy is asymptotically k; these frames can be real if there exist real Hadamard matrices of sizes $jk + 1$ or $jk + 2$, respectively.

We conclude with a few thoughts on Theorems 1 and 2. First, we emphasize that the method of Theorem 1 is a method for constructing some ETFs, and by no means constructs them all. Indeed, as noted above, the redundancy of Steiner ETFs is always strictly greater than 2; while some of those ETFs with $\frac{N}{M} < 2$ will be the Naimark complements of Steiner ETFs, one

Name	M	N	Redundancy	Density	Real?	Restrictions
2-blocks	$\frac{v(v-1)}{2}$	v^2	$2\frac{v}{v-1}$	$\frac{2}{v}$	v	None
3-blocks	$\frac{v(v-1)}{6}$	$\frac{v(v+1)}{2}$	$3\frac{v+1}{v-1}$	$\frac{3}{v}$	$\frac{v+1}{2}$	$v \equiv 1,3 \mod 6$
4-blocks	$\frac{v(v-1)}{12}$	$\frac{v(v+2)}{3}$	$4\frac{v+2}{v-1}$	$\frac{4}{v}$	Never	$v \equiv 1,4 \mod 12$
5-blocks	$\frac{v(v-1)}{20}$	$\frac{v(v+3)}{4}$	$5\frac{v+3}{v-1}$	$\frac{5}{v}$	$\frac{v+3}{4}$	$v \equiv 1,5 \mod 20$
Affine	$q^{n-1}(\frac{q^n-1}{q-1})$	$q^n(1+\frac{q^n-1}{q-1})$	$q(1+\frac{q-1}{q^n-1})$	$\frac{1}{q^{n-1}}$	$1+\frac{q^n-1}{q-1}$	q a prime power, $n \geq 2$
Projective	$\frac{(q^n-1)(q^{n+1}-1)}{(q+1)(q-1)^2}$	$\frac{q^{n+1}-1}{q-1}(1+\frac{q^n-1}{q-1})$	$(q+1)(1+\frac{q-1}{q^n-1})$	$\frac{q^2-1}{q^{n+1}-1}$	$1+\frac{q^n-1}{q-1}$	q a prime power, $n \geq 2$
Unitals	$\frac{q^2(q^3+1)}{q+1}$	$(q^2+1)(q^3+1)$	$(q+1)(1+\frac{1}{q^2})$	$\frac{q+1}{q^3+1}$	Never	q a prime power
Denniston	$\frac{(2^s+1)(2^{r+s}+2^r-2^s)}{2^r}$	$(2^s+2)(2^{r+s}+2^r-2^s)$	$2^r\frac{2^s+2}{2^s+1}$	$\frac{2^r}{2^{r+s}+2^r-2^s}$	Never	$2 \leq r < s$

Table 1. Eight infinite families of Steiner ETFs obtained by applying Theorem 1 to eight known infinite families of $(2,k,v)$-Steiner designs [1, 3]. A complete derivation of these facts is given in [4]. Each family permits both M and N to grow very large, but only a few families—affine, projective and Denniston—give one the freedom to simultaneously control the proportion between M and N, namely the redundancy $\frac{N}{M}$ of the ETF. The column denoted "Real?" indicates the size for which a real Hadamard matrix must exist in order for the resulting ETF to be real; it suffices to have this size be a power of 2; if the Hadamard conjecture is true, it would suffice for this number to be divisible by 4.

must admit that the Steiner method contributes little towards the understanding of those ETFs with $\frac{N}{M} = 2$, such as those arising from Paley graphs [7]. Moreover, Theorem 2 implies that not even every ETF with $\frac{N}{M} > 2$ arises from a Steiner system: though there exists an ETF of 76-elements in $\mathbb{R}^{19}$ [7], the corresponding parameters of the design would be $v = \frac{38}{3}$, $r = 5$ and $k = \frac{10}{3}$, not all of which are integers.

That said, the method of Theorem 1 is truly significant: the Steiner method produces 4 of the 17 real ETFs of dimension 50 or less [7] that have redundancy greater than 2, namely 6×16, 7×28, 28×64 and 35×120 ETFs. Interestingly, an additional 4 of these 17 ETFs can also be produced by the Steiner method, but only in complex form, namely those of 15×36, 20×96, 21×126 and 45×100 dimensions; it is unknown whether this is the result of a deficit in our analysis or the true non-existence of real-valued Steiner-based constructions of these sizes. The plot further thickens when one realizes that an additional 2 of these 17 real ETFs satisfy the necessary conditions of Theorem 2, but that the corresponding $(2,k,v)$-Steiner systems are known to not exist: if a 28×288 ETF was to arise as a result of Theorem 1, the corresponding Steiner system would have $k = 6$ and $v = 36$, while the 43×344 ETF would have $k = 7$ and $v = 43$; in fact, $(2,6,36)$- and $(2,7,43)$-Steiner systems cannot exist [1]. With our limited knowledge of the rich literature on Steiner systems, we were unable to resolve the existence of two remaining candidates: 23×276 and 46×736 ETFs could potentially arise from $(2,10,46)$- and $(2,14,92)$-Steiner systems, respectively, provided they exist.

4. ACKNOWLEDGMENTS

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