Custom Building Finite Frames

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Abstract. After the introduction of *frame potentials* by Benedetto and Fickus, there was an explosion of new results in frame theory. The generalization of this notation to *weighted frame potentials* yields a physical interpretation for finite frames along the lines of Coulomb's law in physics. This allows us to use results from classical mechanics to anticipate new results in frame theory. All of this resulted in the resolution of a large number of important open questions concerning the existence and construction of finite frames.

We will put all of this into the framework of a theory as well as adding many new results along the way. As part of the theory, we include the recent "algorithmic constructions" for finite frames. We will also discuss the related important questions which are still open at this time.

Introduction

Hilbert space frames were introduced by Duffin and Schaeffer in 1952 [**DS**] while working on some deep questions in non-harmonic Fourier series (see [**Y**]). These problems grew out of G.D. Birkhoff's work on Sturm-Liouville functions in 1917; the trigonometric results of Welch in 1921; and results of Paley-Wiener in 1934 (see [**Y**]). For some reason the results of Duffin and Schaeffer were not continued (except in signal/image processing [**G**]) until in 1986 Daubechies, Grossman and Meyer [**DGS**] brought this back to life right at the dawn of the wavelet era.

Recently, many new applications of frames have arisen to internet coding (see [**GKK**] and its references and [**CK**]); multiple antenna coding (see [**HH**] and [**HM**]); Communication Theory; Sampling Theory and more.

Benedetto and Fickus [**BF**] introduced an important tool into frame theory called *frame potentials.* This gave a geometric interpretation for tight frames which resulted in an explosion of important new results in the field including a physical interpretation for tight frames along the lines of Coulomb's law in Physics [**BF, CF**]. As a result, we can now use results from quantum mechanics to anticipate new results on frames. This also led to important advances in the *algorithmic construction* of frames which is critically important for applications of frames.

We will put all of these new advances into the framework of a theory as well as singling out the important open questions at this time. Also, we include may new

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results which did not make it into these papers. For general information concerning frames, we refer the reader to [**C, C1, Ch**].

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1. Preliminaries

A *frame* is a redundant set of vectors in a Hilbert space that allows one "natural" representation for each vector. Formally, a set of vectors $\{\phi_n\}_{n\in I}$ in a Hilbert space H is a *frame* for H if there are numbers $A, B > 0$ so that for all $\phi \in H$,

$$A\|\phi\|^2 \le \sum_{n\in I} |\langle \phi, \phi_n\rangle|^2 \le B\|\phi\|^2. \tag{1.1}$$

We call A (resp. B) a *lower frame bound* (resp. *upper frame bound*) for the frame. If $A = B$ this is an *A-tight frame* and if $A = B = 1$ it is called a *Parseval frame.* If all the frame vectors have the same norm, we call this an *equal-norm frame* and if the frame vectors are all norm one, we call $\{\phi_n\}_{n\in I}$ a *unit norm frame.*

We will work with finite frames on N-dimensional (real $\mathbb{R}^N$ or complex $\mathbb{C}^N$) Hilbert spaces H_N. If $\{\phi_m\}_{m=1}^M$ is a sequence of vectors in H_N, the *analysis operator* is $F : H \to \ell_2^M$ given by

$$(F\phi)_i = \langle \phi, \phi_i\rangle$$

and the *synthesis operator* is F^*. It follows from (1.1) that

$$AI \le F^*F \le BI.$$

We call $S = F^*F$ the *frame operator.* So $\{\phi_m\}_{m=1}^M$ is a frame for H_N if and only if S is a positive, self-adjoint invertible operator on H_N. And it is a A-tight frame if and only if $F^*F = AI$. Also, for every $\phi \in H_N$, we have the *reconstruction formula*:

$$\phi = \sum_{m=1}^M \langle S^{-1}\phi, \phi_m\rangle \phi_m = \sum_{m=1}^M \langle \phi, S^{-\frac{1}{2}}\phi_m\rangle S^{-\frac{1}{2}}\phi_m.$$

That is, $\{S^{-\frac{1}{2}}\phi_m\}_{m=1}^M$ is a Parseval frame.

If $\{\phi_m\}_{m=1}^M$ is a frame for H_N with frame operator S having eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ then

$$\sum_{k=1}^M \lambda_k = \sum_{m=1}^M \|\phi_m\|^2.$$

The frame $\{S^{-1}\phi_m\}_{m=1}^M$ is called the *canonical dual frame* of $\{\phi_m\}_{m=1}^M$. This frame carries out reconstruction. But there are other frames giving reconstruction formulas. A sequence $\{\psi_m\}_{m=1}^M$ is called a *dual frame* for $\{\phi_m\}_{m=1}^M$ if for all $\phi \in H_N$ we have

$$\phi = \sum_{m=1}^M \langle \phi, \phi_m\rangle \psi_m = \sum_{m=1}^M \langle \phi, \psi_m\rangle \phi_m.$$

2. Frame Potentials

Our goal in this section is to introduce the notion of *frame potential* of Benedetto and Fickus [**BF**] and develop their basic properties and applications. The importance of frame potentials is the fact that for many families of frames, the frames of minimal frame potential are precisely the tight frames in the family. This fact has proven to be very useful in the resolution of several important open questions in applications of frames. For most applications of frames, we prefer (or need) tight frames. Now, we can consider the family of frames which are allowed for our application and search for an element of minimal frame potential. Even more important, if our application does not allow tight frames, by minimizing frame potential we obtain the allowable frames which are the closest to being tight.

Definition 2.1. If $\{\varphi_m\}_{m=1}^M$ is a frame for $\mathbb{H}_N$, the **Frame Potential** of $\{\varphi_m\}_{m=1}^M$ is given by:

$$FP(\{\varphi_m\}_{m=1}^M) = \sum_{n,m=1}^M |\langle \varphi_n, \varphi_m\rangle|^2.$$

The frame potential is measuring how close a frame is to being orthogonal. In particular, we will see that if $\mathcal{F}$ is the family of frames with lower frame bound λ then the λ-tight frames are the minimizers of the frame potential over $\mathcal{F}$. This theorem gives us a way to identify tight frames. That is, they are the minimizers of the frame potential on certain families of frames. Our goal is to identify those families of frames which have minimizers of the frame potential and for which these minimizers must be tight.

If $\{\phi_m\}_{m=1}^M$ is a frame for H_N with frame operator S having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N > 0$, then

$$\begin{aligned} FP(\{\phi_m\}_{m=1}^M) &= \sum_{m,n=1}^M |\langle \phi_n, \phi_m\rangle|^2 = Tr(S^2) \\ &= \sum_{n=1}^M \lambda_n^2 = \sum_{m=1}^M \|S^{\frac{1}{2}}\phi_m\|^2. \end{aligned} \tag{2.1}$$

The important point here is the fact that the minimizers of the frame potential are tight frames [**BF**] (the proof below comes from [**CF**]).

Propostion 2.2. *Let $\mathbb{H}_N$ be an N-dimensional Hilbert space, $0 < \lambda$ and let*

$$W = \left\{ \{\phi_m\}_{m=1}^M \Big| \sum_{m=1}^M \|\phi_m\|^2 = \lambda \right\}.$$

1. *If $M \leq N$, the minimum value of the frame potential on W is λ^2/M and the minimizers are orthogonal sequences of vectors all with the same norm $\sqrt{\lambda/M}$.*
2. *If $M \geq N$, the minimum value of the frame potential on W is λ^2/N and the minimizers are the tight frames with tight frame bound λ/N.*

PROOF. 1. We compute:

$$\begin{aligned} FP(\{\phi_m\}_{m=1}^M) &= \sum_{n,m=1}^M |\langle \phi_n, \phi_m\rangle|^2 \\ &= \sum_{m=1}^M \|\phi_m\|^4 + \sum_{n\neq m} |\langle \phi_n, \phi_m\rangle|^2 \qquad (2.2)\\ &\geq \sum_{m=1}^M \|\phi_m\|^4. \end{aligned}$$

Since we assumed that

$$\sum_{m=1}^M \|\phi_m\|^2 = \lambda,$$

by a standard application of Lagrange multipliers, the right-hand side of (2.2) is minimized when

$$\|\phi_m\|^2 = \|\phi_n\|^2, \quad \text{for all } 1 \leq m, n \leq M.$$

In this case $\|\phi_m\|^2 = \lambda/M$, for all $1 \leq m \leq M$ and so

$$\sum_{m=1}^M \|\phi_m\|^4 = \frac{\lambda^2}{M}.$$

This minimum is achieved when we have equality in (2.2), and hence

$$\sum_{n\neq m} |\langle \phi_n, \phi_m\rangle|^2 = 0,$$

showing that $\{\phi_m\}_{m=1}^M$ is an orthogonal sequence.

2. When $M > N$, we cannot use the same approach, since we cannot find M mutually orthogonal vectors ϕ_m. By (2.1), minimizing the frame potential under our constraint means minimizing $\sum_{n=1}^N \lambda_n^2$ under the constraint $\sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|\phi_m\|^2 = \lambda$. Again, a standard application of Lagrange multipliers yields that the minimizers satisfy $\lambda_n = \lambda/N$, for all $1 \leq n \leq N$. Hence, the frame operator for $\{\phi_m\}_{m=1}^M$ is $(\lambda/N)I$. That is, a minimizer of the frame potential is a tight frame with the tight frame bound λ/N. Since there always exist such tight frames and these are clearly minimizers of the frame potential, we have the proof.

□

We have immediately:

COROLLARY 2.3. *Given* $\{\phi_m\}_{m=1}^M \subseteq H_N$,

$$FP(\{\phi_m\}) = \sum_{m=1}^M \sum_{n=1}^M |\langle \phi_n, \phi_m\rangle|^2 \geq \frac{\left(\sum_{m=1}^M \|\phi_m\|^2\right)^2}{N}$$

with equality if and only if $\{\phi_m\}_{m=1}^M$ *is a tight frame.*

These results give a method for finding and even constructing tight frames since the tight frames are minimizers of the frame potential. Suppose in a given application we need tight frames with“extra properties”. We let $\mathcal{F}$ be the family of frames having these properties. Generally, a simple compactness arguement shows that $\mathcal{F}$ is “closed” in almost any choice of topology. This means that $\mathcal{F}$ has elements of minimal frame potential. Now we use our theory of frame potentials to show that the “minimizers” must be tight frames. We will see in several places later in these notes where this strategy works quite well.

Minimizing the frame potential over certain classes of frames classifies tight frames. The following result from [**CF**] shows how frame potentials also identify Parseval frames.

Theorem 2.4. *If* $\{\phi_m\}_{m=1}^M$ *is a frame with frame operator* S *and canonical dual frame* $\{S^{-1}\phi_m\}_{m=1}^M$ *then*

$$FP(\{\phi_m\}_{m=1}^M) + FP(\{S^{-1}\phi_m\}_{m=1}^M)$$

is minimum if and only if $\{\phi_m\}_{m=1}^M$ *is a Parseval frame.*

Proof. If $\{\lambda_n\}_{n=1}^N$ are the eigenvalues of the frame operator S, then we are just minimizing

$$\sum_{n=1}^{N} \lambda_n^2 + \sum_{n=1}^{N} \frac{1}{\lambda_n^2}.$$

□

The following result from [**CF**] shows that for any frame, the canonical dual frame is the closest to being tight of all the dual frames.

This result allows us to give quantative estimates when working with dual frames.

Theorem 2.5. *If* $\{\phi_m\}_{m=1}^M$ *is a frame for* H_N*, then the canonical dual frame* $\{S^{-1}\phi_m\}_{m=1}^M$ *is the unique dual frame of minimal frame potential.*

Important applications of the frame potential come from studying its *local minimizers* (see Proposition 2.8 for the definition). Now we will examine this local behavior of the frame potential.

An important result of Benedetto and Fickus [**BF**] is that local minimizers of the frame potential for unit norm frames are also global minimizers. We present here a generalization of this which is a previously unpublished result of Casazza, Fickus, Kovačević, Leon, and Tremain.

Propostion 2.6. *Let* $\{\varphi_m\}_{m=1}^M$ *be a frame for* $\mathbb{H}_N$ *with frame operator* S *having distinct eigenvalues* $\lambda_1 > \lambda_2 > \cdots > \lambda_N$ *with respective eigenspaces* $\{E_{\lambda_i}\}_{i=1}^L$*. Define*

$$F : S^{\mathbb{K}^N} \to \mathbb{R},$$

(where $\mathbb{K}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$*) by*

$$F(\Phi) = \sum_{m=1}^{M} |\langle \Phi, \varphi_m \rangle|^2.$$

Then the sphere of E_{λ_L} *is the set of local minimizers of* F*. Hence, the local minimizers of* F *are global minimizers.*

Proof. We will do the real case since it is quite illuminating. The complex case follows from the same proof with notational changes. Since F is continuous and nonzero and $\{\varphi_m\}_{m=1}^M$ spans $\mathbb{R}^N$, by Lagrange Multipliers, there is a $\mu \neq 0$ so that the minimizers of F satisfy:

$$\nabla F = \mu \nabla G,$$

where for $\Phi = (\Phi_1, \Phi_2 \ldots, \ldots, \Phi_N)$ in $\mathbb{R}^N$ we have

$$G(\Phi) = \sum_{i=1}^N \Phi_i^2 - 1.$$

Hence,

$$\partial_i \lambda G = 2\mu\Phi_i, \quad \text{for all } 1 \le i \le N.$$

Also,

$$F(\Phi) = \sum_{m=1}^M \left[\sum_{k=1}^N \Phi_k \langle \varphi_m, e_k \rangle \right]^2 .$$

Hence,

$$\partial_i F = \sum_{m=1}^M 2 \left[\sum_{k=1}^N \Phi_k \langle \varphi_m, e_k \rangle \right] \langle \varphi_m, e_i \rangle = 2 \sum_{m=1}^M \langle \Phi, \varphi_m \rangle \langle \varphi_m, e_i \rangle .$$

Finally,

$$\Phi_i = (2\mu)^{-1} \sum_{m=1}^M \langle \Phi, \varphi_m \rangle \langle \varphi_m, e_i \rangle = (2\mu)^{-1} \left\langle \left(\sum_{m=1}^M \langle \Phi, \varphi_m \rangle \varphi_m \right), e_i \right\rangle .$$

Hence,

$$\Phi = (\mu)^{-1} \sum_{m=1}^M \langle \Phi, \varphi_m \rangle \varphi_m .$$

We now have that a minimizer of F is an eigenvector of S. Next we show that these eigenvectors are all in E_{λ_L}. We proceed by way of contradiction. If Φ is a local minimizer of F and $\Phi \in E_{\lambda_i}$ for some $1 \le i < L$, choose $e_1 \in E_{\lambda_L}$ with $\|e_1\| = 1$. Fix $0 < \epsilon < 1$ and let

$$\Psi = \epsilon \Phi + \sqrt{(1-\epsilon^2)} e_1 .$$

So $\|\Psi\| = 1$ and

$$F(\psi) = \sum_{m=1}^M |\langle \Psi, \varphi_m \rangle|^2 = \epsilon^2 \lambda_i + (1-\epsilon^2)\lambda_L < \lambda_i = F(\Phi).$$

This contradiction completes the proof. □

Definition 2.7. A frame $\{\varphi_m\}_{m=1}^M$ for $\mathbb{H}_N$ with frame operator S is called a **FF-critical sequence** if each ϕ_m is an eigenvector for S.

The next two propositions and their corollaries are also previously unpublished results of Casazza, Fickus, Kovačević, Leon, and Tremain.

The main point is to identify the exact structure of local minimizers of the frame potential over quite general familes of frames.

Propostion 2.8. *If* $\{\varphi_m\}_{m=1}^M$*,* $M \geq N$ *is a sequence of vectors in* $\mathbb{H}_N$ *with frame operator* S *over its span, and for a fixed* i*,* ϕ_i *is a local minimizer for the frame potential over the set:*

$$\Omega_i = \{\{\psi_m\}_{m=1}^M \mid \psi_m = \phi_m \textit{ for all } m \neq i \textit{ and } \|\psi_i\| = \|\phi_i\|\}$$

then ϕ_i *is an eigenvector for* S*. Hence, any locally minimal sequence* $\{\phi_m\}_{m=1}^M$ *(i.e.* ϕ_i *is a local minimizer of the frame potential over* Ω_i *for all* $1 \leq i \leq M$*) is an* FF*-critical sequence. Moreover, any minimizer for the frame potential must span* H_N*.*

Proof. An obvious compactness argument guarantees that there is some sequence $\{\varphi_m\}_{m=1}^M \in \Omega_i$ which minimizes the frame potential. It is not clear, and will be addressed at the end of the proof, that the minimizers span the space $\mathbb{H}_N$. We note that:

$$\begin{aligned} FP(\{\{\varphi_m\}_{m\neq i} \cup \{\Phi\}) &= \sum_{n,m\neq i} |\langle \varphi_n, \varphi_m\rangle|^2 + |\langle \Phi, \Phi\rangle|^2 + 2\sum_{m\neq i} |\langle \Phi, \varphi_n\rangle|^2 \\ &= FP(\{\varphi_m\}_{m\neq i}) + 2F(\{\Phi\}) - \|\varphi_i\|^4, \end{aligned}$$

where

$$F(\{\Phi\}) = \sum_{m\neq i} |\langle \Phi, \varphi_m\rangle|^2.$$

It follows that the minimizers of FP over Ω_i are the minimizers of F. By Proposition 2.6, φ_i is an eigenvector for $\{\varphi_m\}_{m\neq i}$ with eigenvalue say λ. Now,

$$\begin{aligned} \sum_{m=1}^M \langle \varphi_i, \varphi_m\rangle \varphi_m &= \sum_{m\neq i} \langle \varphi_i, \varphi_m\rangle \varphi_m + \langle \varphi_i, \varphi_i\rangle \varphi_i \\ &= \lambda\varphi_i + \|\varphi_i\|^2 \varphi_i = (\lambda + \|\varphi_i\|^2)\varphi_i. \end{aligned}$$

Suppose $\{\varphi_m\}_{m=1}^M \in \Omega_i$ is a minimizer for the frame potential over Ω_i. We proceed by way of contradiction. Suppose span $\{\varphi_m\}_{m=1}^M \neq \mathbb{H}_N$. Let S be the frame operator for $\{\varphi_m\}_{m=1}^M$ over its span with eigenspaces $\{E_{\lambda_i}\}_{i=1}^L$. By the first part of the proof, the vectors ϕ_m sit in the E_{λ_i}. For every $1 \leq j \leq L$ let $I_j = \{m \mid \phi_m \in E_{\lambda_j}\}$. If $\{\psi_m\}_{m=1}^M$ is a λ-tight frame, it is known (and a simple calculation to verify) that $\|\psi_i\|^2 \leq \lambda$ and $\|\psi_i\|^2 = \lambda$ if and only if $\psi_i \perp$ span $\{\psi_m\}_{m\neq i}$. Now, since $M \geq N$ and span $\{\varphi_m\}_{m=1}^M \neq H_N$, it follows that there is a $1 \leq j \leq L$ so that $|I_j| > \dim E_{\lambda_j}$. Hence, there is some $i \in I_j$ with $\|\varphi_i\|^2 < \lambda_j$. Now choose $e_1 \in \mathbb{H}_N$ with $\|e_1\| = 1$ and $e_1 \perp \{\varphi_m\}_{m=1}^M$. Define $\{\psi_m\}_{m=1}^M \in \Omega$ by: $\psi_m = \phi_m$ for all $1 \leq m \neq i \leq M$ and let $\psi_i = \|\varphi_i\| e_1$. We will obtain a contradiction by showing that $FP(\{\psi_m\}_{m=1}^M) < FP(\{\varphi_m\}_{m=1}^M)$.

$$\begin{aligned}
FP(\{\psi_m\}_{m=1}^M) &= FP(\{\psi_m\}_{m\neq i} + \|\psi_i\|^4 \\
&= FP(\{\varphi_m\}_{m\neq i}) + \|\varphi_i\|^4 \\
&= \sum_{n,m\neq i} |\langle \varphi_n, \varphi_m\rangle|^2 + \|\varphi_i\|^4 \\
&= \sum_{n,m=1}^M |\langle \varphi_n, \varphi_m\rangle|^2 + \|\varphi_i\|^4 - 2\sum_{n=1}^M |\langle \varphi_i, \varphi_n\rangle|^2 + \|\varphi_i\|^4 \\
&= FP(\{\varphi_m\}_{m=1}^M) + 2\|\varphi_i\|^4 - 2\lambda_j \|\varphi_i\|^2 \\
&= FP(\{\varphi_m\}_{m=1}^M) + 2\|\varphi_i\|^2(\|\varphi_i\|^2 - \lambda_j) < FP(\{\varphi_m\}_{m=1}^M).
\end{aligned}$$

This contradiction completes the proof of the Theorem. □

In some applications of frame potentials we must switch to subsets of our frame and then see how the frame potential changes. The next two results give the basic calculations for the frame potential restricted to a subset of our frame.

PROPOSTION 2.9. *Let* $\{\varphi_m\}_{m=1}^M$ *be a frame for* $\mathbb{H}_N$ *and* $I \subset \{1, 2, \ldots, M\}$. *Then*

$$FP(\{\phi_m\}_{m\notin I}) = FP(\{\varphi_m\}_{m=1}^M) + FP(\{\phi_m\}_{m\in I}) - 2\sum_{n\in I}\sum_{m=1}^M |\langle \phi_m, \phi_m\rangle|^2.$$

PROOF. We compute:

$$FP(\{\varphi_m\}_{m\in I^c}) = \sum_{n,m\in I^c} |\langle \varphi_n, \varphi_m\rangle|^2$$

$$= \sum_{n,m=1}^M |\langle \varphi_n, \varphi_m\rangle|^2 - \sum_{n,m\in I} |\langle \varphi_n, \varphi_m\rangle|^2 - 2\sum_{n\in I}\sum_{m\in I^c} |\langle \varphi_n, \varphi_m\rangle|^2$$

$$= FP(\{\varphi_m\}_{m=1}^M) - FP(\{\varphi_n\}_{n\in I}) - 2\sum_{n\in I}\sum_{m\in I^c} |\langle \varphi_n, \varphi_m\rangle|^2 =$$

$$FP(\{\varphi_m\}_{m=1}^M) + FP(\{\varphi_n\}_{n\in I}) - 2\sum_{n\in I}\sum_{m\in I} |\langle \varphi_n, \varphi_m\rangle|^2 - 2\sum_{n\in I}\sum_{m\in I^c} |\langle \varphi_n, \varphi_m\rangle|^2 =$$

$$FP(\{\varphi_m\}_{m=1}^M) + FP(\{\varphi_m\}_{m\in I}) - 2\sum_{n\in I}\sum_{m=1}^M |\langle \varphi_n, \varphi_m\rangle|^2.$$

□

COROLLARY 2.10. *Let* $\{\varphi_m\}_{m=1}^M$ *be a tight frame for* $\mathbb{H}_N$ *with* $\|\varphi_m\| = 1$, *for all* $1 \le m \le M$. *If* $I \subset \{1, 2, \ldots, M\}$ *with* $|I| = k$, *then*

$$FP(\{\varphi_m\}_{m\in I^c}) = \frac{M^2}{N} - 2k\frac{M}{N} + FP(\{\varphi_m\}_{m\in I}).$$

Hence,

1. *If* $|I| > |I^c|$ *then* $FP(\{\varphi_m\}_{m\in I^c}) < FP(\{\varphi_m\}_{m\in I})$.
2. *If* $|I| < |I^c|$ *then* $FP(\{\varphi_m\}_{m\in I^c}) > FP(\{\varphi_m\}_{m\in I})$.
3. *If* M *is even and* $|I| = |I^c|$ *then* $FP(\{\varphi_m\}_{m\in I^c}) = FP(\{\phi_m\}_{m\in I})$.

Proof. Since $\|\phi_m\| = 1$, $FP(\{\varphi_m\}_{m=1}^M) = \frac{M^2}{N}$ and since the frame is a tight frame,

$$-2\sum_{n\in I}\sum_{m=1}^{M} |\langle \varphi_n, \varphi_m\rangle|^2 = -2\sum_{n\in I}\frac{M}{N} - 2k\frac{M}{N}.$$

$\square$

3. A physical interpretation for finite tight frames

A problem which arose in communication theory is that of classifying those sequences $a_1 \geq a_2 \geq \cdots \geq a_m > 0$ so that there exists a tight frame $\{\phi_m\}_{m=1}^M$ for H_N for which $\|\phi_m\| = a_m$, for all $1 \leq m \leq N$. More importantly, if no tight frames exist for a given sequence $\{a_m\}_{m=1}^M$, can we identify those frames which are the closest to being tight and satisfy $\|\phi_m\| = a_m$? Casazza, Fickus, Kovačević, Leon and Tremain [**CF**] gave a complete solution to this question by building on earlier work of Benedetto and Fickus involving "frame-equivalent" notions to force and potential. In the process, the authors of [**CF**] arrived at the *Fundamental Inequality* for tight frames which governs the distribution of power among the frame vectors. All the results of this section come from [**CF**].

3.1. Coulomb's Law and the frame force. The authors of [**CF**] investigated the common notions of what it means to equally distribute a collection of electrons upon a conductive spherical shell. Physically, in the absence of external forces the charged particles will repel each other according to the inverse-square Coulomb force law

$$CF(\phi_m, \phi_n) = \frac{\phi_m - \phi_n}{\|\phi_m - \phi_n\|^3}.$$

Intuitively, the corresponding optimal arrangements are those which minimize the internal pressure of the points upon each other. Specifically, given M distinct electrons located at points $\{\phi_m\}_{m=1}^M$, one seeks to minimize the total corresponding potential energy of the system,

$$CP(\{\phi_m\}_{m=1}^M) = \sum_{m=1}^{M}\sum_{n\neq m}\frac{1}{\|\phi_m - \phi_n\|}.$$

And though only a global minimizer corresponds to true equidistribution in this context, local minimizers are also of physical interest, in that they correspond to collections of points in equilibrium.

It is known that local minimizers of potential energy need not be global minimizers. For example, a dodecahedron inscribed in a sphere is a local minimizer of potential energy while there are different configurations with smaller total potential energy.

In [**BF**], Benedetto and Fickus defined a central force between real unit norm vectors ϕ_m and ϕ_n (called the *frame force*) by

$$FF(\phi_m, \phi_n) = 2\langle \phi_m, \phi_n\rangle(\phi_m - \phi_n).$$

The important point here is that the frame force for orthogonal vectors is 0. Also, vectors having an acute angle between them are repelling while vectors having an obtuse angle are attracting. So "charged particles" under the frame force are trying to reach equilibrium by becoming as orthogonal as possible. There are some other unnatural aspects to the frame force. For one, it needs a universal reference point—a fixed origin. Also, the force field generated by a point is not conservative. That is, the work required to get from one point to another depends upon the particular path taken. However (and this is all we need for our applications) the frame force is conservative when the points are constricted to lie on a sphere.

3.2. The weighted frame force. Since we need our vectors to lie on possibly different spheres, we need a corresponding *weighted frame force (potential).* This makes the physical systems much more difficult to understand intuitively. That is, visualizing the movements of M charged particles restricted to M concentric spheres can challenge the imagination. However, this can be greatly simplified by "projecting" the dynamics down onto the unit sphere.

Consider two points, each of whose movement is restricted to a sphere of a given, yet arbitrary radius. That is, given $a_m, a_n > 0$, consider ϕ_m with $\|\phi_m\| = a_m$ and ϕ_n with $\|\phi_n\| = a_n$. Note that

$$\begin{aligned}\|\phi_m - \phi_n\|^2 &= \|\phi_m\|^2 - 2\langle \phi_m, \phi_n\rangle + \|\phi_n\|^2,\\ &= a_m^2 - 2\langle \phi_m, \phi_n\rangle + a_n^2,\end{aligned}$$

and so we may rewrite the frame force between these points as

$$\begin{aligned}FF(\phi_m, \phi_n) &= 2\langle \phi_m, \phi_n\rangle(\phi_m - \phi_n),\\ &= (a_m^2 + a_n^2 - \|\phi_m - \phi_n\|^2)(\phi_m - \phi_n).\end{aligned}$$

As in the Coulomb case, the pairwise potential between these points may be found by integrating the "magnitude" of this central force,

$$p(x) = -\int (a_m^2 + a_n^2 - x^2)x\,dx = \frac{1}{4}x^2[x^2 - 2(a_m^2 + a_n^2)],$$

and evaluating at $x = \|\phi_m - \phi_n\|$,

$$P(\phi_m, \phi_n) = p(\|\phi_m - \phi_n\|) = \langle \phi_m, \phi_n\rangle^2 - \frac{1}{4}(a_m^2 + a_n^2)^2.$$

The total potential contained within the physical system is the sum of all pairwise potentials,

$$TP(\{\phi_m\}_{m=1}^M) = \sum_{m,n} |\langle \phi_m, \phi_n\rangle|^2 - \frac{1}{4}\sum_{m,n}(a_m^2 + a_n^2)^2$$

However, we may disregard the additive constant, as it has no physical significance. This then leads to the *frame potential* definition of Benedetto and Fickus [Definition 2.1 above].

Remark 3.1. Given two sequences $\{\phi_m\}_{m=1}^M$ and $\{\psi_m\}_{m=1}^M$ in $\mathbb{R}^N$ with $\|\phi_m\| = a_m = \|\psi_m\|$ for all m, $FP(\{\psi_m\}_{m=1}^M) - FP(\{\phi_m\}_{m=1}^M)$ is the work required to transform $\{\phi_m\}_{m=1}^M$ into $\{\psi_m\}_{m=1}^M$ while remaining on the spheres of radii $\{a_m\}_{m=1}^M$.

Though the frame potential is the potential energy contained within a system of points of equal weight on spheres of varying radii, we form an equivalent situation by considering points of varying weight all lying on a common sphere.

Definition 3.2. Let S denote the unit sphere in $\mathbb{R}^N$, and $\psi_m \in \mathbb{R}^N$, $\|\psi_m\| = 1$ for all $1 \le m \le M$. We associate a weight w_m to each ψ_m.

- The **weighted frame force** between ψ_m and ψ_n is

$$WFF : [0,\infty) \times [0,\infty) \times S \times S \to \mathbb{R}^N,$$
$$WFF(w_m, w_n, \psi_m, \psi_n) = 2w_m w_n \langle \psi_m, \psi_n\rangle(\psi_m - \psi_n).$$

- The **weighted frame potential** of $\{\psi_m\}_{m=1}^M$ is

$$WFP(\{w_m\}_{m=1}^M, \{\psi_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{n=1}^M w_m w_n |\langle \psi_m, \psi_n\rangle|^2.$$

For $\phi_m = a_m \psi_m$, we write,

$$WFP(\{a_m^2\}_{m=1}^M, \{\psi_m\}_{m=1}^M) = FP(\{\phi_m\}_{m=1}^M)$$

Thus, we shall no longer view the frame potential as the total potential energy of a system of points of mass 1 restricted to spheres of radius a_m. Rather, it shall be perceived more naturally as the energy of a system of points of masses a_m^2 restricted to a single sphere of radius 1.

In our situation, the points experiencing the frame force are constrained to move upon spheres. Therefore, only the components of the frame force acting upon a point which are tangent to the surface of the sphere at that point make a contribution to the frame potential. Thus, given $\phi_m, \phi_n \in \mathbb{R}^N$ with $\|\phi_m\| = a_m$, we wish to explicitly find the component of $FF(\phi_m, \phi_n)$ which lies tangent to the sphere of radius a_m at ϕ_m. Of course, to find the tangential component, we need only subtract the normal component from the whole. And since the surface in question is a sphere, the normal component of the force is simply the projection of the force onto the line passing through ϕ_m, i.e.

$$\frac{\langle FF(\phi_m, \phi_n), \phi_m\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m.$$

We therefore simplify

$$\begin{aligned} FF(\phi_m, \phi_n) &- \frac{\langle FF(\phi_m, \phi_n), \phi_m\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m \\ &= \langle \phi_m, \phi_n\rangle(\phi_m - \phi_n) - \langle \phi_m, \phi_n\rangle \frac{\langle \phi_m, \phi_m\rangle - \langle \phi_m, \phi_n\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m \\ &= \langle \phi_m, \phi_n\rangle \left(\frac{\langle \phi_m, \phi_n\rangle}{\langle \phi_m, \phi_m\rangle}\phi_m - \phi_n\right). \end{aligned}$$

This leads us to:

Propostion 3.3. *Given a sequence* $\{\phi_m\}_{m=1}^M \subseteq \mathbb{R}^N$ *with* $\|\phi_m\| = a_m$, *the following are equivalent*

1. $\{\phi_m\}_{m=1}^M$ *is a tight frame for* $\mathbb{R}^N$,
2. $\sum_{m=1}^M FF(\phi, \phi_m) = 0$ *for all* $\phi \in \mathbb{R}^N$,
3. $\sum_{m=1}^M FF(w, a_m^2, \phi, \phi_m) = 0$ *for all* $\phi \in S^{(N-1)}$.

3.3. The fundamental inequality and minimizers of the weighted frame potential.

To answer the question stated at the beginning of this section, we need an observation about decreasing sequences of positive numbers. Basically, this result compares the terms of the sequence to the "dimensional average" of the following terms.

Propostion 3.4. *Given any sequence $\{a_m\}_{m=1}^M \subset \mathbb{R}$ with $a_1 \geq \cdots \geq a_M \geq 0$, and any $N \leq M$, there is a unique index N_0 with $1 \leq N_0 \leq N$, such that the inequality*

$$a_n > \frac{\sum_{m=n+1}^M a_m}{N-n} \tag{3.1}$$

holds for $1 \leq n < N_0$, while the opposite inequality

$$a_n \leq \frac{\sum_{m=n+1}^M a_m}{N-n} \tag{3.2}$$

holds for $N_0 \leq n < N$.

Proof. Consider the set of indices n such that (3.2) holds, namely $\mathcal{I} = \{n : (N-n)a_n \leq \sum_{m=n+1}^M a_m\}$. Clearly $\mathcal{I} \neq \emptyset$, since $N \in \mathcal{I}$. Also, if $n \in \mathcal{I}$, then $n+1 \in \mathcal{I}$ since

$$\begin{aligned}
[N-(n+1)]a_{n+1} &= -a_{n+1} + (N-n)a_{n+1}, \\
&\leq -a_{n+1} + (N-n)a_n, \\
&\leq -a_{n+1} + \sum_{m=n+1}^M a_m, \\
&= \sum_{m=n+2}^M a_m.
\end{aligned}$$

The requisite properties upon N_o are then satisfied by letting N_0 be the minimum index in $\mathcal{I}$. ☐

The general answer to our original problem is contained in the following result from [**CF**]. We will denote by $S(a_1, \ldots, a_m)$ the product of spheres in $\mathbb{R}^N$ of radii $\{a_1, \ldots, a_m\}$.

Theorem 3.5. *Given a sequence $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ and any $N \leq M$, let N_0 be the smallest index n for which*

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_m^2}{N-n}$$

holds. Then any local minimizer of the frame potential

$$FP : S(a_1, \ldots, a_m) \to \mathbb{R}$$

is of the form

$$\{\phi_m\}_{m=1}^M = \{\phi_m\}_{m=1}^{N_0-1} \cup \{\phi_m\}_{m=N_0}^M$$

where $\{\phi_m\}_{m=1}^{N_0-1}$ is an orthogonal set for whose orthogonal complement $\{\phi_m\}_{m=N_0}^M$ forms a tight frame.

The proof of this theorem is a serious piece of work relying heavily on a deep intuitive understanding of the frame potential, and so we refer the reader to [**CF**] for the details. An important consequence of Theorem 3.5 is that we can only get tight frames for H_N when $N_0 = 1$. This case can be summarized as:

Corollary 3.6. *Fix $N \leq M$ and $a_1 \geq a_2 \geq \cdots \geq a_M > 0$. The following are equivalent:*

1. *There is a tight frame* $\{\phi_m\}_{m=1}^M$ *for* H_N *satisfying* $\|\phi_m\| = a_m$, *for* $1 \leq m \leq M$.
2. *For all* $1 \leq n < N$ *we have*

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_m^2}{N-n}.$$

3. *We have:*

$$\sum_{m=1}^M a_m^2 \geq N a_1^2.$$

4. *If*

$$\lambda = \sqrt{\frac{N}{\sum_{m=1}^M a_m^2}},$$

then $\lambda a_m \leq 1$ *for all* $1 \leq m \leq M$.

Proof. The equivalence of (1) and (3) is Theorem 3.5. The equivalence of (2) and (4) is immediate.

(2)⇒(3): Let $n = 1$ in (1) and get

$$\sum_{m=2}^M a_m^2 \geq (N-1)a_1^2.$$

Hence,

$$\sum_{m=1}^M a_m^2 \geq N a_1^2.$$

(3)⇒(2): We assume that

$$\sum_{m=1}^M a_m^2 \geq N a_1^2.$$

Since $a_1 \geq a_m$, for all $1 \leq m \leq M$ we have

$$\begin{aligned}\sum_{m=N+1}^M a_m^2 &\geq N a_1^2 - \sum_{m=1}^N a_m^2 \\ &= (N-n)a_1^2 - \sum_{m=n+1}^N a_m^2 + \sum_{m=1}^d (a_1^2 - a_m^2) \\ &\geq (N-n)a_n^2 - \sum_{m=n+1}^N a_m^2.\end{aligned}$$

Hence,

$$\sum_{m=n+1}^M a_m^2 \geq (N-n)a_n^2,$$

and thus

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_m^2}{N-n}.$$

□

We refer to (3) in the corollary above as the *Fundamental Inequality* for tight frames. Note that what we have shown in Corollary 3.7 is that properties (2)-(4) are equivalent for any sequence of numbers.

As we discussed earlier, local minimizers of the Coulomb potential are not necessarily global minimizers. However, for the frame potential, we do have that local minimizers are global minimizers. Hence, the characterization of Theorem 3.5 applies to all minimizers and yields the following result from [**CF**].

Corollary 3.7. *Given a sequence* $\{a_m\}_{m=1}^M \subset \mathbb{R}$ *with* $a_1 \geq \cdots \geq a_M > 0$*, and any* $N \leq M$*, let* N_0 *denote the smallest index* n *such that*

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_n^2}{N-n}$$

holds. Then, for the frame potential $FP : S(a_1, \ldots, a_M) \to \mathbb{R}$,

1. *The minimal value is* $\sum_{m=1}^{N_0-1} a_m^4 + \frac{(\sum_{m=N_0}^M a_m^2)^2}{N-N_0+1}$.
2. *Any local minimizer is a global minimizer.*
3. *The minimizers are precisely those sequences where* $\{\phi_m\}_{m=1}^{N_0-1}$ *is an orthogonal set for whose orthogonal complement* $\{\phi_m\}_{m=N_0}^M$ *forms a tight frame.*

Proof. To begin, we compute the frame potential for any sequence of the form $\{\phi_m\}_{m=1}^{N_0-1} \cup \{\phi_m\}_{m=N_0}^M$, where $\{\phi_m\}_{m=1}^{N_0-1}$ is an orthogonal sequence for whose orthogonal complement the sequence $\{\phi_m\}_{m=N_0}^M$ is a tight frame. Thus, for $m = 1, \ldots, N_0 - 1$, $\langle \phi_m, \phi_n \rangle = 0$ for $n \neq m$, and so

$$\begin{aligned} FP(\{\phi_m\}_{m=1}^M) &= \sum_{m=1}^M \sum_{n=1}^M |\langle \phi_m, \phi_n \rangle|^2, \\ &= \sum_{m=1}^{N_0-1} |\langle \phi_m, \phi_m \rangle|^2 + \sum_{m=N_0}^M \sum_{n=N}^M |\langle \phi_m, \phi_n \rangle|^2, \\ &= \sum_{m=1}^{N_0-1} a_m^4 + FP(\{\phi_m\}_{m=N_0}^M). \end{aligned}$$

Now the fact that $\{\phi_m\}_{m=N_0}^M$ is a tight frame for the $(N - N_0 + 1)$-dimensional orthogonal complement implies that

$$FP(\{\phi_m\}_{m=1}^M) = \sum_{m=1}^{N_0-1} a_m^4 + \frac{(\sum_{m=N_0}^M a_m^2)^2}{N - N_0 + 1}. \tag{3.3}$$

In light of Theorem 3.5 we observe that any local minimizer of the frame potential is necessarily of this form, and consequently attains the potential value given in (3.3).

Next, we note that $S(a_1, \ldots, a_M)$ is compact, being a Cartesian product of spheres. The continuity of the frame potential upon this compact domain then guarantees that a global minimizer exists. As any global minimizer is also a local minimizer, the value given in (3.3) is indeed the global minimum, yielding the first claim.

Since all local minimizers attain the same value, the second claim follows.

For the third claim, note that any sequence of this form has already been shown to attain the minimal value, while Theorem 3.5 yields the converse implication. □

4. Algorithmic constructions of frames

In this section we will look at algorithms for constructing frames with prescribed properties. This is critically important for applications where we need systematic methods for constructions which can be carried out, for e.g., in MatLab.

One serious inhibitor for algorithmic constructions of tight frames is that we cannot get tight frames at each stage of the construction. That is, if $\{\phi_m\}_{m=1}^{M}$ is a tight frame then $\{\phi_m\}_{m=1}^{M-K+1}$ cannot also be a tight frame for $1 \leq K < N$. That is, assume for all $\phi \in H_N$ we have

$$\lambda\|\phi\|^2 = \sum_{m=1}^{M} |\langle \phi, \phi_m\rangle|^2.$$

Now, if $\{\phi_m\}_{m=1}^{M-K+1}$ is a δ tight frame,

$$\delta\|\phi\|^2 = \sum_{m=1}^{M-K+1} |\langle \phi, \phi_m\rangle|^2,$$

and so

$$(\lambda - \delta)\|\phi\|^2 = \sum_{m=M-K+2}^{M} |\langle \phi, \phi_m\rangle|^2.$$

But $\{\phi_m\}_{m=M-K+2}^{M}$ does not span H_N so this is impossible.

4.1. Nearly equal norm and nearly tight frames.

In this subsection we will give algorithms for turning frames which are nearly tight (respectively, nearly equal norm) into tight frames (respectively, equal norm frames). We wish to thank Beata Randrianantoanina for sharing with us her notes on algorithmic constructions of frames. Propositions 4.1 and 4.3 of this section are an analgam of Randrianantoanina's notes and the author's.

Let $\{\phi_m\}_{m=1}^{M}$ be a frame with frame bounds A, B and satisfying $c-\epsilon \leq \|\phi_m\| \leq c+\epsilon$, for all $1 \leq m \leq M$. After a change of scale we may assume $1-\epsilon \leq \|\phi_m\| \leq 1+\epsilon$ and after a permutation we may as well assume that $1+\epsilon \geq \|\phi_1\| \geq \|\phi_2\| \geq \cdots \geq \|\phi_M\| \geq 1-\epsilon$. We call such a frame a ϵ *nearly unit norm frame.* We can turn $\{\phi_m\}_{m=1}^{M}$ into an equal norm frame by letting

$$\psi_m = x\frac{\phi_m}{\|\phi_m\|}, \quad \text{for } 0 < x.$$

Then

$$\max_{1\leq m\leq M} |x - \|\phi_m\||$$

and

$$\sum_{m=1}^{M} |\|\psi_m\| - \|\phi_m\|| = \sum_{m=1}^{M} |x - \|\phi_m\||$$

are minimized when

$$x = \frac{\|\phi_1\|^2 + \|\phi_M\|^2}{2}.$$

Also,

$$E=\sum_{m=1}^{M}|x-\|\phi_m\||^2$$

is minimized when

$$x=\frac{\sum_{m=1}^{M}\|\phi_m\|}{M}.$$

We need to check how close our new frame is to the old one.

$$\begin{aligned}\sum_{m=1}^{M}\|\psi_m-\phi_m\|^2&=\sum_{m=1}^{M}\left\|x\frac{\phi_m}{\|\phi_m\|}-\phi_m\right\|^2\\&=\sum_{m=1}^{M}|x-\|\phi_m\||^2=E.\end{aligned}$$

In this case we say $\{\psi_m\}_{m=1}^M$ is *E-close* to $\{\phi_m\}_{m=1}^M$. Also, for all $\phi\in H_N$,

$$\begin{aligned}\left(\frac{1-\epsilon}{1+\epsilon}\right)^2A\|\phi\|^2&\le\left(\frac{1-\epsilon}{1+\epsilon}\right)^2\sum_{m=1}^{M}|\langle\phi,\phi_m\rangle|^2\\&\le\sum_{m=1}^{M}\frac{x^2}{\|\phi_m\|^2}|\langle\phi,\phi_m\rangle|^2\\&=\sum_{m=1}^{M}|\langle\phi,\psi_m\rangle|^2\\&\le\left(\frac{1+\epsilon}{1-\epsilon}\right)^2B\|\phi\|^2.\end{aligned}$$

We formulate this as:

Propostion 4.1. *If $\{\phi_m\}_{m=1}^M$ is an ϵ nearly unit norm frame with frame bounds A,B, then $\{\phi_m\}_{m=1}^M$ is E-close to a unit norm frame with frame bounds $\left(\frac{1-\epsilon}{1+\epsilon}\right)^2A$, and $\left(\frac{1+\epsilon}{1-\epsilon}\right)^2B$.*

Now we look at nearly tight frames. After a change of scale, we may restrict our attention to nearly Parseval frames.

We say that a frame $\{\phi_m\}_{m=1}^M$ with frame bounds A,B is ϵ *nearly Parseval* if for all $\phi\in H_N$,

$$(1-\epsilon)\|\phi\|^2\le\sum_{m=1}^{M}|\langle\phi,\phi_m\rangle|^2\le(1+\epsilon)\|\phi\|^2.$$

Let S be the frame operator for a frame $\{\phi_m\}_{m=1}^M$ with eigenvalues $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_N>0$ and a corresponding orthonormal set of eigenvectors $\{e_n\}_{n=1}^N$. The following result can be found in [**CF1**].

Propostion 4.2. *For any $1\le n\ne k\le N$,*

$$\sum_{m=1}^{M}\langle\phi_m,e_n\rangle\overline{\langle\phi_m,e_k\rangle}=0.$$

That is, the column vectors of the matrix $(\langle \phi_m, e_n\rangle)_{m=1,n=1}^{M,N}$ *are orthogonal.*

PROOF. We compute:

$$\begin{aligned}\langle Se_n, e_k\rangle &= \langle \lambda_n e_n, e_k\rangle = \lambda_n\langle e_n, e_k\rangle = \lambda_n\delta_{nk}\\ &= \left\langle \sum_{m=1}^{M}\langle e_n, \phi_m\rangle\phi_m, e_k\right\rangle\\ &= \sum_{m=1}^{M}\overline{\langle \phi_m, e_n\rangle}\langle \phi_m, e_k\rangle\end{aligned}$$

□

Now we want to check that an ϵ nearly Parseval frame is close to a Parseval frame. Given $\{\phi_m\}_{m=1}^M$ ϵ nearly Parseval for H_N with frame bounds A, B, let $\lambda_1, \geq \lambda_2 \geq \cdots \geq \lambda_N > 0$ be the eigenvalues for the frame operator S with orthonormal eigenvectors $\{e_n\}_{n=1}^M$. It is known (see [**B**] and [**J**]) that the "closest" tight frame to $\{\phi_m\}$ is $\{S^{-\frac{1}{2}}\phi_m\}$. We have that $1+\epsilon \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 1-\epsilon$. We construct a new frame $\{\psi_m\}_{m=1}^M$ by

$$S^{-\frac{1}{2}}\phi_m = \psi_m = \sum_{n=1}^{N}\frac{1}{\lambda_n}\langle \phi_m, e_n\rangle e_n.$$

Then

$$\begin{aligned}\sum_{m=1}^{M}\|\psi_m - \phi_m\|^2 &= \sum_{m=1}^{M}\sum_{n=1}^{N}|\langle \phi_m, e_n\rangle|^2\left|\frac{1}{\lambda_n}-1\right|^2\\ &= \sum_{n=1}^{N}\left|\frac{1}{\lambda_n}-1\right|^2\sum_{m=1}^{M}|\langle \phi_m, e_n\rangle|^2\\ &\leq \sum_{n=1}^{N}\left|\frac{1}{\lambda_n}-1\right|(1+\epsilon)\\ &\leq N\frac{\epsilon}{1-\epsilon}(1+\epsilon).\end{aligned}$$

Also, for every $\phi \in H_N$,

$$\begin{aligned}\sum_{m=1}^{M}|\langle \phi, \psi_m\rangle|^2 &= \sum_{m=1}^{M}\left|\sum_{n=1}^{N}\frac{1}{\lambda_n}\langle \phi, e_n\rangle\overline{\langle \phi_m, e_n\rangle}\right|^2\\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{k=1}^{M}\frac{1}{\lambda_n}\frac{1}{\lambda_k}\langle \phi, e_n\rangle\overline{\langle \phi, e_k\rangle}\;\overline{\langle \phi_m, e_n\rangle}\langle \phi_m, e_k\rangle\\ &= \sum_{n=1}^{N}\sum_{k=1}^{N}\frac{1}{\lambda_n}\frac{1}{\lambda_k}\langle \phi, e_n\rangle\overline{\langle \phi, e_k\rangle}\sum_{m=1}^{M}\overline{\langle \phi_m, e_n\rangle}\langle \phi_m, e_k\rangle\\ &= \sum_{n=1}^{N}\frac{1}{\lambda_n}|\langle \phi, e_n\rangle|^2\sum_{m=1}^{M}|\langle \phi_m, e_n\rangle|^2\\ &= \sum_{n=1}^{N}\frac{1}{\lambda_n}|\langle \phi, e_n\rangle|^2\lambda_n = \|\phi\|^2.\end{aligned}$$

In the next to last equality above we used Proposition 4.2. We summarize this as:

PROPOSTION 4.3. *If* $\{\phi_m\}_{m=1}^M$ *is* ϵ *nearly Parseval, then* $\{\phi_m\}_{m=1}^M$ *is* $N\epsilon\frac{1+\epsilon}{1-\epsilon}$ *close to a Parseval frame.*

So, we have seen that nearly unit norm frames are close to unit norm frames and nearly Parseval frames are close to Parseval frames. However, each of these algorithms destroys the other property. Vern Paulsen has raised the question of whether there is a quantitative construction giving both of these properties at once.

PROBLEM 4.4 (Paulsen). If a frame is ϵ nearly equal norm and δ nearly Parseval, is the frame $f(\epsilon,\delta)$ close to a equal norm Parseval frame? The real point here is to identity the closeness function $f(\epsilon,\delta)$.

Related to this is the old and important question of identifying all (equal norm) Parseval frames.

PROBLEM 4.5. Is there an algorithm for constructing all (equal norm) Parseval frames?

There is a simple algorithm for turning a Parseval frame into an equal norm Parseval frame due to Holmes and Paulsen [**HP**]. Unfortunately, at this time we have no way to access the closeness of the new frame to the old under this construction.

PROPOSTION 4.6. *There is an algorithm for turning any frame into an equal norm frame without changing the frame operator.*

PROOF. Let $\{\phi_m\}_{m=1}^M$ be a frame for H_N with frame operator S and analysis operator F. Then

$$\sum_{m=1}^M \|\phi_m\|^2 = TrS,$$

so let $\lambda = \frac{TrS}{M}$. If $\|\phi_m\|^2 = \lambda$ for all m, we are done. Otherwise, there exists $1 \le i \ne j \le M$ with $\|\phi_i\|^2 > \lambda > \|\phi_j\|^2$. For any θ, replace the vectors ϕ_i, ϕ_j by the vectors $\psi_i = (\cos\theta)\phi_i - (\sin\theta)\phi_j$, $\psi_j = (\sin\theta)\phi_i + (\cos\theta)\phi_j$ and let $\psi_k = \phi_k$ for all other k. Then the analysis operator for $\{\psi_m\}_{m=1}^M$ is $F_1 = UF$ for a unitary operator U on ℓ_2^M given by the Givens rotation. Hence, $F_1^*F_1 = F^*U^*UF = F^*F = S$. So our frame operator is unchanged for any value of θ. By choosing θ appropriately, we can obtain $\|\phi_i\|^2 = \lambda$. Repeating this process at most $N-1$ times yields an equal norm frame with the same frame operator as $\{\phi_m\}_{m=1}^M$. Note that this gives an alternative proof of Corollary 4.11. □

D. Hadwin (see [**HP**]) has pointed out that a simple compactness argument shows that a solution to Problem 4.4 exists. Unfortunately, this argument does not give any quantitative estimates on the parameters involved. We want to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if a frame has frame bounds $1-\delta$, $1+\delta$ and the lengths of the frame vectors differ by at most δ, then there is an equal norm tight frame whose vectors are at distance at most $\epsilon > 0$ from the original vectors [**HP**]. So, we assume not. Then there is an $\epsilon > 0$ so that for every $\delta = \frac{1}{n}$, there is a frame $\{\phi_m^n\}_{m=1}^M$ with frame bounds $1-\frac{1}{n}$, $1+\frac{1}{n}$ and whose vectors have lengths differing by at most $\frac{1}{n}$, but this frame is greater than $\epsilon > 0$ from any equal norm tight frame. By compactness (and switching to a subsequence) we may assume that

$$\lim_{n\to\infty} \phi_m^n = \phi_m, \quad \text{for all } 1 \le m \le M.$$

But now, $\{\phi_m\}_{m=1}^M$ is an equal norm tight frame, which is a contradiction.

4.2. Algorithmic constructions for tight frames. A problem which arose in Communication Theory is to identify those sequences of numbers $a_1 \ge a_2 \ge \cdots \ge a_M > 0$ so that there is a tight frame $\{\phi_m\}_{m=1}^M$ for H_N with $\|\phi_m\| = a_m$, for all $1 \le m \le M$. As we saw earlier, this problem was resolved in [**CF**]. But this proof is not implementable in practice. Casazza and Leon [**CL**] gave an algorithmic construction for this class of frames as well as a MatLab program [**CL1**] for carrying this out.

THEOREM 4.7. *For any $M \ge N$ and any sequence of numbers $1 \ge a_1 \ge a_2 \ge \cdots \ge a_M > 0$ satisfying $\sum_{m=1}^M a_m^2 = N$, there is a Parseval frame $\{\phi_m\}_{m=1}^M$ for H_N satisfying $\|\phi_m\| = a_m$, for all $1 \le m \le M$.*

PROOF. We sketch the algorithm here.

A *Givens rotation*

$$\theta(t,j,k) \text{ for } 1 \le j < k \le M$$

is

$$\theta(t,j,k) = \begin{pmatrix} I_{j-1,j-1} & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) & 0 \\ 0 & 0 & I_{M-j-k-2,M-j-k-2} & 0 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 0 & I_{k-1,k-1} \end{pmatrix}$$

It is clear that

$$\theta(t,j,k)^{-1} = \theta(-t,j,k).$$

To construct a tight frame $\{\phi_m\}_{m=1}^M$ with M elements in H_N with $\|\phi_m\| = a_m$, for all $1 \le m \le M$, we need [**CK, CF1**] to construct an $M \times M$ unitary matrix $(m_{ij})_{i,j=1}^M$ so that

$$\sum_{j=1}^N |m_{ij}|^2 = a_i^2, \quad \text{for all } 1 \le i \le M.$$

To do this, we will start with any block unitary matrix of the form

$$O = \begin{pmatrix} R_N & O \\ O & R_{M-N,M-N} \end{pmatrix}$$

where R_K denotes a $K \times K$ unitary matrix. Then by applying Givens notations, we will turn it into a solution to our problem.

For step I, let

$$t_1 = \arg(a_1 + i\sqrt{1 - a_1^2})$$

and let

$$O^1 = \theta(t_1, 1, M)O$$

We will end up with a new unitary matrix O^1 with the values of the squared norms of the partial rows as described by the diagram:

$$\begin{pmatrix} a_1^2 & | & 1-a_1^2 \\ 1 & | & 0 \\ 1 & | & 0 \\ \vdots & | & \vdots \\ 1 & | & 0 \\ --- & | & --- \\ 0 & | & 1 \\ \vdots & | & \vdots \\ 0 & | & 1 \\ 1-a_1^2 & | & a_1^2 \end{pmatrix}$$

Now one of the following applies:

(CASE I) $1 \leq a_1^2 + a_2^2$, or

(CASE II) $1 > a_1^2 + a_2^2$

In CASE I, take t_2 a solution of

$$\sin(t_2)^2 = \frac{1-a_2^2}{a_1^2} \text{ and } O^2 = \theta(t_2, 2, M)O^1.$$

Then O^2 is also a unitary matrix with the values of the squared norms of the partial rows as described by the diagram:

$$\begin{pmatrix} a_1^2 & | & 1-a_1^2 \\ a_2^2 & | & 1-a_2^2 \\ 1 & | & 0 \\ \vdots & | & \vdots \\ 1 & | & 0 \\ ----- & | & ----- \\ 0 & | & 1 \\ \vdots & | & \vdots \\ 0 & | & 1 \\ 2-a_1^2-a_2^2 & | & a_1^2+a_2^2-1 \end{pmatrix}$$

In CASE II, take t_2 a solution of

$$\sin(t_2)^2 = \frac{a_2^2}{1-a_1^2} \text{ and let } O^2 = \theta(t_2, M-1, M)O^1.$$

Then O^2 is a unitary matrix with the values of the squared norms of the partial rows as described by the diagram:

$$\left(\begin{array}{c|c}
a_1^2 & 1-a_1^2 \\
1 & 0 \\
1 & 0 \\
\cdots & \cdots \\
1 & 0 \\
-\,-\,-\,-\,- & -\,-\,-\,-\,- \\
0 & 1 \\
\cdots & \cdots \\
0 & 1 \\
a_2^2 & 1-a_2^2 \\
1-a_1^2-a_2^2 & a_1^2+a_2^2
\end{array}\right)$$

Now it remains to prove that CASE I eventually takes place whenever $k < N-1$ and when $k = N-1$ only CASE II can apply. We refer the reader to [**CL**] for the details of this argument. $\square$

The algorithm above actually proves more than we stated. A sequence $\{a_1 \geq a_2 \geq \cdots \geq a_M > 0\}$ is called *admissible* for N if $a_m \leq 1$, for $1 \leq m \leq M$ and $\sum_{m=1}^{M} a_m^2 = N$. Also, let $O(K)$ denote the family of $K \times K$ unitary matrices. We now have:

Theorem 4.8. *Fix $N < M$ and $\{a_m\}_{m=1}^M$ an admissible sequence. Then*

1. *There is a tight frame $\{\phi_m\}_{m=1}^M$ for H_N with $\|\phi_m\| = a_m$, for all $1 \leq m \leq M$.*
2. *Moreover, there is an embedding from $O(N) \times O(M-N)$ into the space of all solutions for $\{a_j\}_{j=1}^M$. The embedding will take the pair of matrices $R_N \in O(N)$ and $R_{M-N} \in O(M-M)$ into*

$$\Psi \begin{pmatrix} R_N & 0 \\ 0 & R_{M-N,M-N} \end{pmatrix}$$

where the matrix Ψ is unitary and is determined by the admissible *sequence $\{a_j\}_{j=1}^M$.*

It must be pointed out that the embedding of $O(N) \times O(M-M)$ into $O(M)$ does not fill out the space of solutions. Using alternative algorithms it is possible to construct solutions which do no belong to the image of the embedding. That is, it can not be factorized as

$$\psi O = \psi \begin{pmatrix} R_N & 0 \\ 0 & R_{M-N} \end{pmatrix}$$

So we still have the problem:

Problem 4.9. Find an algorithm which constructs all tight frames of the form $\|\phi_m\| = a_m$, $1 \leq m \leq M$, for H_N.

4.3. Frames with a given frame operator. Let S be a positive self-adjoint invertible operator on H_N. We want to find necessary and sufficient conditions on $a_1 \geq a_2 \geq \cdots \geq a_M > 0$ so that there is a frame $\{\phi_m\}_{m=1}^M$ for H_N with frame operator S and satisfying $\|\phi_m\| = a_m$, for all $1 \leq m \leq M$. Again, what we need is an algorithm for constructing such frames. Such an algorithm was given by Casazza and Leon [**CL**] along with a MatLab program for implementing it [**CL2**].

THEOREM 4.10. *Let S be a positive self-adjoint operator on a N-dimensional Hilbert space H_N. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ be the eigenvalues of S. Fix $M \ge N$ and real numbers $a_1 \ge a_2 \ge \cdots a_M > 0$. The following are equivalent:*

1. *There is a frame $\{\varphi_m\}_{m=1}^M$ for H_N with frame operator S and $\|\varphi_m\| = a_m$, for all $m = 1, 2, \ldots, M$.*
2. *For every $1 \le k \le N$,*

$$\sum_{m=1}^{k} a_i^2 \le \sum_{i=1}^{k} \lambda_i, \quad \text{and} \quad \sum_{m=1}^{M} a_m^2 = \sum_{i=1}^{N} \lambda_i.$$

The construction of these frames is a variation of the algorithm given in Section 4.2 [**CL**].

In Section 4.4 we will see a class of equal norm Parseval frames called the *harmonic frames*. As a consequence of Theorem 4.10 above, we conclude that there are equal norm frames for any frame operator.

COROLLARY 4.11. *If S is a positive self-adjoint invertible operator on H_N, then for every $M \ge N$ there is an equal norm frame $\{\varphi_m\}_{m=1}^M$ for H_N whose frame operator equals S.*

PROOF. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ be the eigenvalues of S and let $\lambda = \sum_{i=1}^N \lambda_i$. Fix $M \ge N$. If $a_m = \sqrt{\lambda/M}$ for every m then $\sum_{m=1}^M a_m^2 = \lambda$, and for all $1 \le k \le N$,

$$\sum_{m=1}^{k} a_n^2 = \frac{k}{M}\lambda = \frac{k}{M}\sum_{i=1}^{N}\lambda_i = k\frac{N}{M}\frac{1}{N}\sum_{i=1}^{N}\lambda_i \le k\frac{N}{M}\frac{1}{k}\sum_{i=1}^{k}\lambda_i \le \sum_{i=1}^{k}\lambda_i.$$

In the next to last inequality above we have used the fact that deleting some of the smallest numbers from a set of numbers will increase the average of the numbers. Hence, by Theorem 4.10 there is a frame $\{\varphi_m\}_{m=1}^M$ with $\|\varphi_m\| = \sqrt{\lambda/M}$ for all $n = 1, 2, \ldots, M$ having frame operator S. ☐

However, as in Section 4.2, this algorithm does not construct all frames (or even all equal norm frames) having frame operator S. It would be invaluable to have such a construction.

PROBLEM 4.12. For a given positive, self-adjoint invertible operator S on H_N, and $a_1 \ge a_2 \ge \cdots \ge a_m > 0$ which satisfies Theorem 4.10, find an algorithm for constructing all frames $\{\phi_m\}_{m=1}^M$ for H_N with frame operator S and satisfying $\|\phi_m\| = a_m$, for all $1 \le m \le M$.

4.4. Grassmannian frames.

An area which seriously needs algorithmic constructions is that of Grassmannian frames introduced by Heath and Strohmer [**HS**].

DEFINITION 4.13. For a given unit norm frame $\{\phi_m\}_{m=1}^M$ in H_N we define the **maximal frame correlation** $\mathcal{M}(\{\phi_m\}_{m=1}^M)$ by

$$\mathcal{M}(\{\phi_m\}_{m=1}^M) = \max_{1\le m\ne k\le M}\{|\langle \phi_m, \phi_k\rangle|\}.$$

The assumption that our frames are unit norm is for convenience above since we could normalize the frame elements in the inner product above instead.

DEFINITION 4.14. A family $\{\phi_m\}_{m=1}^M$ in H_N is called a **Grassmannian frame** if it is a solution to

$$\min\{\mathcal{M}(\{\phi_m\}_{m=1}^M)\}$$

where the min is taken over all unit norm frames $\{\phi_m\}_{m=1}^M$ in H_N.

So a Grassmannian frame minimizes the maximum correlation between frame elements. Grassmannian frames are the best frames for many applications. Unfortunately, very few are known. Even the very useful *harmonic frames* are rarely Grassmannian. The elements of a harmonic tight frame $\{\phi_m\}_{m=1}^M$ for $\mathbb{C}^N$ are given by (see [**CK**]):

$$\phi_m = \frac{1}{\sqrt{N}}(w_1^m, w_2^m, \ldots, w_N^m)_{k=1}^N$$

where the w_k are M-th roots of unity. It is known that harmonic tight frames are always Grassmannian frames when $M = N$ or $N+1$. H. König [**K**] has given examples of harmonic tight frames in his work with spherical designs.

For p a prime number, set $k = p^\ell + 1$ for $\ell \in \mathbb{N}$ and let $K = n^2 - n + 1$. Then there exist integers $0 \le d_1 < \cdots < d_k < M$ such that all numbers $1, 2, \ldots, M-1$ occur as residues $\mod M$ of the $n(n-1)$ differences $d_i - d_j$, $i \ne j$. For $m = 1, 2, \ldots, M$ we define

$$\phi_m = \left\{ \frac{1}{\sqrt{n}} e^{2\pi i m d_j / N} \right\}_{j=1}^k .$$

Then $\{\phi_m\}_{m=1}^M$ is a harmonic tight Grassmannian frame with $\mathcal{M}(\{\phi_m\}_{m=1}^M) = \frac{\sqrt{n-1}}{n}$.

Definition 4.15. A unit norm frame $\{\phi_m\}_{m=1}^M$ for H_N is **equiangular** if there is a $c > 0$ so that

$$|\langle \phi_n, \phi_m \rangle| = c \text{ for all } n \ne m.$$

There are bounds for the maximal frame correlation function and the maximal number of elements in a Grassmannian frame [**HS**].

Theorem 4.16. *Let $\{\phi_m\}_{m=1}^M$ be a unit norm frame for H_N. then*

$$\mathcal{M}(\{\phi_m\}_{m=1}^M) \ge \sqrt{\frac{M-N}{N(M-1)}}, \tag{4.1}$$

with equality if and only if $\{\phi_m\}_{m=1}^M$ is an equiangular tight frame.

Furthermore,

1. *If $H_N = \mathbb{R}^N$, to have equality in (4.1) it is necessary that $M \le \frac{N(N+1)}{2}$.*
2. *If $H_N = \mathbb{C}^N$, to have equality in (4.1) it is necessary that $M \le N^2$.*

We call unit norm fames which produce equality in (4.1) *optimal Grassmannian frames.*

There are few known examples of optimal Grassmannian frames. But they arise in a variety of areas such as Grassmannian spaces, spherical codes, graph theory and the study of equiangular line sets [**HS**]. It would be a major advance for all these subjects and more if we could find an algorithmic method for constructing Grassmannian frames. At this time we have no general methods for producing Grassmannian frames—with or without algorithms. Moreover, frames giving the optimal upper bound number of elements are known (or more accurately, are approximately given by computer programs) for $N \le 15$.

The bound given in Theorem 4.16 can be generalized to non-unit norm frames. This was observed by Waldron [**W**] in relation to Welsh bounds in signal processing.

Theorem 4.17. *If* $\{\phi_m\}_{m=1}^M$ *is a frame for* H_N *then*

$$\max_{1\le m\ne k\le M} |\langle \phi_m, \phi_k\rangle|^2 \ge \frac{\left(\sum_{m=1}^M \|\phi_m\|^2\right)^2 N - \sum_{m=1}^M \|\phi_m\|^4}{M(M-1)},$$

with equality if and only if $\{\phi_m\}_{m=1}^M$ *is an equiangular tight frame.*

5. Concluding remarks

Some of the results in this paper have been extended in [**CF2**] to infinite frames. First for infinite frames on finite dimensional Hilbert spaces.

Theorem 5.1. *Given* $a_1 \ge a_2 \ge \cdots \ge 0$ *and a Hilbert space* $\mathbb{H}_N$*, the following are equivalent:*

1. *There is a normalized tight frame* $\{\phi_m\}_{m=1}^\infty$ *for* $\mathbb{H}_N$ *with* $\|\varphi_m\| = a_m$*, for all* $m = 1, 2, 3, \dots$.
2. *We have that* $\sum_{m=1}^\infty a_m^2 < \infty$ *and*

$$\sum_{m=1}^\infty a_m^2 \ge N a_1^2.$$

Moreover, if (2) fails and $1 \le N_0 < N$ *is the largest number for which (2) fails, then the frame* $\{\psi_m\}_{m=1}^\infty$ *which is closest to being tight (in the sense of minimizing potential energy) with* $\|\varphi_m\| = a_m$ *is:*

$$\{c_m a_m e_m\}_{m=1}^{N_0} \cup \{\varphi_m\}_{m=N_0+1}^\infty$$

where $\{e_i\}_{i=1}^N$ *is an orthonormal basis for* $\mathbb{H}_N$*,* $|c_i| = 1$*, for all* $1 \le i \le N_0$*, and* $\{\varphi_m\}_{m=N_0+1}^\infty$ *is a tight frame for span* $\{e_i\}_{i=N_0+1}^N$.

Also in [**CF2**], we have the corresponding result for infinite dimensional Hilbert spaces.

Theorem 5.2. *Let* $\mathbb{H}$ *be an infinite dimensional Hilbert space and* $a_m > 0$ *for all* $m = 1, 2, \dots$. *The following are equivalent:*

1. *There is a frame* $\{\psi_m\}_{m=1}^\infty$ *for* $\mathbb{H}$ *with* $\|\psi_m\| = a_m$*, for all* $m = 1, 2, \dots$.
2. *There is a* λ*-tight frame* $\{\varphi_m\}_{m=1}^\infty$ *for* H *with* $\|\varphi_m\| = a_m$*, for all* $m = 1, 2, \dots$ *and* $\lambda = \sup_m a_m^2$.
3. *The sequence* $\{a_m\}_{m=1}^\infty$ *is bounded and* $\sum_{m=1}^\infty a_m^2 = \infty$.

Moreover, λ *is the smallest real number satisfying (2). That is if* $\{\psi_m\}_{m=1}^\infty$ *is any* λ_1*-tight frame for* $\mathbb{H}$ *with* $\|\psi_m\| = a_m$*, for all* $m = 1, 2, \dots$*, then* $\lambda_1 \ge \lambda$.

Finally, let us point out that the proofs in sections 2 and 3 can be done in much more generality showing that we really can *custom build* tight frames for most applications.

Definition 5.3. If $\{\varphi_m\}_{m=1}^M$ is a frame for $\mathbb{H}_N$ with frame operator S and each φ_m is an eigenvector for S, we call $\{\varphi_m\}_{m=1}^M$ an **FF-critical sequence**.

Definition 5.4. Let $\mathcal{F}$ be a family of frames and $\{\varphi_m\}_{m=1}^M \in \mathcal{F}$ have frame operator S.

1. The family $\mathcal{F}$ **locally allows small perturbations** of $\{\varphi_m\}_{m=1}^M$ if for every $1 \le k \le M$ there is an $\epsilon > 0$ so that whenever $\psi_m = \varphi_m$ for all $m \ne k$, $\|\psi_k\| = \|\varphi_k\|$ and $\|\psi_k - \varphi_k\| < \epsilon$, then $\{\psi_m\}_{m=1}^M \in \mathcal{F}$.

2. If $\{\varphi_m\}_{m=1}^M \in \mathcal{F}$ is an FF-critical sequence, we say the family $\mathcal{F}$ **allows eigenspace perturbations** of $\{\varphi_m\}_{m=1}^M$ if given any set $I \subset \{1, 2, \ldots, M\}$ for which $\{\varphi_m\}_{m\in I}$ are in one eigenspace of S, there exists an $\epsilon > 0$ satisfying: Whenever $0 < \epsilon_m < \epsilon$, $\psi_m = \varphi_m$ for all $m \notin I$ and for $m \in I$, $\|\psi_m\| = \|\varphi_m\|$ and $\|\psi_m - \varphi_m\| < \epsilon_m$, then $\{\psi_m\}_{m=1}^M \in \mathcal{F}$.

Now, the proof of Proposition 2.8 yields:

THEOREM 5.5. *Let $\mathcal{F}$ be a family of frames for $\mathbb{H}_N$. Let $\{\varphi_m\}_{m=1}^M \in \mathcal{F}$ have frame operator S. If $\{\varphi_m\}_{m=1}^M$ is a local minimizer of the frame potential on $\mathcal{F}$ and $\mathcal{F}$ allows small perturbations of $\{\varphi_m\}_{m=1}^M$, then $\{\varphi_m\}_{m=1}^M$ is an FF-critical sequence.*

In general, if a family $\mathcal{F}$ allows small perturbations of its elements, it still may not contain a tight frame. For example, if $\mathcal{F}$ is the family of two element frames for $\mathbb{R}^2$, $\{\varphi_m\}_{m=1}^2$ with $\|\varphi_1\| = 1$ and $\|\varphi_2\| = 2$, then by Theorem 3.6 this family cannot contain any tight frames.

Now, the proof of Theorem 3.6 yields:

THEOREM 5.6. *Let $\mathcal{F}$ be a family of M-element frames for $\mathbb{H}_N$. Let $\{\varphi_m\}_{m=1}^M \in \mathcal{F}$ be an FF-critical sequence which is a minimizer of the frame potential on $\mathcal{F}$ and assume that $\mathcal{F}$ allows eigenspace perturbations of $\{\varphi_m\}_{m=1}^M$. Without loss of generality, assume $\|\varphi_1\| \geq \|\varphi_2\| \geq \cdots \geq \|\varphi_M\|$. Then there exists a natural number $0 \leq N_0 < N$ so that $\{\varphi_m\}_{m=1}^M$ has the form:*

$$\{\varphi_m\}_{m=1}^M = \{c_m\|\varphi_m\|e_m\}_{m=1}^{N_0} \cup \{\varphi_m\}_{m=N_0+1}^M\},$$

where $\{e_i\}_{i=1}^N$ is an orthonormal basis for $\mathbb{H}_N$, $|c_m| = 1$ for $1 \leq m \leq N_0$ and $\{\varphi_m\}_{m=N_0+1}^M$ is a λ-tight frame for span $\{e_i\}_{i=N_0+1}^N$.

Finally, if $\{\|\varphi_m\|\}_{m=1}^M$ satisfies the fundamental inequality then $N_0 = 0$ and so $\{\varphi_m\}_{m=1}^M$ is a tight frame.

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