EVERY HILBERT SPACE FRAME HAS A NAIMARK COMPLEMENT

PETER G. CASAZZA, MATT FICKUS, DUSTIN MIXON, JESS PETERSON AND IHAR SMALYANAU

ABSTRACT. Naimark complements for Hilbert space Parseval frames are one of the most fundamental and useful results in the field of frame theory. We will show that actually all Hilbert space frames have Naimark complements which possess all the usual properties for Naimark complements with one notable exception. So these complements can be used for equiangular frames, RIP property, fusion frames etc. Along the way, we will correct a mistake in a recent fusion frame paper where chordal distances for Naimark complements are computed incorrectly.

1. INTRODUCTION

Naimark complements for Hilbert space Parseval frames are one of the most fundamental and useful results in the field (See, e.g. [4]).

Naimark's Theorem 1.1. *A family of vectors* $\{f_i\}_{i=1}^M$ *is a Parseval frame for* $\mathcal{H}_N$ *if and only if there is a Hilbert space* $\mathcal{H}_N \subset \mathcal{K}_M$ *with an orthonormal basis* $\{e_i\}_{i=1}^M$ *so that the orthogonal projection* $P : \mathcal{K}_M \to \mathcal{H}_N$ *satisfies:* $Pe_i = f_i$, *for all* $i = 1, 2, \cdots, M$.

In this case, $\{(I-P)e_i\}_{i=1}^M$ *is a Parseval frame for a* $(M-N)$*-dimensional Hilbert space called the* **Naimark complement** *of* $\{f_i\}_{i=1}^M$.

It is known that most standard properties of the frame carry over to the Naimark complement including: 1. Equal norm; 2. Equiangular; 3. RIP property; 4. Orthogonality; and more. This theorem is one of the most used theorems in frame theory.

In this paper we will show that all frames actually have a natural Naimark complement which also carries all of the basic properties of the frame to the complement, with one notable exception. That is, the lower frame bound of the Naimark complement may be quite different from the lower frame bound of the frame. However, we calculate this lower frame bound exactly in terms of the eigenvalues of the frame operator of the original frame. These complements preserve equiangularity, equal norm, RIP property and fusion frame properties. Recently, an incorrect calculation was made [1] in computing the chordal distance between subspaces of a fusion frame and those of the Naimark complements. We will actually give a much more general calculation of the *principal angles* between the subspaces and compare each of them to the *principal angles* of the Naimark complement subspaces.

We start with the definition of a Hilbert space frame.

Definition 1.2. *A family of vectors* $\{f_i\}_{i=1}^M$ *is called a frame for* $\mathcal{H}_N$ *if there are constants* $0 < A \leq B < \infty$ *satisfying*

$$A\|f\|^2 \leq \sum_{i=1}^{M} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \text{ for all } f \in \mathcal{H}_N. \tag{1}$$

The numbers A, B are called *lower* (respectively, *upper*) *frame bounds* of the frame. If we only require the upper frame bound, we call this a *B-Bessel sequence*. If $A = B$ we call this an *A-tight frame* and if $A = B = 1$, this is a *Parseval frame*. When we give the frame bounds A, B for a frame, we will assume they are the *optimal values*. That is, A is the largest number satisfying Inequality (1) and B will be the smallest number satisfying the inequality. The *analysis operator* of the frame is the operator $T : \mathcal{H}_N \to \ell_2(M)$ given by $T(f) = (\langle f, f_i\rangle)_{i=1}^M$. The *synthesis operator* is $T^* : \ell_2(M) \to \mathcal{H}_N$ given by $T(e_i) = f_i$, where $\{e_i\}_{i=1}^M$ is the natural orthonormal basis for $\ell_2(M)$. The frame operator of the frame is give by $S = T^*T : \mathcal{H}_N \to \mathcal{H}_N$. That is,

$$S(f) = \sum_{i=1}^{M} \langle f, f_i\rangle f_i, \text{ for all } f \in \mathcal{H}_N.$$

The frame operator is a positive, self-adjoint invertible operator on $\mathcal{H}_N$.

From the matrix point of view, the synthesis operator is the matrix

$$\mathbf{F}^* = \begin{pmatrix} | & | & \cdots & | \\ f_1 & f_2 & \cdots & f_M \\ | & | & \cdots & | \end{pmatrix}$$

From the matrix completion point of view, an $M \times N$ Parseval matrix F^* can be extended by Naimark's theorem to an $M \times M$ unitary matrix by appending $(M - N)$-rows to F^* to obtain

$$\mathbf{A} = \begin{pmatrix} | & | & \cdots & | \\ f_1 & f_2 & \cdots & f_M \\ | & | & \cdots & | \\ g_1 & g_2 & \cdots & g_M \\ | & | & \cdots & | \end{pmatrix}$$

The

2. The Construction of General Naimark Complements

In this section we will give our construction for Naimark complements for arbitrary Hilbert space frames. But first, let us recall for later reference the proof of Naimark's theorem.

Proof of Naimark's Theorem: Given a Parseval frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$, the analysis operator $T:\mathcal{H}_N \to \ell_2(M)$ is the co-isometry

$$Tf = (\langle f, f_1\rangle, \langle f, f_2\rangle, \cdots, \langle f, f_M\rangle).$$

If P is the orthogonal projection of $\ell_2(M)$ onto $T(\mathcal{H}_N)$ then for any Tf we have:

$$\begin{aligned}\langle Tf, Pe_i\rangle &= \langle Tf, e_i\rangle \\ &= \langle f, T^* e_i\rangle \\ &= \langle f, f_i\rangle \\ &= \langle Tf, Tf_i\rangle.\end{aligned}$$

It follows that $Pe_i = Tf_i$, which completes the proof.

We will now give the construction for general Naimark complements for arbitrary frames. This involves a simple technique for using the fewest number of elements to turn a B-Bessel sequence into a B-tight frame. For this we need to recall a standard result from frame theory [3].

Theorem 2.1. *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, $\{e_i\}_{i=1}^N$ is an orthonormal basis for $\mathcal{H}_N$ and $\{\lambda_j\}_{j=1}^N$ non-negative real numbers. The following are equivalent:*

(1) *The $\{e_i\}_{i=1}^N$ are the eigenvectors of the frame operator for $\{f_i\}_{i=1}^M$ with respective eigenvalues $\{\lambda_i\}_{i=1}^N$*

(2) *The matrix*

$$\begin{pmatrix} \langle f_1, e_1\rangle & \langle f_2, e_1\rangle & \cdots & \langle f_M, e_1\rangle \\ \langle f_1, e_2\rangle & \langle f_2, e_2\rangle & \cdots & \langle f_M, e_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle f_1, e_N\rangle & \langle f_2, e_N\rangle & \cdots & \langle f_M, e_N\rangle \end{pmatrix}$$

has orthogonal rows and the sum of the squared elements of the j^{th} row equals λ_j.

Now we are ready for the construction we will use for Naimark complements.

Theorem 2.2. *If $\{f_i\}_{i=1}^M$ is a B-Bessel sequence in $\mathcal{H}_N$ and its frame operator has eigenvectors $\{e_j\}_{j=1}^N$ with respective eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_K > \lambda_{K+1} \geq \cdots \geq \lambda_N \geq 0$, then there exist vectors $\{h_i\}_{i=M+1}^{M+N-K}$ so that*

$\{f_i\}_{i=1}^M \bigcup \{h_i\}_{i=M+1}^{M+N-K}$ *is a* $\lambda_1 = B$*-Tight frame with eigenvectors* $\{e_j\}_{j=1}^N$ *and* $K \leq N-1$

Proof. Construct a matrix as follows:

$$\begin{pmatrix} \langle f_1, e_1\rangle & \langle f_2, e_1\rangle & \cdots & \langle f_M, e_1\rangle & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \langle f_1, e_K\rangle & \langle f_2, e_K\rangle & \cdots & \langle f_M, e_K\rangle & 0 & 0 & \cdots & 0 \\ \langle f_1, e_{K+1}\rangle & \langle f_2, e_{K+1}\rangle & \cdots & \langle f_M, e_{K+1}\rangle & \sqrt{\lambda_1 - \lambda_{K+1}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle f_1, e_N\rangle & \langle f_2, e_N\rangle & \cdots & \langle f_M, e_N\rangle & 0 & 0 & \cdots & \sqrt{\lambda_1 - \lambda_N} \end{pmatrix}$$

And the result follows from the fact that the rows of the constructed matrix are orthogonal and the sum of the squared row elements is equal to λ_1. And in this case,

$$h_{M+j} = \sqrt{\lambda_1 - \lambda_{K+j}}\ e_{K+j}, \text{ for } j = 1, 2, \cdots, N-K.$$

□

Remark 2.3. (1) *To obtain a tight frame in* $\mathcal{H}_N$ *we have to add a maximum of* $N-1$ *vectors to the frame.*
(2) *We can construct a tight frame from the initial frame with the new frame operator having the same eigenvectors.*

Definition 2.4. *Let* $\{f_i\}_{i=1}^M$ *be a* B*-Bessel sequence in* $\mathcal{H}$ *with a frame operator* S *having eigenvectors* $\{\phi_j\}_{j=1}^N$ *and respective eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. *Choose* K *so that* $B = \lambda_1 = \lambda_2 = \cdots = \lambda_K > \lambda_{K+1} \geq \cdots \geq \lambda_N$. *We add* $\{h_j\}_{j=K+1}^N$ *to the Bessel sequence to make it a* $\lambda_1 = B$*-tight, where* $h_j = \sqrt{B - \lambda_j}\ \phi_j$. *So* $\{\frac{1}{\sqrt{B}} f_i\}_{i=1}^M \cup \{\frac{1}{\sqrt{B}} h_j\}_{j=K+1}^N$ *is a Parseval frame.*
Now, there is a projection $P : \ell_2(M+N-K) \to \mathcal{H}$ *with*

$$Pe_i = \frac{1}{\sqrt{B}} f_i \qquad 1 \leq i \leq M$$

and

$$Pe_i = \frac{1}{\sqrt{B}} h_{K+l} \qquad i = M+l \qquad 1 \leq l \leq N-K,$$

$\{e_i\}_{i=1}^{M+N-K}$ *the unit vector basis for* $\ell_2(M+N-K)$. *So* $\{f_i \oplus \sqrt{B}(I-P)e_i\}_{i=1}^M$ *is an orthogonal set with*

$$\|f_i \oplus \sqrt{B}(I-P)e_i\|^2 = B, \text{ for all } i = 1, 2, \cdots, M.$$

We call $\{g_i\}_{i=1}^M =: \{\sqrt{B}(I-P)e_i\}_{i=1}^M$ *the* **Naimark Complement** *of* $\{f_i\}_{i=1}^M$

The next theorem formalizes the new Naimark complements.

Theorem 2.5. *If $\{f_i\}_{i=1}^M$ is a Bessel sequence in $\mathcal{H}_N$ with Bessel bound B, the Naimark Complement $\{g_i\}_{i=1}^M$ satisfies:*

(1) *$\{f_i \oplus g_i\}_{i=1}^M$ is orthogonal with $\|f_i \oplus g_i\|^2 = B$*

(2) *We have:*

$$span\{g_i\}_{i=1}^M = \mathcal{H}_{M-K}$$

(3) *Moreover, we have* **uniqueness** *in the sense that if $\{\psi_i\}_{i=1}^M$ satisfies*

$$\{f_i \oplus \psi_i\}_{i=1}^M, \quad \text{is orthogonal, and } \|f_i \oplus \psi_i\|^2 = B,$$

then $\{\psi_i\}_{i=1}^M$ is unitarily equivalent to $\{g_i\}_{i=1}^M$.

Proof. (1) is obvious from our construction.

(2) Given

$$F = \begin{pmatrix} | & | & \cdots & | \\ f_1 & f_2 & \cdots & f_M \\ | & | & \cdots & | \end{pmatrix}$$

Let $F' = \begin{pmatrix} F & F_1 \end{pmatrix}$ be our B-tight frame. Choose the Naimark Complement where

$$H = \begin{pmatrix} F & F_1 \\ G & G_1 \end{pmatrix}$$

Hence

$$I_{M+N-K} = \frac{1}{B} H^* H = \frac{1}{B} \begin{pmatrix} F^* & G^* \\ F_1^* & G_1^* \end{pmatrix} \begin{pmatrix} F & F_1 \\ G & G_1 \end{pmatrix} =$$

$$= \frac{1}{B} \begin{pmatrix} F^*F + G^*G & 0 \\ 0 & F_1^*F_1 + G_1^*G_1 \end{pmatrix}$$

Thus

$$\begin{aligned}
diag(G^*G) &= diag(BI_M - F^*F) \quad (2)\\
&= diag\left[BI_M - \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_N & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}\right] \quad (3)\\
&= diag\begin{pmatrix} B-\lambda_1 & & & & & \\ & \ddots & & & & \\ & & B-\lambda_N & & & \\ & & & B & & \\ & & & & \ddots & \\ & & & & & B \end{pmatrix} \quad (4)\\
&= diag\begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & B-\lambda_{K+1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & B-\lambda_N & & & \\ & & & & & & B & & \\ & & & & & & & \ddots & \\ & & & & & & & & B \end{pmatrix} \quad (5)
\end{aligned}$$

So we have that $rank\ G^*G = M - K$, hence $span\{g_i\}_{i=1}^M = \mathcal{H}_{M-K}$.

(3) The families $\{f_i \oplus g_i\}_{i=1}^M$ and $\{f_i \oplus \psi_i\}_{i=1}^M$ are orthogonal bases for their spans with the squared norms of their vectors all equal to B. Hence, for all scalars $\{a_i\}_{i=1}^M$ we have

$$B\sum_{i=1}^M |a_i|^2 = \| \sum_{i=1}^M a_i f_i\|^2 + \| \sum_{i=1}^M a_i g_i\|^2$$

and

$$B\sum_{i=1}^M |a_i|^2 = \| \sum_{i=1}^M a_i f_i\|^2 + \| \sum_{i=1}^M a_i \psi_i\|^2.$$

Hence, for all scalars $\{a_i\}_{i=1}^M$ we have

$$\|\sum_{i=1}^M a_i g_i\|^2 = \|\sum_{i=1}^M a_i \psi_i\|^2$$

□

Remark 2.6. *The main thing to be careful about here is that* $\{g_i\}_{i=1}^M$ *does not span the orthogonal complement of the span of our Parseval frame* $\{f_i\}_{i=1}^M \cup \{h_i\}_{i=K+1}^N$ *in general.*

Theorem 2.7. *If* $\{f_i\}_{i=1}^M$ *is a frame for* $\mathcal{H}_N$ *with frame bounds A,B, then the Naimark Complement is a frame with lower frame bound* $B - \lambda_{K+1}$ *and upper frame bound B, if* $M > N$*, and upper frame bound* $B - \lambda_N$ *if* $M = N$*.*

Proof. Immediate from Equations (1) - (4) above. □

Remark 2.8. *The previous theorem shows that in general the Naimark complement of a frame may not have a lower frame bound which is comparable to the lower frame bound of the original frame. However, we do know exactly what this lower frame bound is. Also, a simple adjustment of the construction will give complements which have comparable frame bounds. Namely, add N vectors to the frame producing a tight frame with tight frame bound* $C > B$ *and then the lower frame bound of the complement is* $C - B$*. Finally, if our original frame is a B-tight frame, then our Naimark complement is exactly equal to the usual Naimark complement.*

It is important for the uniqueness of our Naimark complements that we use the specific minimal method for turning a frame into a tight frame so we can get general Naimark complements. Otherwise, even the dimension of the span of the Naimark complements are not unique as the next example shows.

Example 2.9. *Let*

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

be a frame matrix. Then

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$F_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & \sqrt{2} \\ 1 & 0 \end{pmatrix},$$

are Parseval frames. Grammians of these Parseval frames are

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$G_2 = \begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{\sqrt{2}}{3} \\ 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ 0 & \frac{\sqrt{2}}{3} & \frac{2}{3} & 0 \\ \frac{\sqrt{2}}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix},$$

respectfully. Now,

$$I - G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$I - G_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{\sqrt{2}}{3} \\ 0 & \frac{2}{3} & -\frac{\sqrt{2}}{3} & 0 \\ 0 & -\frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ -\frac{\sqrt{2}}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix}.$$

Note that the two complements of $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ *are not unitary equivalent. In fact, one has dimension one and the other has dimension two.*

3. Properties of the Naimark Complement

In this section we will check the basic properties of the new Naimark complements.

Proposition 3.1. *Given a B-Bessel sequence* $\{f_i\}_{i=1}^M$ *for* $\mathcal{H}_N$ *with Naimark Complement* $\{g_i\}_{i=1}^M$*, the following properties hold:*

1. *We have*

$$\langle g_i, g_j \rangle = -\langle f_i, f_j \rangle, \text{ for all } 1 \le i \ne j \le M.$$

   In particular, if $\{f_i\}_{i=1}^M$ *is an equiangular frame, so is* $\{g_i\}_{i=1}^M$*.*
2. *If* $\{f_i\}_{i=1}^M$ *is equal norm, then so is* $\{g_i\}_{i=1}^M$*.*
3. *If* $J \subset \{1, 2, \cdots, M\}$ *and* $\{f_i\}_{i\in J}$ *is an orthogonal set, then so is* $\{g_i\}_{i\in J}$*.*

Proof. (1) Since $\{f_i \oplus g_i\}_{i=1}^M$ is an orthogonal set, for all $1 \le i \ne j \le M$ we have:

$$0 = \langle f_i \oplus g_i, f_j \oplus g_j \rangle = \langle f_i, f_j \rangle + \langle g_i, g_j \rangle.$$

(2) Set $\|f_i\|= c$, for all $i = 1, 2, \cdots, M$. Now

$$\begin{aligned}\|e_i\|^2 &= 1\\ &= \frac{1}{B}(\|f_i\|^2+\|g_i\|^2)\\ &= \frac{1}{B}(c^2+\|g_i\|^2).\end{aligned}$$

Hence,

$$\|g_i\|^2 = B - c^2, \quad \text{for all } i = 1, 2, \cdots, M.$$

(3) This is immediate from (1). □

Although the lower frame bound of the Naimark complement is not controllable in general, we still are able to get optimal bounds for the Restricted Isometry Property.

Definition 3.2. *Restricted Isometry Property(RIP)*
A family of unit norm vectors $\{f_i\}_{i\in I}$ in $\mathcal{H}_N$ has RIP with constants $\epsilon > 0$ and $K < N$ if for every $J \subseteq I$, $|J| \le K$ and all scalars $\{a_i\}_{i\in J}$ we have

$$(1-\epsilon)\sum_{i\in J}|a_i|^2 \le \|\sum_{i\in J}a_if_i\|^2 \le (1+\epsilon)\sum_{i\in J}|a_i|^2$$

Theorem 3.3. *If $\{f_i\}_{i=1}^M$ is RIP with constants $\epsilon > 0$ and $K < N$, with Naimark compement $\{g_i\}_{i=1}^M$, then $\{\frac{1}{\sqrt{B-1}}g_i\}_{i=1}^M$ is RIP with constants:*

$$\left(1-\frac{\epsilon}{B-1}\right) \text{ and } \left(1+\frac{\epsilon}{B-1}\right).$$

Proof. Let $\{g_i\}_{i=1}^M$ be the Naimark complement of $\{f_i\}_{i\in I}$. Now, $\{\frac{1}{\sqrt{B-1}}g_i\}_{i=1}^M$ is unit norm since

$$B = \|f_i \oplus g_i\|^2 = \|f_i\|^2 + \|g_i\|^2 = 1 + \|g_i\|^2.$$

For any $J \subset \{1, 2, \cdots, M\}$ and any scalars $\{a_i\}_{i\in J}$ we have

$$B\sum_{i\in J}|a_i|^2 = \|\sum_{i\in J}a_if_i\|^2 + \|\sum_{i\in J}a_ig_i\|^2.$$

Hence,

$$\begin{aligned}\|\sum_{i\in J}a_ig_i\|^2 &= B\sum_{i\in J}|a_i|^2 - \|\sum_{i\in J}a_if_i\|^2\\ &\ge B\sum_{i\in J}|a_i|^2 - (1+\epsilon)\sum_{i\in J}|a_i|^2\\ &= [(B-1)-\epsilon]\sum_{i\in J}|a_i|^2.\end{aligned}$$

Dividing through this inequality by $B-1$ yields

$$\|\sum_{i\in J} a_i \frac{g_i}{\sqrt{B-1}}\|^2 \geq \left(1-\frac{\epsilon}{B-1}\right)\sum_{i\in J}|a_i|^2.$$

Similarly,

$$\|\sum_{i\in J} a_i \frac{g_i}{\sqrt{B-1}}\|^2 \leq \left(1+\frac{\epsilon}{\sqrt{B-1}}\right)\sum_{i\in J}|a_i|^2.$$

□

4. Fusion Frames

In this section we will examine the new Naimark complements for fusion frames. Fusion frames were introduced in [2] and have quickly turned into an industry (See www.fusionframes.org).

Definition 4.1. *Let $\{W_i\}_{i=1}^K$ be subspaces of $\mathcal{H}_N$ and let $v_i > 0$, $i = 1, 2, \cdots, K$ be positive weights. Then $\{W_i, v_i\}_{i=1}^K$ is a* fusion frame *for $\mathcal{H}_N$ if there are constants $0 < A \leq B < \infty$ so that*

$$A\|f\|^2 \leq \sum_{i=1}^{K} v_i^2 \|P_i f\|^2 \leq B\|f\|, \text{ for all } f \in \mathcal{H}_N,$$

where P_i is the orthogonal projection onto W_i. We call A, B the fusion frame bounds *and if $A = B = 1$, this is a* Parseval fusion frame.

Definition 4.2. *Let $\{W_i, v_i\}_{i=1}^K$ be a fusion frame for $\mathcal{H}_N$. A* Naimark complement *fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^K$ is defined as: Choose orthonormal bases $\{e_{ij}\}_{j=1}^{L_i}$ for W_i and consider the frame $\{v_i e_{ij}\}_{i=1,j=1}^{K,\ L_i}$ for $\mathcal{H}_N$ which has frame bounds equal to the fusion frame bounds. Construct the "new" Naimark complement $\{g_{ij}\}_{i=1,j=1}^{K,\ L_i}$ for this frame. Then for each $i = 1, 2, \cdots, K$, $\{g_{ij}\}_{j=1}^{L_i}$ is an equal norm orthogonal set. Let*

$$W_i' = span\{g_{ij} : j = 1, 2, \cdots, L_i\}.$$

Remark 4.3. *Note that we need to pick an orthonormal basis for W_i in order to get the Naimark complement W_i'. This makes it look like we may have many Naimark complements for any given fusion frame. Actually, all of these complements are unitarily equivalent. That is, if we choose two different orthonormal bases for our W_i, to get the Naimark complement, we will take the two different analysis operators (which are co-isometries) T_1 and T_2 and look at the corresponding fusion frames $\{T_1(W_i), v_i\}_{i=1}^K$ and $\{T_2(W_i), v_i\}_{i=1}^K$ in $\ell_2(M)$. But now, $T_1 T_2^{-1}$ is a unitary operator moving one family of fusion subspaces onto the other, and this property carries over to their complements.*

We also recall a result from [1].

Theorem 4.4. *Let* $\{W_i, v_i\}_{i=1}^K$ *be a Parseval fusion frame for* $\mathcal{H}_N$. *Then the Naimark complements fusion frame* $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^K$ *is also a Parseval fusion frame.*

There are many ways to measure the distance between subspaces of a Hilbert space. The most exact measure comes from the *principal angles.* Here, we find the unit norm vectors in each subspace which are the closest. Then switch to the orthogonal complements of these vectors in their respective subspaces and find the closest unit norm vectors in these orthogonal subspaces, and continue. The formal definition follows. For notation, if W is a subspace of $\mathcal{H}_N$, we write

$$S_W = \{f \in W : \|f\| = 1\}.$$

Definition 4.5. *Given two subspaces* W_1, W_2 *of* $\mathcal{H}_N$ *with dim* $W_1 =: k \leq$ *dim* $W_2 = \ell$, *the* principal angles $(\theta_1, \theta_2, \cdots \theta_k)$ *between the subspaces are defined as:*

$$\theta_1 = min\{arccos\ \langle f, g\rangle : f \in S_{W_1}, g \in S_{W_2}\}.$$

Two vectors f_1, g_1 *are called* principal vectors *if they give the minimum above.*

The other principal angles and vectors are then defined recursively via:

$$\theta_i = min\{arccos\ \langle f, g\rangle : f \in S_{W_1}, g \in S_{W_2},\ and\ f \perp f_j, g \perp g_j, 1 \leq j \leq i-1.\}$$

Now we will check how principal angles are passed to complementary subspaces.

To check the principal angles between our Naimark complements, we need a lemma.

Lemma 4.6. *Let* $\{W_i, v_i\}_{i=1}^K$ *be a Parseval fusion frame for* $\mathcal{H}_N$ *with dim* $W_i = k$ *for all* $i = 1, 2, \cdots, K$, *and let* $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^K$ *be its Naimark complement. Assume these subspaces have been embedded into* $\ell_2(L)$ *(see the proof of Naimark's theorem) with* P *the orthogonal projection of* $\ell_2(L)$ *onto* $\mathcal{H}_N$ *so that* $\{e_{ij}\}_{i=1,j=1}^{k\ \ K}$ *is an orthonormal basis for* $\ell_2(L)$ *and it satisfies:*

1. $W_j = span\ \{Pe_{ij}\}_{i=1}^k$, *for all* $j = 1, 2, \cdots, K$.
2. *For say* $j = 1, 2$, *the vectors* $\{\frac{1}{v_j} Pe_{ij}\}_{i=1}^K$ *are the principle vectors for* W_j *with principle angles* $\{\theta_i\}_{i=1}^k$.

Then, $\{\frac{1}{\sqrt{1-v_i^2}}(I-P)e_{ij}\}_{i=1}^k$ *are the principle vectors for* W_j', $j = 1, 2$, *with principle angles*

$$\left\{ arccos \left[\frac{v_1}{\sqrt{1-v_1^2}} \frac{v_2}{\sqrt{1-v_2^2}} cos(\theta_i) \right] \right\}_{i=1}^k .$$

Proof. We will check the first principal angles and the rest will follow by iteration of the argument. To identify the first principal vectors we need to maximize

$$\{\langle f, g\rangle : f \in S_{W_1'}, g \in S_{W_2'}\}.$$

That is, we need to maximize

$$\left\{\left\langle \sum_{i=1}^{k} \frac{a_i}{\sqrt{1-v_1^2}}(I-P)e_{i1}, \sum_{i=1}^{k} \frac{b_i}{\sqrt{1-v_2^2}}(I-P)e_{i2} \right\rangle\right\}, \tag{1}$$

subject to the constraints

$$\sum_{i=1}^{k} |a_i|^2 = \sum_{i=1}^{k} |b_i|^2 = 1.$$

For Equation 1, we need to maximize

$$\frac{1}{\sqrt{1-v_1^2}} \frac{1}{\sqrt{1-v_2^2}} \sum_{i=1}^{k} a_i \overline{b_i} \langle (I-P)e_{i1}, (I-P)e_{i2} \rangle. \tag{2}$$

But, Equation 2 equals

$$-\frac{1}{\sqrt{1-v_1^2}} \frac{1}{\sqrt{1-v_2^2}} \sum_{i=1}^{k} a_i \overline{b_i} \langle Pe_{i1}, Pe_{i2} \rangle, \tag{3}$$

which in turn equals

$$-\frac{v_1}{\sqrt{1-v_1^2}} \frac{v_2}{\sqrt{1-v_2^2}} \left\langle \sum_{i=1}^{k} \frac{a_i}{v_1} Pe_{i1}, \frac{b_i}{v_2} Pe_{i2} \right\rangle. \tag{4}$$

However, since $\{\frac{1}{v_j} Pe_{ij}\}_{i=1}^{k}$, $j=1,2$, are the principle vectors for W_1, W_2, the max in Equation 3 is

$$\frac{v_1}{\sqrt{1-v_1^2}} \frac{v_2}{\sqrt{1-v_2^2}} cos(\theta_1).$$

We complete the proof by observing that the inner product below yields precisely this value and hence these are the principal vectors:

$$\left\langle \frac{1}{\sqrt{1-v_1^2}}(I-P)e_{11}, \frac{1}{\sqrt{1-v_2^2}}(I-P)e_{12} \right\rangle.$$

□

Remark 4.7. *It is not actually necessary to have the subspaces of equal dimension above since the same proof works in complete generality.*

Corollary 4.8. *Let $\{W_i, v_i\}_{i=1}^{K}$ be a Parseval fusion frame for $\mathcal{H}_N$ with Naimark complement Parseval fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^{K}$. Let $\{\theta_{\ell,ij}\}_{\ell=1}^{k}$ be the* *principal angles for the subspaces W_i, W_j. Then the principal angles for the subspaces W_i', W_j' are:*

$$\left\{ arccos\left[\frac{v_i}{\sqrt{1-v_i^2}} \frac{v_j}{\sqrt{1-v_j^2}} cos(\theta_{\ell,ij}) \right] \right\}_{\ell=1}^{k}.$$

Proof. For a fixed i, j we can use the Naimark embedding of our fusion frame into $\ell_2(L)$ using the principal vectors for W_i, W_j and apply Lemma 4.6. But as we observed, all such embeddings yield the same Naimark complements up to unitary equivalence and hence preserve the principal angles. □

Remark 4.9. *The above corollary also holds for our new Naimark complements. When we take a fusion frame $\{W_i, v_i\}_{i=1}^K$ with fusion frame bounds A, B, we take an orthonormal basis for each subspace weighted by the v_i and this is a frame with frame bounds A, B. Then we add at most $(N-1)$ vectors to make this a B-tight frame. Normalizing by $\frac{1}{\sqrt{B}}$, this becomes a Parseval frame giving a Naimark complement which is a fusion frame since the individual vectors we added are actually weighted one dimensional subspaces. So we get the same principal angles as above except that we need to replace the v_i by $\frac{v_i}{\sqrt{B}}$.*

Another measure of *distance* for subspaces of a Hilbert space is the **chordal distance**. There are many equivalent forms for this distance. We will use the following equivalent forms from [5].

Definition 4.10. *If W_1, W_2 are subspaces of $\mathcal{H}_N$ of dimension k, the* chordal distance *$d_c(W_1, W_2)$ between the subspaces is given by:*

$$d_c^2(W_1, W_2) = k - tr[P_1, P_2] = k - \sum_{\ell=1}^{k} cos^2\theta_{\ell,ij},$$

where P_i is the orthogonal projection onto W_i and $\{\theta_{\ell,ij}\}_{\ell=1}^k$ are the principal angles for W_i, W_j.

Now we are ready to correct Theorem 3.6 of [1] which computes the chordal distances for the Naimark complement of a fusion frame incorrectly.

Theorem 4.11. *Let $\{W_i, v_i\}_{i=1}^K$ be a Parseval fusion frame for $\mathcal{H}_N$. Then there is a complementary Parseval fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^K$ satisfying:*

$$d_c^2(W_i', W_j') = \left[1 - \frac{v_i^2}{1-v_i^2}\frac{v_j^2}{1-v_j^2}\right] k + \left[\frac{v_i^2}{1-v_i^2}\frac{v_j^2}{1-v_j^2}\right] d_c^2(W_i, W_j).$$

Proof. By Corollary 4.8 we have

$$\begin{aligned} d_c^2(W_i', W_j') &= k - \sum_{\ell=1}^{k} \frac{v_i}{1-v_i^2}\frac{v_j}{1-v_j^2} cos^2\theta_{\ell,ij} \\ &= k - \left[\frac{v_i}{1-v_i^2}\frac{v_j}{1-v_j^2}\right]\sum_{\ell=1}^{k} cos^2\theta_{\ell,ij} \\ &= k - \left[\frac{v_i}{1-v_i^2}\frac{v_j}{1-v_j^2}\right]\left[k - d_c^2(W_i, W_j)\right] \\ &= \left[1 - \frac{v_i^2}{1-v_i^2}\frac{v_j^2}{1-v_j^2}\right]k + \left[\frac{v_i^2}{1-v_i^2}\frac{v_j^2}{1-v_j^2}\right] d_c^2(W_i, W_j). \end{aligned}$$

□

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Casazza/Peterson/Smalyanau: Department of Mathematics, University of Missouri, Columbia, MO 65211-4100, USA

Fickus: Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio 45433, USA

Mixon: Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544, USA

E-mail address: Casazzap@missouri.edu; Matthew.Fickus@afit.edu; dmixon@princeton.edu; jdpq6c@mail.missouri.edu; ivsvhd@mizzou.edu