Frame expansions in separable Banach spaces

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Abstract

Banach frames are defined by straightforward generalization of (Hilbert space) frames. We characterize Banach frames (and $X_d$-frames) in separable Banach spaces, and relate them to series expansions in Banach spaces. In particular, our results show that we can not expect Banach frames to share all the nice properties of frames in Hilbert spaces.

1 Introduction

Let $X$ denote a separable Banach space and $\{g_i\}$ be a sequence in the dual $X^\star$. A central question is whether we can find a sequence $\{f_i\}$ in $X$ such that the reconstruction property

$$ f = \sum g_i(f) f_i $$

holds for all $f \in X$.

Banach frames were introduced by Gröchenig as a terminology to express expansions like (1). Banach frames for $X$ are defined with respect to certain sequence spaces $X_d$ (see Definition 1.3). In this paper we characterize pairs of spaces $(X, X_d)$ for which Banach frames (and the related $X_d$-frames) exist. Furthermore we reveal the connections between Banach frames and the reconstruction property. We also prove that if $\{g_i\} \subset X^\star$ is total on $X$, then we can always find a sequence space $X_d$ such that $\{g_i\}$ is a Banach frame for $X$ w.r.t. $X_d$. In particular, this leads to a (somewhat unwanted) example of a Banach frame for a Hilbert space, which is not a Hilbert frame.
Our starting point is the concept of $p$-frames, which was introduced by Aldroubi et al. [1] as a tool to obtain series expansions in shift-invariant spaces. An analysis of $p$-frames in general Banach spaces appeared in [6]. In order to gain more flexibility, we extend the definition to more general sequence spaces in Definition 1.2.

In the rest of this introduction we state the main definitions. Then, in Section 2 we discuss the main results and their implications. Most proofs, and further remarks, are finally collected in Section 3.

**Definition 1.1** A sequence space $X_d$ is called a BK-space, if it is a Banach space and the coordinate functionals are continuous on $X_d$, i.e. the relations $x_n = \{\alpha_j^{(n)}\}, x = \{\alpha_j\} \in X_d$, $\lim_{n \to \infty} x_n = x$ imply $\lim_{n \to \infty} \alpha_j^{(n)} = \alpha_j$ ($j = 1, 2, ...$).

**Definition 1.2** Let $X$ be a Banach space and $X_d$ be a BK-space. A countable family $\{g_i\}_{i \in I}$ in the dual $X^*$ is called an $X_d$-frame for $X$ if

(i) $\{g_i(f)\} \in X_d$, $\forall f \in X$;

(ii) the norms $\|f\|_X$ and $\|\{g_i(f)\}\|_{X_d}$ are equivalent, i.e. there exist constants $A, B > 0$ such that

$A\|f\|_X \leq \|\{g_i(f)\}\|_{X_d} \leq B\|f\|_X$, $\forall f \in X$. (2)

$A$ and $B$ are called $X_d$-frame bounds. If at least (i) and the upper condition in (2) are satisfied, $\{g_i\}$ is called an $X_d$-Bessel sequence for $X$.

If $X$ is a Hilbert space and $X_d = \ell^2$, (2) means that $\{g_i\}$ is a frame, and in this case it is well known that there exists a sequence $\{f_i\}$ in $X$ such that

$$f = \sum \langle f, f_i \rangle g_i = \sum \langle f, g_i \rangle f_i.$$  

Similar reconstruction formulas are not always available in the Banach space setting. This is the reason behind the following definition:

**Definition 1.3** Let $X$ be a Banach space and $X_d$ a sequence space. Given a bounded linear operator $S : X_d \to X$, and an $X_d$-frame $\{g_i\} \subset X^*$, we say that $(\{g_i\}, S)$ is a Banach frame for $X$ with respect to $X_d$ if

$S(\{g_i(f)\}) = f$, $\forall f \in X$. (3)
Note that (3) can be considered as some kind of “generalized reconstruction formula”, in the sense that it tells how to come back to \( f \in X \) based on the coefficients \( \{ g_i(f) \} \). The condition, however, does not imply reconstruction via an infinite series, as we will see later.

The \( X_d \)-frame condition implies that we can define an isomorphism

\[
U : X \to X_d, \quad U f := \{ g_i(f) \} \quad f \in X.
\]

The extra condition in Definition 1.3 means that \( S \) is a left-inverse of \( U \), and thus \( US \) is a bounded linear projection of \( X_d \) onto the range \( R(U) \) of the operator \( U \).

## 2 The main results

In this section we state the most important results. In order not to interrupt the flow, only very short proofs are included here; the more technical proofs are given in Section 3.

We first characterize the Banach spaces \( X \) which have an \( X_d \)-frame w.r.t. a given BK-space \( X_d \):

**Theorem 2.1** Let \( X \) be a Banach space and \( X_d \) a BK-space. Then there exists an \( X_d \)-frame for \( X \) if and only if \( X \) is isomorphic to a subspace of \( X_d \).

**Proof:** From the definition, if \( \{ g_i \} \) is an \( X_d \)-frame for a Banach space \( X \) then the mapping \( U : X \to X_d \) given by \( U(f) = \{ g_i(f) \} \) is an isomorphism of \( X \) into \( X_d \).

For the converse, let \( X \) be a subspace of \( X_d \) and \( \{ f_i \} \) the coordinate functionals (which are assumed to be continuous). Let \( g_i = f_i|_X \). Then for all \( f \in X \), \( \{ g_i(f) \} = f \in X_d \) and \( \| f \|_X = \| \{ g_i(f) \} \|_{X_d} \). □

Given an \( X_d \)-frame \( \{ g_i \} \), where \( X_d \) is a BK-space for which the canonical unit vectors form a basis, the next result clarifies which extra condition we need in order to ensure that \( \{ g_i \} \) is a Banach frame. A more detailed result is given in Proposition 3.4.

**Proposition 2.2** Suppose that \( X_d \) is a BK-space and that \( \{ g_i \} \subset X^* \) is an \( X_d \)-frame for \( X \). If the canonical unit vectors \( \{ e_i \} \) form a basis for \( X_d \), then the following conditions are equivalent:

1. \( R(U) \) is complemented in \( X_d \).
(ii) There exists a linear bounded operator $S$, such that $(\{g_i\}, S)$ is a Banach frame for $X$ with respect to $X_d$.

(iii) There exists an $X^*_d$-Bessel sequence $\{f_i\} \subset X \subseteq X^{**}$ for $X^*$ such that

$$f = \sum g_i(f) f_i, \quad \forall f \in X.$$ 

In case $X_d$ does not have the canonical unit vectors as a basis, the reconstruction property might hold without $R(U)$ being complemented in $X_d$:

Example 2.3 Let $X = c_0, X_d = \ell_\infty$, and $\{g_i\}$ be the canonical unit vector basis of $\ell_1$. Then $\{g_i\}$ is an $X_d$-frame for $c_0$, and by [10], $R(U) = c_0$ is not complemented in $X_d = \ell_\infty$. However, the reconstruction property holds, e.g., via the canonical unit vector basis $\{e_i\}$ of $c_0$.

A reformulation of Proposition 2.2 gives a characterization of spaces $X$ possessing Banach frames:

Theorem 2.4 A Banach space $X$ has a Banach frame with respect to a given sequence space $X_d$ if and only if $X$ is isomorphic to a complemented subspace of $X_d$.

Proof: The result follows from Proposition 3.4, but let us show how a Banach frame can be constructed if we assume that $X$ is isomorphic to a complemented subspace of $X_d$. Let $T: X \rightarrow X_d$ be an isomorphism and let $P: X_d \rightarrow R(T)$ be a projection of $X_d$ onto $R(T)$. Define $S: X_d \rightarrow X$ by $Sx = T^{-1}Px$. Let $\{e_i\}$ be the coordinate functionals of $X_d$ and $y_i = T^*e_i$. For each $x \in X$ we have

$$y_i(x) = T^*e_i(x) = e_i(Tx)$$

Hence, $Tx = \{y_i(x)\}$.

Since $T$ is an isomorphism, it follows that $(\{y_i\}, S)$ is a Banach frame with respect to $X_d$. □

It is known [3] that every separable Banach space has a Banach frame. We will now describe a way to obtain such a Banach frame; for this we need the following definition.

Definition 2.5 A family $\{g_i\}_{i \in I} \subset X^*$ is total on $X$, if

$$g_i(x) = 0, \forall i \Rightarrow x = 0.$$
By [14], p. 189, when the family \( \{g_i\} \subset X^* \) is total on \( X \), the linear space
\[
Z_d := \{ \{g_i(x)\} \mid x \in X \}, \quad \|\{g_i(x)\}\|_{Z_d} := \|x\|_X
\] (4)
is a BK-space, isometrically isomorphic to \( X \).

Every total system \( \{g_i\}_{i \in I} \subset X^* \) is a Banach frame for \( X \) with respect to the corresponding BK-space \( Z_d \):

**Lemma 2.6** Let \( \{g_i\}_{i \in I} \subset X^* \) be a total system. Then there exists an operator \( S : Z_d \rightarrow X \) such that \( (\{g_i\}, S) \) is a Banach frame for \( X \) with respect to \( Z_d \).

**Proof:** The operator \( G : X \rightarrow Z_d \) defined by \( G(x) := \{g_i(x)\} \) is an isometrical isomorphism between \( X \) and \( Z_d \) and hence \( (\{g_i\}, G^{-1}) \) is a Banach frame for \( X \) with respect to \( Z_d \). \( \Box \)

Lemma 2.6 has the following consequence, proved in Section 3.

**Proposition 2.7** For every separable Banach space \( X \) there exists a total system \( \{g_i\}_{i \in I} \subset X^* \) such that the finite sequences are dense in the space \( Z_d \) given in (4), and an operator \( S : Z_d \rightarrow X \) such that \( (\{g_i\}, S) \) is a Banach frame for \( X \) with respect to \( Z_d \).

It is well known that there exist separable Banach spaces having no basis; in this light, it is nice to know that there always exist Banach frames. However, the next example demonstrates that the Banach frames might not have the properties we are used to for Hilbert space frames, i.e., we might not gain what we want. In fact, we prove the existence of a Banach frame for a Hilbert space, which is not a frame, and which does not even have the reconstruction property. In order to exclude pathologies like this, it is necessary to exclude BK-spaces like \( Z_d \) in (4) from the definition of Banach frames.

**Example 2.8** Let \( \{e_i\}_{i=1}^\infty \) be an orthonormal basis for a separable Hilbert space \( \mathcal{H} \). Consider the family \( \{e_i + e_{i+1}\}_{i=1}^\infty \), which is complete, but not a frame for \( \mathcal{H} \) ([4]). In fact, \( e_1 \) does not have a representation as an infinite sum \( \sum_{i=1}^\infty c_i(e_i + e_{i+1}) \). Moreover, there exists no family \( \{f_i\} \subset \mathcal{H} \) such that \( f = \sum \langle f, e_i + e_{i+1} \rangle f_i \) holds for all \( f \in \mathcal{H} \). However, by Lemma 2.6 \( \{e_i + e_{i+1}\}_{i=1}^\infty \) is a Banach frame for \( \mathcal{H} \) with respect to the BK-space
\[
Z_d = \{ \{\langle h, e_i + e_{i+1} \rangle\} \mid h \in \mathcal{H} \} = \{\{c_i + c_{i+1}\} \mid \{c_i\} \in \ell^2\},
\]
\[
\|\{c_i + c_{i+1}\}\|_{Z_d} = \|\{c_i\}\|_{\ell^2}.
\]
For a given sequence \( \{g_i\} \subset X^* \) we now present an equivalent condition for the existence of a sequence \( \{f_i\} \subset X \) such that \( f = \sum_i g_i(f) f_i \) for all \( f \in X \).

**Proposition 2.9** Let \( \{g_i\} \subset X^* \). The following are equivalent:

(i) There exists a sequence \( \{f_i\} \subset X \) such that \( f = \sum_i g_i(f) f_i \), for all \( f \in X \).

(ii) There is a BK-space \( X_d \) with the canonical unit vectors \( \{e_i\} \) as a basis so that \( \{g_i\} \) is an \( X_d \)-frame for \( X \) and an operator \( S : X_d \to X \) so that \((\{g_i\}, S)\) is a Banach frame for \( X \) with respect to \( X_d \).

If the conditions are satisfied, a choice of \( \{f_i\} \) in (i) is \( f_i = S(e_i) \).

The importance of this proposition is the following. We know we cannot hope to get reconstruction just from the existence of an \( X_d \)-frame since there are spaces with no reconstruction for any family \( \{g_i\} \) (i.e. spaces failing the approximation property [2]). So if we are going to be able to use the existence of an \( X_d \)-frame for reconstruction, we must have some connection between \( X_d \) and \( X \). That is, we need some type of operator going back from \( X_d \) to \( X \). The above proposition formalizes this fact.

In general, having reconstruction is much different from having a basis. Assume for example that \( \{f_i\} \) is a basis for a Banach space \( X \), with the biorthogonal sequence \( \{g_i\} \). If \( P \) is any bounded linear projection on \( X \), \( P \neq I \), then for every \( f \in P(X) \) we have:

\[
f = \sum_i g_i(f) f_i = \sum_i P^*(g_i)(f) P(f_i).
\]

That is, the reconstruction property (1) holds with \( g_i \) replaced by \( P^*(g_i) \) and \( f_i \) replaced by \( P(f_i) \); however, \( \{P(f_i)\} \) is not a basic sequence. Also, the existence of reconstruction families \( \{f_i, g_i\} \) does not imply that the space \( X \) needs to have any basis at all. For example, there is a Banach space with a basis \( \{f_i\} \) (see the discussion following Definition 1.11, page 279 in [2]) and a bounded linear projection \( P \) on \( \text{span}\{f_i\} \) so that \( P(\text{span}\{f_i\}) \) fails to have a basis (see Proposition 6.7, page 301 of [2]). However, as we saw above \( P(\text{span}\{f_i\}) \) does have a countable family of reconstruction functions.

Our final result below expresses the key problem if we want to obtain reconstruction via a given sequence \( \{g_i\} \): we can always satisfy the conditions in Proposition 2.9(ii) for a certain choice of \( X_d \), except that the canonical unit vectors might only be a basis for a subspace of \( X_d \).
Theorem 2.10 If $X$ is a separable Banach space then there is a family \( \{g_i\} \subset X^* \) and a separable BK-space $X_d$ containing the canonical unit vectors as a basic sequence so that $\{g_i\}$ is a $X_d$-frame for $X$ and $R(U)$ is complemented in $X_d$.

3 Proofs and auxiliary results

3.1 General results

We need a general result about continuous linear functionals on $X_d$; for its proof we refer to [9], page 201. Let us denote the dual space $(X_d)^*$ by $X_d^*$. Let $X_d$ be a BK-space for which the canonical unit vectors $\{e_i\}$ form a Schauder basis. Then the space $Y_d := \{\{h(e_i)\} \mid h \in X_d^*\}$ with the norm $||\{h(e_i)\}||_{Y_d} := ||h||_{X_d^*}$ is a BK-space isometrically isomorphic to $X_d^*$. Also, every continuous linear functional $\Phi$ on $X_d$ has the form

$$\Phi\{c_i\} = \sum c_i d_i,$$

where $\{d_i\} \in Y_d$ is uniquely determined by $d_i = \Phi(e_i)$, and

$$||\Phi|| = ||\{\Phi(e_i)\}||_{Y_d}.$$

Throughout the paper when we use the dual $X_d^*$ of a BK-space $X_d$ having the canonical unit vectors as a basis, we will identify $X_d^*$ with its isometrically isomorphic BK-space constructed by the above Lemma.

Proposition 3.2 Let $X_d$ be a BK-space, for which the canonical unit vectors form a basis. Then $\{g_i\} \subset X^*$ is an $X_d^*$-Bessel sequence for $X$ with bound $B$ if and only if the operator

$$T: \{d_i\} \rightarrow \sum d_i g_i$$

is well-defined (hence bounded) from $X_d$ into $X^*$ and $\|T\| \leq B$.

Proof: First, let $\{g_i\} \subset X^*$ be an $X_d^*$-Bessel sequence for $X$ with bound $B$ and let $\{e_i\}$ be the canonical unit vector basis of $X_d$. Define $R: X \rightarrow X_d^*$ by $R(f) = \{g_i(f)\}$; then $\|R\| \leq B$. The linear bounded operator $R^*: X_d^{**} \rightarrow X^*$ satisfies

$$R^*(e_j)(f) = e_j(R(f)) = g_j(f), \quad \forall f \in X,$$
and thus $R^*e_j = g_j$. Letting $T = R^*_d$ we have that $\|T\| \leq \|R^*\| = \|R\| \leq B$. Finally, $T(\{d_i\}) = T(\sum_i d_ i e_i) = \sum_i d_ig_i$.

Now suppose that $T : X_d \to X^*$ given by $T(\{d_i\}) = \sum_i d_ig_i$ is well-defined and thus bounded by the Banach–Steinhaus theorem. Then $T(e_i) = g_i$ and for every $f \in X$ the bounded operator $T^* : X^{**} \to X_d^*$ satisfies

$$T^*(f)(e_i) = f(T(e_i)) = f(g_i).$$

That is, $\{g_i(f)\} = \{T^*(f)(e_i)\}$ which is identified with $T^*(f)$ by Lemma 3.1. So $\{g_i\}$ is an $X_d^*$-Bessel sequence for $X$ with a bound $\|T^*\| = \|T\| \leq B$. □

**Corollary 3.3** Let $X_d$ be a BK-space, whose dual $X_d^*$ has the canonical unit vectors as a basis. If $\{g_i\} \subset X^*$ is an $X_d$-Bessel sequence for $X$ with bound $B$ then the operator

$$T : \{d_i\} \to \sum_i d_ig_i \quad (6)$$

is well-defined (hence bounded) from $X_d^*$ into $X^*$ and $\|T\| \leq B$. If $X_d$ is reflexive, the converse is true.

**Proof:** By Proposition 3.2, $\{g_i\} \subset X^*$ is a $X_d^*$-Bessel sequence for $X$ with bound $B$ if and only if the operator $T : \{d_i\} \to \sum_i d_ig_i$ is well defined from $X_d^*$ into $X^*$ and $\|T\| \leq B$. Clearly, every $X_d$-Bessel sequence for $X$ with bound $B$ is an $X_d^{**}$-Bessel sequence for $X$ with bound $B$, and the converse is true when $X_d$ is reflexive. □

The following result relates $X_d$-frames to Banach frames and the question of discrete expansions in $X$ and $X^*$. It extends Proposition 2.2.

**Proposition 3.4** Suppose that $X_d$ is a BK-space and that $\{g_i\} \subset X^*$ is an $X_d$-frame for $X$. Then the following conditions are equivalent:

(i) $R(U)$ is complemented in $X_d$.

(ii) The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : X_d \to X$.

(iii) There exists a linear bounded operator $S$, such that $(\{g_i\}, S)$ is a Banach frame for $X$ with respect to $X_d$.

Also, the condition
(iv) There exists a family \( \{f_i\} \subset X \) such that \( \sum c_i f_i \) is convergent for all \( \{c_i\} \in X_d \) and \( f = \sum g_i(f) f_i, \forall f \in X \).

implies each of (i)-(iii). If we also assume that the canonical unit vectors \( \{e_i\} \) form a basis for \( X_d \), (iv) is equivalent to the above (i)-(iii) and to the following condition (v):

(v) There exists an \( X_d^* \)-Bessel sequence \( \{f_i\} \subset X \subseteq X^{**} \) for \( X^* \) such that

\[
f = \sum g_i(f) f_i, \forall f \in X.
\]

If the canonical unit vectors form a basis for both \( X_d \) and \( X_d^* \), (i)-(v) is equivalent to

(vi) There exists an \( X_d^* \)-Bessel sequence \( \{f_i\} \subset X \subseteq X^{**} \) for \( X^* \) such that

\[
g = \sum g(f_i) g_i, \forall g \in X^*.
\]

In each of the cases (v) and (vi), \( \{f_i\} \) is actually an \( X_d^* \)-frame for \( X^* \).

**Proof:** For convenience, we index \( \{f_i\} \) and \( \{g_i\} \) by the natural numbers. Suppose that \( X_d \) is a BK-space. (i) \( \Rightarrow \) (ii) is trivial. For the converse, assume (ii) and let \( V : X_d \to X \) be a linear bounded extension of \( U^{-1} \). Now consider the bounded operator \( P : X_d \to R(U) \) defined by \( P = UV \). Using the fact that \( VU = I \) (on \( X \)), we get \( P^2 = P \). For every \( f \in X \), we have

\[
Uf = UVUf = P(Uf) \in R(P).
\]

Hence \( R(P) = R(U) \), i.e., the range of \( U \) equals the range of a bounded projection. Thus, \( R(U) \) is complemented (see [12], p. 127). The equivalence (ii) \( \Leftrightarrow \) (iii) is clear.

We now relate the condition (iv) to (i)–(iii). First, assume that (iv) is satisfied. By assumption we can define an operator

\[
V : X_d \to X \quad \text{by} \quad V : \{c_i\} \to \sum c_i f_i.
\]

By the Banach–Steinhaus theorem, \( V \) is bounded. Let \( \{g_i(f)\} \in R(U) \). Furthermore,

\[
V\{g_i(f)\} = \sum g_i(f) f_i = f = U^{-1}f = U^{-1}\{g_i(f)\},
\]
i.e., $V$ is an extension of $U^{-1}$. That is, (ii) holds; according to the equivalences proved so far, this means that (i)–(iii) holds.

Assume now that the canonical unit vectors $\{e_i\}$ form a basis for $X_d$. Assuming that (ii) is satisfied, we will show that (iv) holds. Let $f_i := Ve_i$. Since $V$ is linear and bounded, for all $\{c_i\} \in X_d$ we have

$$
\sum_{i=1}^{n} c_i f_i = V\left(\sum_{i=1}^{n} c_i e_i\right) \rightarrow V\{c_i\}.
$$

That is, $\sum c_i f_i$ is convergent. Also, by construction, for all $f \in X$ we have

$$
f = VUf = \sum g_i(f) f_i,
$$

(7)

Thus (iv) holds, as claimed.

Still assuming that the canonical unit vectors $\{e_i\}$ form a basis for $X_d$, we now prove the equivalence of (iv) and (v). First, assume that (iv) holds. Due to the equivalence of (ii) and (iv), we can (as before) define $f_i := Ve_i$, and the equation (7) is available. By Lemma 3.1, for every $g \in X^*$ we have

$$
\{g(f_i)\} = \{gVe_i\} \in X_d^* \quad \text{and} \quad ||\{g(f_i)\}||_{X_d^*} = ||gV|| \leq ||V|| \ ||g||_{X^*},
$$

hence $\{f_i\}$, considered as a sequence in $X^{**}$, is a $n X_d^*$-Bessel sequence for $X^*$. Thus, we have proved the claims in (v). On the other hand, if (v) is valid then Proposition 3.2 shows that $\sum c_i f_i$ is convergent for all $\{c_i\} \in X_d$ and hence (iv) holds.

Assume now that the canonical unit vectors form a basis for both $X_d$ and $X_d^*$; in this case, we want to prove the equivalence of (v) and (vi). We will let $B$ denote a Bessel bound for the $X_d$-Bessel sequence $\{g_i\}$. Denote the canonical basis for $X_d$ by $\{e_i\}$ and the canonical basis for $X_d^*$ by $\{z_i\}$. Assume that (v) is valid. Let $g \in X^*$; given $n \in \mathbb{N},$

$$
||g - \sum_{i=1}^{n} g(f_i) g_i||_{X^*} = \sup_{f \in X, ||f||=1} |g(f) - \sum_{i=1}^{n} g(f_i) g_i(f)|
$$

$$
= \sup_{f \in X, ||f||=1} \left| \sum_{i=1}^{\infty} g(f_i) g_i(f) - \sum_{i=1}^{n} g(f_i) g_i(f) \right|
$$

$$
= \sup_{f \in X, ||f||=1} \left| \sum_{i=n+1}^{\infty} g(f_i) g_i(f) \right|
$$

$$
\leq B \left| \sum_{i=n+1}^{\infty} g(f_i) z_i \right| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
$$
and hence (vi) holds. Assume now (vi). Let \( K \) be an \( X_d^* \)-Bessel bound for \( \{f_i\} \). For every \( g \in X^* \), \( \{g(f_i)\} \) belongs to \( X_d^* \), which by Lemma 3.1 is isometrically isomorphic to the space \( \{G(e_i) : G \in X_d^* \} \), and hence \( \{g(f_i)\} \) can be identified with \( \{G_g(e_i)\} \) for a unique \( G_g \in X_d^* \). Then for every \( f \in X \),

\[
\|f - \sum_{i=1}^{n} g_i(f) f_i\|_X = \sup_{g \in X^*, \|g\| = 1} |g(f) - \sum_{i=1}^{n} g(f_i) g_i(f)|
\]

\[
= \sup_{g \in X^*, \|g\| = 1} \left| \sum_{i=1}^{\infty} g(f_i) g_i(f) - \sum_{i=1}^{n} g(f_i) g_i(f) \right|
\]

\[
= \sup_{g \in X^*, \|g\| = 1} \left| \sum_{i=n+1}^{\infty} g(f_i) g_i(f) \right|
\]

\[
= \sup_{g \in X^*, \|g\| = 1} |G_g \left( \sum_{i=n+1}^{\infty} g_i(f) e_i \right)|
\]

\[
\leq \sup_{g \in X^*, \|g\| = 1} \|G_g\| \left\| \sum_{i=n+1}^{\infty} g_i(f) e_i \right\|
\]

\[
= \sup_{g \in X^*, \|g\| = 1} \|\{g(f_i)\}\| \left\| \sum_{i=n+1}^{\infty} g_i(f) e_i \right\|
\]

\[
\leq K \left\| \sum_{i=n+1}^{\infty} g_i(f) e_i \right\| \to 0 \text{ as } n \to \infty.
\]

Hence (v) is valid. Moreover, by a similar calculations as above, for every \( g \in X^* \) we have

\[
\|g\| = \sup_{f \in X, \|f\| = 1} |g(f)| = \sup_{f \in X, \|f\| = 1} \left| \sum_{i=1}^{\infty} g(f_i) g_i(f) \right| \leq B \|\{g(f_i)\}\|_{X_d^*},
\]

and hence \( \{f_i\} \) is an \( X_d^* \)-frame for \( X^* \).

\[\square\]

We now consider certain special choices of the BK-space in the definition of a Banach frame. It turns out that some of them lead to undesired properties in the sense that there exist Banach frames without the properties one usually associates with frames.
3.2 Proof of Proposition 2.7

It is well known (see e.g., [14], p. 219, 226) that every separable Banach space \( X \) has a \( M \)-basis, i.e., there exists a biorthogonal system \( \{x_i, g_i\} \subset X \times X^* \) such that \( \{g_i\} \) is total on \( X \) and \( \text{span}\{x_i\} = X \).

**Proof of Proposition 2.7:** Let \( \{x_i, g_i\} \subset X \times X^* \) be a \( M \)-basis for \( X \).

The operator \( G : X \to Z_d \) defined by \( G(x) := \{g_i(x)\} \) is an isometrical isomorphism between \( X \) and \( Z_d \) and hence \( \{g_i, G^{-1}\} \) is a Banach frame for \( X \) with respect to \( Z_d \). Since \( \{g_i\} \) has a biorthogonal sequence, all the canonical unit vectors, and hence all finite sequences, belong to \( Z_d \). Let now \( x \in X \) and fix an arbitrary \( \epsilon > 0 \). Then there exist \( c_{i_1}, c_{i_2}, \ldots, c_{i_N} \) such that

\[
\|x - \sum_{k=1}^{N} c_{i_k} x_{i_k}\| < \epsilon.
\]

Then for the finite sequence

\[
\{g_i\left(\sum_{k=1}^{N} c_{i_k} x_{i_k}\right)\}_{i = 1}^{N} = \{\sum_{k=1}^{N} c_{i_k} \delta_{i,i_k}\} = \{0, \ldots, c_{i_1}, 0, \ldots, c_{i_N}, 0\ldots\}
\]

we have

\[
\|\{g_i(x)\} - \{g_i(\sum_{k=1}^{N} c_{i_k} x_{i_k})\}\|_{Z_d} = \|\{g_i(x - \sum_{k=1}^{N} c_{i_k} x_{i_k})\}\|_{Z_d} = \|x - \sum_{k=1}^{N} c_{i_k} x_{i_k}\|_{X} < \epsilon.
\]

Thus the finite sequences are dense in \( Z_d \). \( \square \)

3.3 Proof of Proposition 2.9

We need a lemma before we give the proof.

**Lemma 3.5** Let \( \{f_i\} \subset X \setminus \{0\} \). Then the sequence space

\[
X_d = \{\{c_i\} : \sum c_i f_i \text{ converges in } X\}
\]

with the norm

\[
\|\{c_i\}\|_{X_d} := \sup_N \|\sum_{i=1}^{N} c_i f_i\|_X
\]

is a Banach space, for which the canonical unit vectors form a basis.
Proof: It is well known that $X_d$ is a Banach space (see e.g., [13] p.18). By the definition of $X_d$, all canonical unit vectors $e_i$ belong to $X_d$. To show that they form a basis for $X_d$, it is enough to prove that \{$e_i$\} is complete and that there exists a constant $C \geq 1$ such that for every $m \geq n$ and scalars $c_1, c_2, \ldots, c_m$, the inequality \[ \| \sum_{i=1}^{n} c_i e_i \| \leq C \| \sum_{i=1}^{m} c_i e_i \| \] holds. Clearly, for every $m \geq n$ and every $c_1, c_2, \ldots, c_m$, we have

\[ \| \sum_{i=1}^{n} c_i e_i \|_{X_d} = \sup_{N \leq n} \| \sum_{i=1}^{N} c_i f_i \|_{X} \leq \sup_{N \leq m} \| \sum_{i=1}^{N} c_i f_i \|_{X} = \| \sum_{i=1}^{m} c_i e_i \|_{X_d}. \]

Choose now arbitrary \{$c_i$\} from $X_d$ and fix arbitrary $\epsilon > 0$. Since $\sum c_i f_i$ converges in $X$, there exists $N_0$ such that $\| \sum_{i=n+1}^{m} c_i f_i \|_{X} < \frac{\epsilon}{2}$, $\forall m > n > N_0$ and therefore $\sup_{N > n} \| \sum_{i=n+1}^{N} c_i f_i \|_{X} \leq \frac{\epsilon}{2} < \epsilon$, $\forall n > N_0$. Thus, for every $n > N_0$ we have

\[ \| \{c_i\} - \sum_{i=1}^{n} c_i e_i \|_{X_d} = \sup_{N > n} \| \sum_{i=n+1}^{N} c_i f_i \|_{X} < \epsilon. \]

Hence \{$e_i$\} is complete in $X_d$, which concludes the proof. \(\square\)

For a given sequence \{$g_i$\} \(\subset X^*\), an equivalent condition for the existence of a sequence \{$f_i$\} \(\subset X\) such that $f = \sum_i g_i(f) f_i$ for all $f \in X$ is given in Proposition 2.9. In case such a representation is possible for a sequence where $f_i \neq 0$, the appearing sequence space equals the one defined in Lemma 3.5, but in the general case a slightly more involved definition is needed:

Proof of Proposition 2.9: (i) \(\Rightarrow\) (ii): First we divide the indices $\mathbb{N}$ into two sets:

\[ A = \{i : f_i = 0\}, \]

and $B$ is the rest of the indices. We define

\[ c_0(A) = \{ \{c_i\}_{i \in A} : \lim_i c_i = 0 \} \]

and norm this space with the sup norm. The canonical unit vectors \{$e_i$\}$_{i \in A}$ form an unconditional basis for $c_0(A)$. Hence, \(\{ \frac{1}{i(\|g_i\|+1)} e_i \}_{i \in A}\) is also a basis for $c_0(A)$. Let

\[ Z_d := \{ \{c_i\}_{i \in A} | \sum_{i \in A} \frac{c_i}{i(\|g_i\|+1)} e_i \text{ converges in } c_0(A) \}, \]

\(\Rightarrow\) the proof.
with the norm
\[ \|\{c_i\}_{i \in A}\|_{Z_d} = \| \sum_{i \in A} \frac{c_i}{i(\|g_i\| + 1)} e_i \|_{c_0(A)}. \]

So \( Z_d \) has the canonical unit vectors \( \{e_i\}_{i \in A} \) as a basis and thus \( Z_d \) is a BK-space. Let \( Y_d \) be the BK-space defined in Lemma 3.5 for the indices in \( B \). Let \( X_d = Y_d \oplus Z_d \) with norm
\[ \|y \oplus z\|_{X_d} = \|y\|_{Y_d} + \|z\|_{Z_d}. \]

For every \( f \in X \), \( \sum_{i \in B} g_i(f) f_i \) converges in \( X \) and so \( \{g_i(f)\}_{i \in B} \in Y_d \). Also, for every \( f \in X \) and every \( i \in A \) we have
\[ |\frac{g_i(f)}{i(\|g_i\| + 1)}| \leq \frac{\|g_i\||f|}{i(\|g_i\| + 1)} \leq \frac{|f|}{i} \quad \text{and thus} \quad \lim_{i \in A} \frac{g_i(f)}{i(\|g_i\| + 1)} = 0, \]

which implies that \( \{g_i(f)\}_{i \in A} \) is an element of \( Z_d \). Therefore, for all \( f \in X \), \( \{g_i(f)\}_{i \in B} \oplus \{g_i(f)\}_{i \in A} \) belongs to \( X_d \).

For the proof of the \( X_d \)-frame inequalities we use an idea from [8]. For convenience, assume that \( \{g_i\}_{i \in B} \) is indexed by \( \mathbb{N} \). Consider the linear bounded operators \( S_n : X \rightarrow X, \ n \in \mathbb{N}, \) defined by \( S_nf = \sum_{i=1}^{\infty} g_i(f) f_i \). For every \( f \in X \) we have \( f = \sum_{i=1}^{\infty} g_i(f) f_i \) and thus the sequence \( \{S_n(f)\} \) is convergent and hence bounded, which implies that \( \sup_n ||S_n(f)|| < \infty \). Therefore, by the Uniform Boundedness Principle, \( \sup_n ||S_n|| < \infty \), and for every \( f \in X \) (and taking into account that \( f_i = 0 \) for all \( i \in A \)),
\[ ||f||_{X} = \lim_{n \to \infty} || \sum_{i=1}^{n} g_i(f) f_i ||_{X} \leq \sup_n || \sum_{i=1}^{n} g_i(f) f_i ||_{X} = ||\{g_i(f)\}_{i \in B}||_{Y_d}. \]

So,
\[ ||f||_{X} \leq \|\{g_i(f)\}_{i \in B}\| + \|\{g_i(f)\}_{i \in A}\| = \|\{g_i(f)\}_{i \in B} \oplus \{g_i(f)\}_{i \in A}\|_{X_d}. \]
Also,
\[ \left\| \{ g_i(f) \}_{i \in B} \oplus \{ g_i(f) \}_{i \in A} \right\|_{X_d} = \left\| \{ g_i(f) \}_{i \in B} \right\|_{Y_d} + \left\| \{ g_i(f) \}_{i \in A} \right\|_{Z_d} \]
\[ = \sup_n \left\| S_n(f) \right\| + \sup_{i \in A} \frac{g_i(f)}{i(\|g_i\| + 1)} \]
\[ \leq \|f\| \sup_n \left\| S_n \right\| + \|f\| \sup_{i \in A} \frac{\|g_i\|}{i(\|g_i\| + 1)} \]
\[ \leq \left( \sup_n \left\| S_n \right\| + 1 \right) \|f\|_X. \]

Finally, define \( S : X_d \to X \) by \( S(e_i) = f_i \) for all \( i \), where \( \{ e_i \}_{i \in B} \) are the canonical unit vectors in \( Y_d \). For every \( \{ c_i \}_{i \in B} \oplus \{ d_i \}_{i \in A} \in X_d \), we have
\[ \sum_{i \in A} d_i f_i = 0, \sum_{i \in B} c_i f_i \text{ converges in } X \text{ and} \]
\[ \left\| S(\{ c_i \}_{i \in B} \oplus \{ d_i \}_{i \in A}) \right\|_X = \left\| \sum_{i \in B} c_i f_i \right\|_X \leq \left\| \{ c_i \}_{i \in B} \right\|_{Y_d} \]
\[ \leq \left\| \{ c_i \}_{i \in B} \oplus \{ d_i \}_{i \in A} \right\|_{X_d}. \]

Thus \( S \) is bounded.

(ii) \( \Rightarrow \) (i): This is immediate from Proposition 3.4. \( \square \)

### 3.4 Proof of Theorem 2.10

Choose a countable dense subset \( \{ x_i \} \) of the unit sphere of \( X \). By the Han-Banach theorem, for each \( i \), choose \( g_i \in X^* \) with \( \|g_i\| = 1 \) and \( g_i(x_i) = 1 \). Thus for every \( f \in X \), \( \sup_i |g_i(f)| \leq \|f\| \). Define \( U : X \to \ell_\infty \) by \( U(f) = \{ g_i(f) \} \). If \( x \in X \) with \( \|x\| = 1 \), there is a sequence of distinct \( n_i \) so that \( x_{n_i} \to x \). Then the inequalities
\[ 1 \geq |g_{n_i}(x)| \geq |g_{n_i}(x_{n_i})| - |g_{n_i}(x - x_{n_i})| \geq 1 - \|x - x_{n_i}\| \]

imply that
\[ \lim_i |g_{n_i}(x)| = 1 \quad (9) \]

and hence \( \sup_i |g_i(x)| \geq 1 \). Thus we obtain
\[ \|U(f)\| = \|f\|, \quad \forall f \in X. \quad (10) \]

Let now \( X_d \) be the closed linear span of \( R(U) \) and the canonical unit vectors in the sup-norm. Since the canonical unit vectors span \( c_0 \), we have them as a
basic sequence in $X_d$. By (10), $\{g_i\}$ is an $X_d$-frame for $X$ and $R(U)$ is closed in $X_d$. The Banach space $X_d$ is separable and thus $c_0$ is complemented in $X_d$, cf. [10]. Let $P$ be a bounded projection from $X_d$ onto $c_0$. For every $x \in X$ with $\|x\| = 1$, $P(Ux)$ belongs to $c_0$ and (9) is valid, which implies that $\|(I - P)(Ux)\| \geq 1$. Therefore for every $f \in X$ we have

$$\|(I - P)(Uf)\| \geq \|f\| = \|Uf\|.$$ 

That is, if $T = (I - P)|_{R(U)}$, then $T : R(U) \rightarrow (I - P)(R(U))$ is an isomorphism (and hence has closed range). Also, if $y \in (I - P)(X_d)$ then by the definition of $X_d$, there are sequences $y_n \in R(U)$ and $z_n \in c_0$ so that $y_n + z_n \rightarrow y$. Hence,

$$y = (I - P)y = \lim_n (I - P)(y_n + z_n) = \lim_n (I - P)(y_n),$$

i.e. $y$ is in the closure of $(I - P)(R(U))$. Hence, $T(R(U)) = (I - P)(X_d)$ and therefore $T^{-1}(I - P)$ is a projection of $X_d$ onto $R(U)$. □

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