Frames and Schauder bases.

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Abstract

We present a rather surprising example of a frame which does not contains a Schauder basis. We also mention a special class of frames which always contain a Scauder basis.

1 Introduction.

Recall that a family of elements $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a *Schauder basis* if, for every $f \in \mathcal{H}$ there exists unique coefficients $\{c_i\}$ such that $f = \sum_{i=1}^{\infty} c_i f_i$.
A *Riesz basis* is a set of the form $\{Te_i\}_{i=1}^{\infty}$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and T is a bounded invertible operator on $\mathcal{H}$. As shown in [5], a Schauder basis is a Riesz basis if and only if $\{f_i\}_{i=1}^{\infty}$ is unconditional and

$$\exists A, B > 0: \quad A \leq ||f_i|| \leq B, \ \forall i \in N.$$

A *frame* is a family of elements $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ such that

$$\exists A, B > 0: \quad A||f||^2 \leq \sum_{i=1}^{\infty} |< f, f_i >|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}.$$

The invertibility and positivity of the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \ Sf = \sum_{i=1}^{\infty} < f, f_i > f_i$$

leads to a representation of any $f \in \mathcal{H}$ as an infinite linear combination of the frame elements:

$$f = SS^{-1}f = \sum_{i=1}^{\infty} < f, S^{-1}f_i > f_i, \quad \forall f \in \mathcal{H}.$$

A frame can be *overcomplete*, i.e., given $f \in \mathcal{H}$,there might exist coefficients $\{c_i(f)\} \neq \{< f, S^{-1}f_i >\}$ such that $f = \sum_{i=1}^{\infty} c_i(f) f_i$. Thus it is natural to look at a frame as a kind of "generalized basis".

Riesz bases can be characterized as those frames which are ω-independent, i.e., for which

$$\sum_{i=1}^{\infty} c_i f_i = 0, \ \{c_i\}_{i=1}^{\infty} \in \ell^2(N) \Rightarrow c_i = 0, \ \forall i.$$

The argument is not hard. That a Riesz basis $\{Te_i\}_{i=1}^{\infty}$ is a frame follows directly from the Parseval equality and the fact that T is invertible. On the other hand, if a frame $\{f_i\}_{i=1}^{\infty}$ is ω-independent, we clearly get an incomplete set if we delete any of the elements, and by [5] this implies that $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis.

The question whether a frame contains a Riesz basis has attracted some attension recently. Seip [4] found conditions on a frame of exponentials $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ for $L^2(-\pi, \pi)$ implying that the frame contains a Riesz basis. He also gave an example of a frame of exponentials without this property. Casazza and Christensen [1] gave a sufficient condition for a frame to contain a Riesz basis, and constructed a frame which does not contains a Riesz basis [2], both in the setting of a general Hilbert space.

The present note investigates the question whether there exists a subfamily of a frame which is a Schauder basis. It turns out that the frame constructed in [2] does not contain a Schauder basis. For a special class of frames (which always contains a Riesz basis) we show that a subfamily is a Schauder basis if and only if it is a Riesz basis.

2 The results.

We say that a frame $\{f_i\}_{i=1}^{\infty}$ has the *subframe property* if any subfamily $\{f_i\}_{i \in J}, J \subseteq N$ is a frame for $\overline{span}\{f_i\}_{i \in J}$. It was proved in [2] that every frame with the subframe property contains a Riesz basis. As pointed out by Lennard, this actually leads to an equivalent characterization of frames with the subframe property:

Theorem 1 *Let $\{f_i\}_{i=1}^{\infty}$ be a frame. The following is equivalent:*

(1) $\{f_i\}_{i=1}^{\infty}$ has the subframe property.

(2) Every subfamily $\{f_i\}_{i \in J}$ contains a subset which is a Riesz basis for $\overline{span}\{f_i\}_{i \in J}$.

Proof.
(1) $\Rightarrow$ (2) : Let $J \subseteq N$. If (1) is satisfied, $\{f_i\}_{i\in J}$ is a frame for the Hilbert space $\overline{span}\{f_i\}_{i\in J}$ having the subframe property; therefore $\{f_i\}_{i\in J}$ contains a Riesz basis for $\overline{span}\{f_i\}_{i\in J}$.

(2) $\Rightarrow$ (1) : Let again $J \subseteq N$. If (2) is satisfied, there exists a set $K \subseteq J$ such that $\{f_i\}_{i\in K}$ is a Riesz basis for $\overline{span}\{f_i\}_{i\in J}$. In particular, $\{f_i\}_{i\in K}$ is a frame for $\overline{span}\{f_i\}_{i\in J}$. Since $\{f_i\}_{i=1}^{\infty}$ satisfies the upper frame bound, it is now clear that $\{f_i\}_{i\in J}$ is a frame for $\overline{span}\{f_i\}_{i\in J}$.

Corollary 2 *Assume that a frame $\{f_i\}_{i=1}^{\infty}$ has the subframe property. Then a subfamily $\{f_i\}_{i\in J}$ is a Schauder basis if and only $\{f_i\}_{i\in J}$ is a Riesz basis.*

Proof. Let $\{f_i\}_{i\in J}$ be a Schauder basis; then $\{f_i\}_{i\in J}$ is ω-independent. By the subframe property, $\{f_i\}_{i\in J}$ is also a frame, so we conclude that $\{f_i\}_{i\in J}$ is a Riesz basis.

We observe that the conclusion in Corollary 2 is false for general frames. For example, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and let

$$\{f_i\}_{i=1}^{\infty} := \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, ...\},$$

(that is, for $n \in N$ the element $\frac{1}{\sqrt{n}}e_n$ appears n times). Then $\{f_i\}_{i=1}^{\infty}$ is a frame with bounds equal to one. The frame contains a Schauder basis but not a Riesz basis.

The main purpose of our note is to present an example of a frame not containing a Schauder basis. We will use the following characterization of a Schauder basis, which can be found in [3], p.2 : a sequence $\{f_i\}_{i=1}^{\infty}$ of non-zero vectors spanning $\mathcal{H}$ is a Schauder basis for $\mathcal{H}$ if and only if the basis constant is finite. Recall that the basis constant of $\{f_i\}_{i=1}^{\infty}$ is the smallest constant K satisfying for all natural numbers n,m and all sequences $\{a_i\}$, the following inequality:

$$\|\sum_{i=1}^{n} a_i f_i\| \leq K \|\sum_{i=1}^{n+m} a_i f_i\|.$$

The main part of the proof of our result is contained in the following lemma:

Lemma 3 *Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis for a finite dimensional space $\mathcal{H}_n$. Define*

$$f_i = e_i - \frac{1}{n}\sum\nolimits_{j=1}^{n} e_j \text{ for } i = 1, 2, ..., n, \text{ and } f_{n+1} = \frac{1}{\sqrt{n}}\sum\nolimits_{j=1}^{n} e_j.$$

Then $\{f_i\}_{i=1}^{n+1}$ is a frame for $\mathcal{H}_n$ with bounds $A = B = 1$. Furthermore, any subset of the frame which spans $\mathcal{H}_n$ has basis constant greater than or equal to $\frac{\sqrt{n-2}}{4}$.

Proof. The proof that $\{f_i\}_{i=1}^{n+1}$ is a frame for $\mathcal{H}_n$ with bounds $A = B = 1$ can be found in [2]. The vectors $\{f_i\}_{i=1}^{n}$ are dependent, so any subset of the frame $\{f_i\}_{i=1}^{n+1}$ which spans $\mathcal{H}_n$ must contain $\frac{\sum_{i=1}^n e_i}{\sqrt{n}}$ and $n-1$ of the other terms. Given a subset Δ of $\{1,2,\dots,n\}$ with $|\Delta| = n-1$, and $\Delta^c = \{k\}$, we have

$$\|\sum_{i\in\Delta}\left(e_i - \frac{\sum_{j=1}^n e_j}{n}\right)\| = \|\frac{1}{n}\sum_{i\in\Delta} e_i - \frac{n-1}{n}e_k\|$$

$$= \left[\frac{n-1}{n^2} + \left(\frac{n-1}{n}\right)^2\right]^{1/2} = \sqrt{\frac{n-1}{n}}.$$

Let $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$. For any subset $A \subset \{1,2,\dots,n\}$ with $|A| = [\frac{n-1}{2}]$, we have, $|A| \le \frac{n}{2}$ and $|A| \ge \frac{n-1}{2} - \frac{1}{2} = \frac{n}{2} - 1$; therefore

$$\|\sum_{i\in A}\left(e_i - \frac{\sum_{j=1}^n e_j}{n}\right)\| = \|\sum_{i\in A}\left(1 - \frac{|A|}{n}\right)e_i + \sum_{i\notin A}\frac{-|A|}{n}e_i\|$$

$$\ge \left(\sum_{i\in A}(1-\frac{|A|}{n})^2\right)^{1/2} = |A|^{1/2}(1-\frac{|A|}{n}) \ge \frac{|A|^{1/2}}{2} \ge \frac{\sqrt{\frac{n}{2}-1}}{2}.$$

Let $\{g_i\}_{i=1}^{n}$ be any spanning subset of the frame $\{f_i\}_{i=1}^{n+1}$. Choose m so that $\{g_i\}_{i=1}^{m}$ contains exactly $[\frac{n-1}{2}]$ of the elements of the form $e_i - \frac{\sum_{j=1}^n e_j}{n}$. Now, there are 2 possibilities.

Case 1 $\frac{\sum_{j=1}^n e_j}{\sqrt{n}}$ does not appear in the list $\{g_i\}_{i=1}^{m}$.

Then as we saw above,

$$\|\sum_{i=1}^m g_i\| \ge \frac{1}{2}\sqrt{\frac{n}{2}-1},$$

while

$$\|\sum_{i=1}^n g_i\| = \left(\|\frac{\sum_{j=1}^n e_j}{\sqrt{n}}\|^2 + \frac{n-1}{n}\right)^{1/2} = \left(1 + \frac{n-1}{n}\right)^{1/2} \le \sqrt{2}.$$

Hence, the basis constant for $\{g_i\}_{i=1}^n$ is greater than or equal to $\frac{\sqrt{\frac{n}{2}-1}}{2\sqrt{2}} = \frac{\sqrt{n-2}}{4}$.

Case 2 $\frac{\sum_{j=1}^n e_j}{\sqrt{n}}$ appears in the list $\{g_i\}_{i=1}^m$.

Now, since $\frac{\sum_{j=1}^n e_j}{\sqrt{n}} \perp f_i, i = 1, 2, ..n$, we have

$$\|\sum_{i=1}^m g_i\| \geq \left(\|\frac{\sum_{j=1}^n e_j}{\sqrt{n}}\|^2 + \left[\frac{1}{2}\sqrt{\frac{n}{2}-1}\right]^2\right)^{1/2} \geq \frac{\sqrt{\frac{n}{2}-1}}{2},$$

while we still have $\|\sum_{i=1}^n g_i\| \leq \sqrt{2}$. So again, the basis constant of $\{g_i\}_{i=1}^n$ is greater than or equal to $\frac{\sqrt{n-2}}{4}$.

We are now ready to prove

Proposition 4 *There is a tight frame $\{f_i\}_{i=1}^\infty$ for which no subset is a Schauder basis for $\mathcal{H}$.*

Proof. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$ and define

$$\mathcal{H}_n := span\{e_{\frac{(n-1)n}{2}+1}, e_{\frac{(n-1)n}{2}+2}, ..., e_{\frac{(n-1)n}{2}+n}\}.$$

By construction, $\mathcal{H} = (\sum_{n=1}^\infty \oplus \mathcal{H}_n)_{\mathcal{H}}$; that is, $g \in \mathcal{H} \Leftrightarrow g = \sum_{n=1}^\infty g_n,\ g_n \in \mathcal{H}_n$, and $||g||^2 = \sum_{n=1}^\infty ||g_n||^2$. We refer to [3] for details about such constructions.
For each space $\mathcal{H}_n$ we construct the sequence $\{f_i^n\}_{i=1}^{n+1}$ as in Lemma 3, starting with the orthonormal basis $\{e_{\frac{(n-1)n}{2}+1}, e_{\frac{(n-1)n}{2}+2}, ..., e_{\frac{(n-1)n}{2}+n}\}$. As shown in [2] , $\{f_i^n\}_{i=1,n=1}^{n+1,\infty}$ is a frame for $\mathcal{H}$ with bounds $A = B = 1$. Let $\{g_i\}_{i=1}^\infty$ be any spanning subset of this frame. Then the basis constant of $\{g_i\}_{i=1}^\infty$ is greater than or equal to the basis constant of any subset of $\{g_i\}_{i=1}^\infty$. By choosing subfamilies as in Lemma 3, we get basis constants greater than or equal to $\frac{\sqrt{n-2}}{4}$ for all n, implying that the basis constant of $\{g_i\}_{i=1}^\infty$ is infinite. Thus $\{g_i\}_{i=1}^\infty$ is not a Schauder basis for $\mathcal{H}$.

Acknowledgment The authors would like to thank Christopher Lennard and Richard Zalik for stimulating conversations.

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