

# FRAMES FOR BANACH SPACES

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ABSTRACT. We use several fundamental results which characterize frames for a Hilbert space to give natural generalizations of Hilbert space frames to general Banach spaces. However, we will see that all of these natural generalizations (as well as the currently used generalizations) are equivalent to properties already extensively developed in Banach space theory. We show that the dilation characterization of framing pairs for a Hilbert spaces generalizes (with much more effort) to the Banach space setting. Finally, we examine the relationship between frames for Banach spaces and various forms of the Banach space approximation properties. We also consider Hilbert space frames. Here we classify the alternate dual frames for a Hilbert space frame by a natural manifold of operators on the Hilbert space. Most of the basic properties of alternate dual frames follow immediately from this characterization. We also answer a question of Han and Larson by showing that the dual frame pairs are compressions of Riesz basis and their dual bases. We also show that a family of frames for the same Hilbert space have the simultaneous dilation property if and only if they have the same deficiency.

## 1. INTRODUCTION AND BACKGROUND RESULTS

In 1946 D. Gabor [18] introduced a fundamental approach to signal decomposition in terms of elementary signals. In 1952, while addressing some difficult problems from the theory of nonharmonic Fourier series, Duffin and Schaeffer [12] abstracted Gabor's method to define *frames* for a Hilbert space. For some reason the work of Duffin and Schaeffer was not continued until 1986 when the fundamental work of Daubechies, Grossman and Meyer [11] brought this all back to life, right at the dawn of the "wavelet era". Today, "frame theory" is a central tool in many applied areas, especially signal analysis, where **Gabor frames** or as they are often now called **Weyl-Heisenberg frames** have become a paradigm for the spectral analysis associated with time-frequency methods.

A **(standard) frame** for a Hilbert space  $H$  is a family of vectors  $f_i \in H$ , for  $i \in \mathbb{N}$  for which there are constants  $A, B > 0$  satisfying:

$$(1.1) \quad A\|f\|^2 \leq \sum_{i \in \mathbb{N}} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \text{ for all } f \in H.$$

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1991 *Mathematics Subject Classification.* Primary 46B15, 46B28, 46B45, 46C05, 47B38, 47A20.  
The first Author was supported by NSF DMS-970618 and the Workshop on Linear Analysis and Probability, Texas A& M University. The third author was supported by NSF DMS-9706810

We call  $A, B$  **frame bounds** for the frame, with  $A$  the **lower frame bound** and  $B$  the **upper frame bound**. A sequence  $(f_i)$  in  $H$  with a finite upper frame bound  $B$  is called a **Hilbertian sequence** in  $H$ . If  $A = B$ , this is called a **tight frame** and if  $A = B = 1$  a **normalized tight frame**. Let  $(f_i)_{i \in \mathbb{N}}$  be a sequence of elements in a Hilbert space  $H$  and  $(e_i)_{i \in \mathbb{N}}$  be an orthonormal basis for  $\ell_2$ . We define the (densely defined) **preframe operator**  $A : \ell_2 \rightarrow H$  by  $Ae_i = f_i$ , for all  $i \in \mathbb{N}$ . If this is bounded (hence extends to  $\ell_2$ ) then for all  $f \in H$  we have

$$\langle A^*f, e_i \rangle = \langle f, Ae_i \rangle = \langle f, f_i \rangle.$$

It follows that

$$A^*f = \sum_{i \in \mathbb{N}} \langle f, f_i \rangle e_i,$$

and so

$$\|A^*f\|^2 = \sum_{i \in \mathbb{N}} |\langle f, f_i \rangle|^2.$$

In light of equation (1.1) we have that  $(f_i)$  is a frame if and only if  $\theta = A^*$  is a (possibly into) isomorphism (called the **frame transform**) which in turn is equivalent to  $A$  being a bounded, linear, onto operator. The **deficiency** of the frame is the dimension of the kernel of  $A$ . In particular,  $(f_i)$  is a normalized tight frame if and only if  $A$  is a quotient map (i.e.  $A$  is a co-isometry). It follows that  $S = \theta^*\theta : H \rightarrow H$  is an invertible operator on  $H$  called the **frame operator**. A direct calculation yields:

$$(3.2) \quad Sf = \sum_i \langle f, f_i \rangle f_i, \text{ for all } f \in H,$$

with the series in (3.2) converging unconditionally. It follows that

$$\langle Sf, f \rangle = \sum_i |\langle f, f_i \rangle|^2, \text{ for all } f \in H.$$

So the frame transform  $S$  is a positive, self-adjoint invertible operator on  $H$ . This yields the **reconstruction formula** for all  $f \in H$ ,

$$f = SS^{-1}f = \sum_i \langle S^{-1}f, f_i \rangle f_i = \sum_i \langle f, S^{-1}f_i \rangle f_i.$$

We call  $\{\langle f, S^{-1}f_i \rangle\}$  the **frame coefficients** of the vector  $f \in H$ . The frame  $(S^{-1}f_i)$  is called the **canonical dual frame** of the frame  $(f_i)$  and is denoted  $(f_i^*)$ . A frame  $(g_i)$  for  $H$  is called an **alternate dual frame** (or a **pseudo-dual**) for  $(f_i)$  if it satisfies the following equality:

$$(3.3) \quad f = \sum_i \langle f, g_i \rangle f_i, \text{ for all } f \in H.$$

Recall that  $(u_i)$  is called a **Riesz basis** for a Hilbert space  $H$  if it is a bounded unconditional basis for  $H$ . This is equivalent to the existence of constants  $C, D$  satisfying for all  $(a_i) \in \ell_2$ ,

$$C\left(\sum_i |a_i|^2\right)^{1/2} \leq \left\| \sum_i a_i u_i \right\| \leq D\left(\sum_i |a_i|^2\right)^{1/2}.$$

Riesz bases have unique natural dual Riesz bases (the elements are called the **dual functionals**) defined by:  $u_i^* f = \langle f, u_i \rangle$ , for all  $f \in H$ . So we generally express a Riesz basis along with its dual as a pair  $(u_i, u_i^*)$ . One advantage of frames over wavelets is that orthogonal projections map frames to frames - and even with the same frame bounds.

**Proposition 1.1.** *If  $(f_i)$  is a frame for a Hilbert space  $H$  with frame bounds  $A, B$  and  $P$  is an orthogonal projection on  $H$ , then  $(Pf_i)$  is a frame for  $PH$  with frame bounds  $A, B$ . Moreover, if  $(f_i, f_i^*)$  is a Riesz basis for  $H$  then  $(Pf_i^*)$  is an alternate dual frame for  $(Pf_i)$ .*

*Proof.* For any  $f \in PH$  we have

$$\sum_i |\langle f, Pf_i \rangle|^2 = \sum_i |\langle Pf, f_i \rangle|^2 = \sum_i |\langle f, f_i \rangle|^2.$$

This proves the first part of the theorem. For the second part, let  $f \in PH$  and compute

$$\sum_i \langle f, Pf_i^* \rangle Pf_i = \sum_i \langle f, f_i^* \rangle Pf_i = P \left( \sum_i \langle f, f_i^* \rangle f_i \right) = Pf = f.$$

□

Han and Larson [22] asked if the converse of Theorem 1.1 holds? In Section 7, Theorem 7.3 we give a positive answer to this question by showing that whenever  $(f_i)$  and  $(g_i)$  are alternate dual frames for a Hilbert space  $H$ , then there is a larger Hilbert space  $K \supset H$  and a Riesz basis  $(u_i, u_i^*)$  for  $K$  with  $P_H u_i = f_i$  and  $P_H u_i^* = g_i$ . We say in this case that  $(f_i)$  is a **compression** of the Riesz basis and the Riesz basis is a **dilation** of the frame  $(f_i)$ . We also show in section 7 that a family of frames for the same Hilbert space are the compressions of a family of Riesz bases for the same larger Hilbert space if and only if all these frames have the same deficiency. In Section 6, we will give a complete classification of the alternate dual frames of a frame in terms of a natural manifold of operators. Most of the standard properties of alternate dual frames follow immediately from this classification.

The main body of the paper deals with frames for Banach spaces. In 1989 Grochenig [21] generalized frames to Banach spaces and called them **atomic decompositions**. The main feature of frames that Grochenig was trying to capture

in a general Banach space was the unique association of a vector in a Hilbert space with the natural set of *frame coefficients*. Grochenig also defined a more general notion for Banach spaces called a *Banach frame*. These ideas had actually been around for several years. By 1985, even before smooth orthonormal wavelet bases were discovered for the Hilbert space  $L^2(\mathbb{R})$ , Frazier and Jawerth [17] had already constructed wavelet atomic decompositions for Besov spaces. They called these  $\phi$ -transform's. After Grochenig formalized atomic decompositions, there followed a flurry of activity in this area. Feichtinger [13] constructed Gabor atomic decompositions for the modulation spaces. Simultaneously, Feichtinger and Grochenig [14,15] developed a general theory for a large class of function spaces and group representations. This was particularly important since Walnut [38] had extended this construction to weighted Hilbert spaces. He had already shown that there are no Gabor frames for a weighted Hilbert space and that Gabor atomic decompositions in  $L^2_\omega(\mathbb{R})$  need not be Hilbert space frames. Further work on atomic decompositions via group representations appeared in 1996 by Christensen [7] and perturbation theory for atomic decompositions was done by Christensen and Heil [8].

In Section 2 we study atomic decompositions in Banach spaces. We will see that these are exactly compressions of Schauder bases for a larger Banach space. This allows us to relate these concepts to several other Banach space notions including the bounded approximation property and finite dimensional expansions of the identity.

In Section 3 we give three new generalizations of frames for a Hilbert space to Banach spaces. The first arises naturally from Proposition 1.1. That is, we define a **frame** for a Banach space as a compression of an unconditional basis for a larger Banach space (equivalently is an inner direct summand of an unconditional basis). This definition captures the unconditional convergence property of the frame transform which seems central to frame theory for Hilbert spaces. We will also use the frame transform notion itself to define a **framing** for a Banach space. Finally, we will use the preframe operator and its adjoint to define a **framing model**. We will see that all of these generalizations of frames for Banach spaces are related to concepts which are well known and well developed in Banach space theory. In Section 4 we will show that an alternate dual frame for a Banach space can be dilated to an unconditional basis for a larger space. As we will see, the proof of this result is much more difficult than the proof of the corresponding result for a Hilbert space frame given in Section 7. In Section 5 we will give a detailed analysis of how atomic decompositions, frames etc. for Banach spaces relate to classical approximation properties. For an introduction to frame theory we refer the reader to Han and Larson [22] or Young [39]. For the standard results on Banach space

theory we refer to the classical books of Lindenstrauss and Tzafriri [30,31].

## 2. ATOMIC DECOMPOSITIONS

A **Banach space of scalar valued sequences** (often called a **BK-space**) is a linear space of sequences with a norm which makes it a Banach space (i.e. it is complete in the norm) and for which the coordinate functionals are continuous. In a Banach space of scalar valued sequences the **unit vectors** are the elements  $e_i$  defined by  $e_i(j) = \delta_{ij}$  ( $\delta_{ij}$  the Kronecker delta). Grochenig [21] first generalized frames to Banach spaces.

**Definition 2.1.** *Let  $X$  be a Banach space and let  $X_d$  be an associated Banach space of scalar valued sequences indexed by  $\mathbf{N}$ . Let  $(y_i)$  be a sequence of elements from  $X^*$  and  $(x_i)$  a sequence of elements of  $X$ . If:*

- (a)  $\langle x, y_i \rangle \in X_d$ , for each  $x \in X$ ,
- (b) The norms  $\|x\|_X$ , and  $\|\langle x, y_i \rangle\|_{X_d}$  are equivalent,
- (c)  $x = \sum_{i=1}^{\infty} \langle x, y_i \rangle x_i$ , for each  $x \in X$ ,

*then  $((y_i), (x_i))$  is an **atomic decomposition** of  $X$  with respect to  $X_d$ . If the norm equivalence is given by  $A\|x\| \leq \|\langle x, y_i \rangle\|_{X_d} \leq B\|x\|_X$ , then  $A, B$  are a choice of **atomic bounds** for  $((x_i), (y_i))$ .*

For a more general setting, Grochenig [21] defined:

**Definition 2.2.** *Let  $X$  be a Banach space and let  $X_d$  be an associated Banach space of scalar valued sequences indexed by  $\mathbf{N}$ . Let  $(y_i)$  be a sequence of elements from  $X^*$  and  $S : X_d \rightarrow X$  be given. If:*

- (a)  $\langle x, y_i \rangle \in X_d$ , for each  $x \in X$ ,
- (b) The norms  $\|x\|_X$  and  $\|\langle x, y_i \rangle\|_{X_d}$  are equivalent,
- (c)  $S$  is bounded and linear, and  $S(\langle x, y_i \rangle) = x$ , for each  $x \in X$ ,

*then  $((y_i), S)$  is a **Banach frame for  $X$  with respect to  $X_d$** . The mapping  $S$  is the **reconstruction operator**. If the norm equivalence is given by*

$$A\|x\|_X \leq \|\langle x, y_i \rangle\|_{X_d} \leq B\|x\|_X,$$

*then  $A, B$  are a choice of **frame bounds** for  $((y_i), S)$ .*

It turns out there is a natural relationship between these two definitions. Namely, a Banach frame is an atomic decomposition if and only if the unit vectors form a basis for the space  $X_d$ . We state this result formally in the next Proposition.

**Proposition 2.3.** *Let  $X$  be a Banach space and  $X_d$  be an associated Banach space of scalar valued sequences indexed by  $\mathbf{N}$ . Let  $(y_i)$  be a sequence of elements from  $X^*$*

and  $S : X_d \rightarrow X$  be given. Let  $(e_i)$  be the unit vectors in  $X_d$ . Then the following are equivalent:

(1)  $((y_i), S)$  is a Banach frame for  $X$  with respect to  $X_d$  and  $(e_i)$  is a Schauder basis for  $X_d$ .

(2)  $((y_i), (S(e_i)))$  is an atomic decomposition for  $X$  with respect to  $X_d$ .

*Proof.* If the unit vectors  $(e_i)$  form a basis of  $X_d$  in Definition 2.2, then let  $x_i = S(e_i) \in X$ , for all  $n$ . Then (a), (b), (c) of Definition 2.2 are now equivalent to the corresponding conditions in Definition 2.1.  $\square$

Banach frames are quite general. In fact, every Banach space has a Banach frame defined on it as the next result shows.

**Proposition 2.4.** *Every separable Banach space has a Banach frame with frame bounds  $A = B = 1$ .*

*Proof.* If  $X$  is a separable Banach space, we can choose a sequence  $y_i \in X^*$  with  $\|y_i\| = 1$ , for  $n = 1, 2, 3, \dots$ , and so that for every  $x \in X$  we have

$$(2.1) \quad \|x\| = \sup_i |y_i(x)|.$$

Let  $X_d$  be the subspace of  $\ell^\infty$  given by:

$$X_d = \{ \langle x, y_i \rangle : x \in X \}.$$

Let  $S : X_d \rightarrow X$  be given by  $S(\langle x, y_i \rangle) = x$ . Now, by equality (2.1),  $S$  is an isometry of  $X_d$  onto  $X$  and  $((y_i), S)$  is a Banach frame for  $X$  with respect to  $X_d$ .  $\square$

The notion of an atomic decomposition was studied in Banach space theory in the 1960's under a different name. This doesn't seem to have been noticed before, so we will next consider these connections. First we need a few definitions. We recall that a sequence  $(x_i)_{n \in \mathbb{N}}$  in a Banach space  $X$  is called a **Schauder basis** (or just a **basis**) for  $X$  if for each  $x \in X$  there is a unique sequence of scalars  $(a_i)_{n \in \mathbb{N}}$  so that  $x = \sum_i a_i x_i$ . The unique elements  $x_i^* \in X^*$  satisfying

$$(2.2) \quad x = \sum_{n \in \mathbb{N}} \langle x, x_n^* \rangle x_n, \quad \text{for all } x \in X,$$

are called the **dual** (or **biorthogonal**) functionals for  $(x_i)$ . If the series in (2.2) converges unconditionally for every  $x \in X$ , we call  $(x_i, x_i^*)$  an **unconditional basis** for  $X$ . There is a simple criterion for checking when a sequence is a basis for a Banach space  $X$  [28], Proposition 1.a.3.

**Proposition 2.5.** *Let  $(x_i)$  be a sequence of vectors in  $X$ . Then  $(x_i)$  is a Schauder basis of  $X$  if and only if the following three conditions hold.*

(i)  $x_i \neq 0$  for all  $n$ .

(ii) There is a constant  $K$  so that for every choice of scalars  $(a_i)$  and integers  $n < m$ , we have

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq K \left\| \sum_{i=1}^m a_i x_i \right\|.$$

(iii) The closed linear span of  $(x_i)$  is all of  $X$ .

The smallest constant  $K$  satisfying (ii) of Proposition 2.5 is called the **basis constant** of  $X$ . We also have a **unconditional basis constant** for an unconditional basis given by:

$$UBC(x_i) = \sup \left\{ \left\| \sum_i b_i x_i \right\| : \left\| \sum_i a_i x_i \right\| = 1, |b_i| \leq a_i, \forall i \right\}.$$

If  $(x_i, x_i^*)$  is an unconditional basis for  $X$ , we can define an equivalent norm on  $X$  by:

$$\| \sum_i a_i x_i \| = \sup \left\{ \left\| \sum_i a_i b_i x_i \right\| : |b_i| \leq 1, \forall i \right\}.$$

Then  $(x_i, x_i^*)$  is an unconditional basis for  $X$  with  $UBC(x_i) = 1$ . In this case we just call  $(x_i)$  a 1-unconditional basis for  $X$ .

If  $X$  is a Banach space with a basis  $(e_i, e_i^*)$ , there is a natural **associated Banach space of scalar valued sequences** given by:

$$X_d = \left\{ (a_i) : \sum_i a_i e_i \text{ converges in } X \right\}.$$

If we equip  $X_d$  with the norm:

$$\|(a_i)\|_{X_d} = \left\| \sum_i a_i e_i \right\|_X,$$

then  $X_d$  becomes a Banach space of scalar valued sequences which is isometric to  $X$ . It follows that  $(e_i, e_i^*, X_d)$  is an atomic decomposition for  $X$ . That is, if  $X$  has a basis, then  $X$  has an atomic decomposition. The converse of this is false (i.e. there are Banach spaces with atomic decompositions which fail to have bases) as we will see in Section 5.

Now we want to classify atomic decompositions in terms of bases for Banach spaces. We start with a classical construction of Pelczynski [33].

**Theorem 2.6.** *Let  $X$  be a Banach space,  $x_i \in X$  and  $y_i \in X^*$ . The following are equivalent:*

(1) There is a Banach space of scalar valued sequences  $X_d$ , so that  $((x_i, (y_i), X_d)$  satisfies definition 2.1 (i.e. is an atomic decomposition for  $X$ ).

(2) There is a Banach space  $Z$  with a basis  $(e_i, e_i^*)$  so that  $X \subset Z$  and there is a bounded linear projection  $P : Z \rightarrow X$  with  $Pe_i = x_i$ , for all  $i$ .

*Proof.*

(1)  $\Rightarrow$  (2): Let  $((x_i, (y_i), X_d)$  be an atomic decomposition for the Banach space  $X$ . We let  $c_{00}$  denote the linear space of scalar sequences with only finitely many non-zero terms. Let  $(e_i, e_i^*)$  be the unit vectors in  $c_{00}$  and define a norm on this space by:

$$(2.1) \quad \left\| \sum_i a_i e_i \right\| = \sup_n \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

for every sequence of scalars  $(a_i) \in c_{00}$ . There is a technical problem here in the case that some of the  $x_i$  are zero in that this is not a norm. For now, we will assume that the  $x_i$  are non-zero and address the general case at the end. It follows immediately from Proposition 2.5 that  $(e_i)$  is a basis for its Banach space completion, which we denote by  $Z$ . We define an operator  $\theta : X \rightarrow Z$  by  $\theta x = \sum_i \langle x, y_i \rangle e_i$ . By assumption (c) of Definition 2.1, it follows easily that  $\theta$  is an isomorphism taking  $X$  onto a subspace of  $Z$ . Also, if we define  $\Gamma : Z \rightarrow X$  by  $\Gamma(\sum_i a_i e_i) = \sum_i a_i x_i$ , then by (2.1) we have that  $\Gamma$  is a bounded linear operator from  $Z$  to  $X$ , and by (c) of Definition 2.1, we have that  $\Gamma\theta = I_X$ . It follows that  $P = \theta\Gamma$  is a bounded linear projection of  $Z$  onto  $\theta X$  which takes  $e_i$  to  $\theta x_i$ . After a renorming, we may assume that  $\theta$  is an (into) isometry and then associate  $X$  with  $\theta X$  and  $x_i$  with  $\theta x_i$ .

This completes the proof except for the technical question we skipped over earlier when some of the  $x_i$  are zero. To adjust for this, we would let  $J = \{i : x_i \neq 0\}$ . Then we can do our above construction on  $(x_i)_{i \in J}$ . For  $i \in J^c$ , we certainly may assume that  $y_i = 0$ . So we consider the larger space  $Z \oplus \ell_2^M$  with  $M = |J^c|$ . If  $(f_i)_{i \in M}$  is any orthonormal basis for  $\ell_2^M$  we define a projection  $Q$  on  $Z \oplus \ell_2^M$  by  $Q(z \oplus y) = P(z) \oplus 0$ . Now, we have the desired space  $Z \oplus \ell_2^M$  with a basis so that  $x_i$  is the image under a projection of a basis for this space.

(2)  $\Rightarrow$  (1): With the notation of (2), if we let  $x_i = Pe_i$ ,  $y_i = P^*e_i^*$  and  $X_d$  the sequence space associated with the basis  $(e_i, e_i^*)$ , then we can easily see that  $(x_i, y_i, X_d)$  is an atomic decomposition for  $X$ .  $\square$

Han and Larson [22] defined a **Schauder frame** for a Banach space  $X$  to be an inner direct summand (i.e. a compression) of a Schauder basis for  $X$ . Similarly, a **unconditional Schauder frames** is a compression of an unconditional Schauder basis for  $X$ . An examination of Definition 2.1 and Theorem 2.6 shows that a sequence  $(x_i)$  is a Schauder frame for a Banach space if and only if there is a

sequence  $(y_i)$  of elements of  $X^*$  and a Banach sequence space of scalar valued sequences  $X_d$  such that  $((y_i), (x_i), X_d)$  is an atomic decomposition for  $X$ . An examination of the proof of Theorem 2.6 further shows that if  $(x_i)$  is a **bounded** sequence then it is an inner direct summand of a **bounded** Schauder basis for a larger Banach space.

Now we proceed to examine the general relationship between atomic decompositions and several forms of the approximation property studied in the late 1960's and early 1970's in Banach space theory.

**Definition 2.7.** *A sequence of non zero finite rank operators  $(A_i)$  from a Banach space  $X$  into itself is called an **(unconditional) finite dimensional expansion of the identity** if for all  $x \in X$ ,*

$$x = \sum_i A_i x,$$

*(and the series converges unconditionally). Moreover, if  $A_i A_n = 0$ , for all  $i \neq n$ , then  $(A_i)$  is called an **(unconditional) orthogonal expansion of the identity** of  $X$ .*

Notes that if a sequence  $(A_i)$  of finite dimensional operators is an (unconditional) finite dimensional expansion of the identity of  $X$  then the span of  $\cup_i A_i X$  is dense in  $X$ .

We need one final approximation property to complete our classification of atomic decompositions.

**Definition 2.8.** *A separable Banach space  $X$  has the  **$\lambda$ -bounded approximation property** (i.e.  $\lambda$ -BAP) if there is a sequence of finite rank operators  $(T_i)$  defined on  $X$  so that for every  $x \in X$ ,  $T_i x \rightarrow x$  in norm. We say that  $X$  has the **Bounded approximation property** (denoted BAP) if  $X$  has the  $\lambda$ -BAP, for some  $\lambda$ .*

Pelczynski [33] and independently Johnson, Rosenthal and Zippin [26] established an important relationship between bases and the BAP.

**Theorem 2.9.** *A Banach space  $X$  has the bounded approximation property if and only if  $X$  is isomorphic to a complemented subspace of a Banach space with a basis.*

Putting this altogether, we get the classification of atomic decompositions in terms of the approximation property for Banach spaces.

**Theorem 2.10.** *For a Banach space  $X$ , the following are equivalent:*

- (1)  $X$  has an atomic decomposition.
- (2)  $X$  has a finite dimensional expansion of the identity.
- (3)  $X$  is complemented in a Banach space with a basis.

(4)  $X$  has the bounded approximation property.

*Proof.*

(1)  $\Leftrightarrow$  (3): This is Theorem 2.6 above.

(3)  $\Leftrightarrow$  (4): This is Theorem 2.9 above.

(2)  $\Rightarrow$  (4): If  $(A_i)$  is a finite dimensional expansion of the identity on  $X$ , then

$$T_n = \sum_{i=1}^n A_i,$$

is a sequence of finite rank operators on  $X$  with the property that  $T_n x \rightarrow x$ , for all  $x \in X$ . Hence,  $X$  has the BAP.

(4)  $\Rightarrow$  (2): If  $(T_n)$  is a sequence of finite rank operators on  $X$  with  $T_n x \rightarrow x$ , for all  $x \in X$ , then it is immediate that  $A_n = T_{n+1} - T_n$  is a finite dimensional expansion of the identity on  $X$ .  $\square$

### 3. FRAMES FOR BANACH SPACES

In this section, we will give three natural formulations for the notion of a frame for a Banach space. Then, we will show that all three definitions give the same “frames”. Finally, we will show that these definitions also are related to some forms of the approximation property and were studied in Banach space theory over twenty-five years ago. Our first candidate will come from a characterization of normalized tight frames due to Han and Larson [22]. This characterization also exhibits the geometric nature of Hilbert space frames.

**Theorem 3.1.** *A sequence  $(f_i)_{i \in I}$  in a Hilbert space  $H$  is a normalized tight frame for  $H$  if and only if there is a Hilbert space  $H \subset K$  and an orthonormal basis  $(e_i)_{i \in I}$  for  $K$  so that if  $P$  is the orthogonal projection of  $K$  onto  $H$  then  $Pe_i = f_i$ , for all  $i \in I$ .*

This result is generalized in Proposition 6.8 to:

**Theorem 3.2.** *A sequence  $(f_i)_{i \in I}$  is a frame for a Hilbert space  $H$  if and only if there is a Hilbert space  $H \subset K$  with an orthonormal basis  $(e_i)_{i \in I}$  and a projection  $P : K \rightarrow H$  with  $Pe_i = f_i$ , for all  $i \in I$ .*

So the difference between a frame and a normalized tight frame is just whether we can build a containing Hilbert space so that the orthogonal projection down onto our space maps an orthonormal basis to our sequence, or whether we can just find some projection (not necessarily an orthogonal projection) with this property.

If  $(x_i, x_i^*)$  is a Schauder basis for a Banach space  $X$  then it is easily checked that  $(x_i), (x_i^*)$  is an atomic decomposition of  $X$  with respect to the associated Banach

space of scalar valued sequences  $X_d$ . We see then that, even for a Hilbert space  $H$ , an atomic decomposition need not be a frame for  $H$  since an atomic decomposition does not require the  $x_i$  to be bounded above in norm and it does not require that the convergence given in Definition 2.1 (c) be unconditional. We will now use Theorems 3.1 and 3.2 as a definition for a frame in a Banach space. This will allow us to recapture the unconditional convergence of the frame representation.

**Definition 3.3.** *A sequence  $(x_i)_{i \in \mathbb{N}}$  in a Banach space  $X$  is a **frame** for  $X$  if there is a Banach space  $Z$  with an unconditional basis  $(z_i, z_i^*)$  with  $X \subset Z$  and a (onto) projection  $P : Z \rightarrow X$  so that  $Pz_i = x_i$  for all  $i \in \mathbb{N}$ . If  $(z_i)$  is a 1-unconditional basis for  $Z$  and  $\|P\| = 1$ , we will call  $(x_i)$  a **normalized tight frame** for  $X$ .*

In this case, we have for all  $x \in X$  that

$$x = \sum_i \langle x, z_i^* \rangle z_i = Px = \sum_i \langle x, z_i^* \rangle Pz_i = \sum_i \langle x, z_i^* \rangle x_i,$$

and this series converges unconditionally in  $X$ . So this definition recaptures the unconditional convergence from the Hilbert space definition. Normalized tight frames in Banach spaces have some surprising properties which we will discuss in section 5. Note that our definition also does not require that the frame elements be bounded above in norm. So even in a Hilbert space a frame in the sense of Definition 3.3 need not be a standard frame since the compression of an unbounded Riesz basis for a Hilbert space  $H$  is not a Hilbert space frame but is a frame in the sense of Definition 3.3. We will see in Example 3.9 that even a bounded sequence satisfying Definition 3.3 need not be a standard frame for a Hilbert space.

With this definition of frame, we can quickly derive many of the standard properties of frames for a Hilbert space. Since a projection on a Banach space is one to one if and only if it is the identity operator, we have immediately that a frame for a Banach space is  $\omega$ -independent if and only if it is an unconditional basis for  $X$ .

Recall that if  $(f_i)$  is a frame for  $H$  with frame transform  $S$  then we get a **reconstruction operator**  $S^{-1}$  satisfying:

$$f = \sum_i \langle f, S^{-1}f_i \rangle f_i,$$

and this series converges unconditionally for all  $f \in H$ . Our next definition for a frame in a Banach space comes naturally from this representation.

**Definition 3.4.** *A **framing** for a Banach space  $X$  is a pair of sequences  $(x_i, y_i)$  with  $x_i \in X$  and  $y_i \in X^*$  satisfying for all  $u \in X$ ,*

$$u = \sum_i \langle u, y_i \rangle x_i,$$

*and this series converges unconditionally for all  $u \in X$ .*

Our final version of a frame in a Banach space comes from unconditionalizing Theorem 2.6.

**Definition 3.5.** A framing model is a Banach space  $Z$  with a fixed unconditional basis  $(e_i)$  for  $Z$ . A framing modeled on  $(Z, (e_i)_{i \in J})$  for a Banach space  $X$  is a pair of sequences  $(y_i)$  in  $X^*$  and  $(x_i)$  in  $X$  so that the operator  $\theta : X \rightarrow Z$  defined by

$$\theta u = \sum_{i \in J} \langle u, y_i \rangle e_i,$$

is an into isomorphism and  $\Gamma : Z \rightarrow X$  given by

$$\Gamma\left(\sum_{i \in J} a_i e_i\right) = \sum_{i \in J} a_i x_i,$$

is bounded and  $\Gamma\theta = I_X$ .

In this setting,  $\Gamma\theta$  becomes the reconstruction operator for the frame. The first thing we want to show is that these three methods for defining a frame on a Banach space are really all the same.

**Theorem 3.6.** Let  $X$  be a Banach space and  $(x_i)$  be a sequence of elements of  $X$ . The following are equivalent:

- (1)  $(x_i)$  is a frame for  $X$ .
- (2) There is a sequence  $y_i \in X^*$  so that  $(x_i, y_i)$  is a framing for  $X$ .
- (3) There is a sequence  $y_i \in X^*$  and a framing model  $(Z, (e_i))$  so that  $(x_i, y_i)$  is a framing modeled on  $(Z, (e_i))$ .

*Proof.* (1)  $\Rightarrow$  (3): Since  $(x_i)$  is a frame for  $X$ , there is a Banach space  $Z$  with a unconditional basis  $(e_i, e_i^*)$  and a bounded linear projection  $P : Z \rightarrow X$  so that  $Pe_i = x_i$ . Now, letting  $y_i = P^*e_i^*$  we have for all  $u \in X$ ,

$$\sum_i \langle u, y_i \rangle x_i = \sum_i \langle u, P^*e_i^* \rangle Pe_i = \sum_i \langle u, e_i^* \rangle Pe_i = \sum_i \langle u, e_i^* \rangle e_i = u.$$

So if we let  $\theta$  be the injection  $i : X \rightarrow Z$  and let  $\Gamma = P$ , we see that  $(Z, x_i, y_i)$  is a framing modeled on  $(Z, (e_i))$  for  $X$ .

(3)  $\rightarrow$  (2): This is obvious by the definitions.

(2)  $\Rightarrow$  (1): First we need to construct our framing model. To do this, define a new Banach space  $C_X$  by:

$$(3.2) \quad C_X = \{(a_i) : \sup_{\epsilon_i = \pm 1} \left\| \sum_i \epsilon_i a_i x_i \right\|_X < \infty\},$$

with the norm on  $C_X$  being given by:

$$\|(a_i)\| = \sup_{\epsilon_i = \pm 1} \left\| \sum_i \epsilon_i a_i x_i \right\|_X.$$

Let  $(e_i)$  be the natural unit vectors in  $C_X$  and let  $Z$  denote the Banach space spanned by the  $e_i$ . Now,  $Z$  is just the sequences of scalars for which the sum

$\sum a_i x_i$  converges unconditionally in  $X$ . Note that  $\|e_i\| = \|x_i\|$ , so it is possible that some of the  $e_i$  are zero. We will adjust for this fact at the end of the proof. Now, we define:  $\theta : X \rightarrow Z$  by

$$\theta u = (\langle u, y_i \rangle) = \sum_i \langle u, y_i \rangle e_i.$$

Also, define a map  $\Gamma : Z \rightarrow X$  by  $\Gamma(a_i) = \sum_i a_i x_i$ . This series clearly converges - even unconditionally - in  $X$ . Also,  $\Gamma\theta = I_X$ , and so  $P = \theta\Gamma$  is a projection from  $Z$  onto  $\theta X$ . So this produces our frame model. Also, we get immediately that

$$Pe_i = \theta x_i,$$

and

$$P^* e_i^* = (\theta^*)^{-1} y_i,$$

since, for all  $\theta x \in \theta X$ , we have

$$\begin{aligned} \langle P^* e_i^*, \theta x \rangle &= \langle e_i^*, P\theta x \rangle = \langle e_i^*, \theta x \rangle \\ &= \langle x, y_i \rangle = \langle \theta x, (\theta^*)^{-1} y_i \rangle. \end{aligned}$$

The technical problem we jumped over in the middle of the proof is handled exactly the same as we did in the proof of Theorem 2.6.  $\square$

Combining Theorem 3.6 with results of Pelczynski and Wojtaszczyk [34], Pelczynski [33] and Johnson, Rosenthal and Zippin [26], we have:

**Proposition 3.7.** *For a Banach space  $X$ , the following are equivalent:*

- (1)  $X$  has a frame.
- (2)  $X$  has a unconditional finite dimensional expansion of the identity.
- (3)  $X$  is complemented in a Banach space with a unconditional basis.

One advantage of the more abstract notion of a framing is that some results which become tedious in the more rigid definition of a framing modeled on a particular space, become elementary in this setting. One such result is the invariance of these notions under isomorphisms on the space.

**Proposition 3.8.** *If  $(x_i), (y_i)$  is a framing for  $X$ , and  $S$  is an isomorphism of  $X$  onto itself, then  $(Sx_i), ((S^{-1})^* y_i)$  is a framing.*

*Proof.* For any  $u \in X$ ,

$$u = SS^{-1}u = S \sum_i \langle S^{-1}u, y_i \rangle x_i = \sum_i \langle u, (S^{-1})^* y_i \rangle Sx_i.$$

$\square$

We end this section by considering a natural question: Is every framing for a Hilbert space  $H$  a Hilbert space frame for  $H$ ? The answer is *no* as the following example shows.

**Example 3.9.** *There is a bounded framing for a Hilbert space  $H$  which is not a standard frame for  $H$ .*

*Proof.* Fix any  $p \neq 2$  and any natural number  $n$ . Consider  $\ell_p^{2^n}$  with unit vectors  $(e_i^n)_{i=1}^{2^n}$ . We denote the Rademachers in  $\ell_p^{2^n}$  by  $(r_i)_{i=1}^{2^n}$  where

$$r_i = \frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} e_j,$$

and  $\epsilon_{ij} = \pm 1$  are appropriately chosen. The Rademachers span a good complemented copy of  $\ell_2^n$  inside  $\ell_p^{2^n}$ . Denote the span of the Rademachers by  $H_i$ . The projection  $P$  of  $\ell_p^{2^n}$  onto  $H_i$  is given by the dual Rademachers:

$$r_i^* = \frac{1}{2^{n/q}} \sum_{j=1}^{2^n} \epsilon_{ij} e_j^*.$$

So,

$$Pe_i = \frac{1}{2^{n/q}} \sum_{j=1}^n \epsilon_{ij} e_i^*(e_i) r_j = \frac{1}{2^{n/q}} \sum_{j=1}^n \epsilon_{ij} r_j.$$

Hence,

$$\sum_{i=1}^{2^n} |\langle r_1, Pe_i \rangle|^2 = \frac{1}{2^{2n/q}} \sum_{i=1}^{2^n} 1 = 2^{n(1-2/q)}.$$

Since  $\|r_i\| = 1$ , in order that  $(Pe_i)$  be a Hilbert space frame for  $H_i$  with frame bounds  $A, B$ , we must have

$$A \leq \sum_{i=1}^{2^n} |\langle r_1, Pe_i \rangle|^2 \leq B,$$

From our previous calculation, we see that this can occur only if

$$(3.3) \quad A \leq 2^{n(1-2/q)} \leq B.$$

Now we do the above construction for each  $n = 1, 2, 3, \dots$  to obtain 1-unconditional bases  $(e_i^n)_{i=1}^{2^n}$  for  $\ell_p^{2^n}$  and projections  $P_i : \ell_p^{2^n} \rightarrow H_i$  as above. Let

$$Z = \left( \sum_{n=1}^{\infty} \oplus \ell_p^{2^n} \right)_{\ell_2}$$

Now,

$$\begin{aligned} Z &\cong \left( \sum_{n=1}^{\infty} \oplus [\ell_2^{2^n} \oplus \ell_p^{2^n-n}] \right)_{\ell_2} \cong \left( \sum_{n=1}^{\infty} \oplus \ell_2^{2^n} \oplus \sum \ell_p^{2^n-n} \right)_{\ell_2} \\ &\cong \left( \sum_{n=1}^{\infty} \oplus \ell_2^{2^n} \right)_{\ell_2} \oplus \left( \sum_{n=1}^{\infty} \oplus \ell_p^{2^n-n} \right)_{\ell_2} \cong H \oplus \left( \sum_{n=1}^{\infty} \ell_p^{2^n-n} \right)_{\ell_2}. \end{aligned}$$

Now,  $(Pe_i^n \oplus (I-P)e_i^n)$ , for  $n = 1, 2, 3, \dots$  and  $i = 1, 2, \dots, 2^n$  is an unconditional basis for  $Z$  so  $(Pe_i^n)$  is a frame for  $H$ . But, by inequality (3.3), this is not a Hilbert space frame for  $H$ . It is a non-Hilbertian frame.  $\square$

#### 4. ALTERNATE DUAL FRAMES

Now we introduce the notion of an alternate dual frame for a frame in a Banach space. In Hilbert spaces alternate dual frames were studied by Han and Larson [22] and have also been called **pseudo-duals** in the literature.

Given two sequences  $(x_i)$  and  $(y_i)$  in Banach spaces  $X$  and  $Y$ , we say that  $(x_i)$  **dominates**  $(y_i)$  if  $Tx_i = y_i$  is a well defined bounded linear operator.

**Definition 4.1.** Let  $Z$  be a Banach space with a unconditional basis  $(e_i, e_i^*)$ ,  $X \subset Z$  and  $P$  a bounded linear projection of  $Z$  onto  $X$  so that  $(Pe_i) = (x_i)$  is a frame in  $X$ . A sequence  $(y_i)$  in  $X^*$  which is dominated by  $(e_i^*)$  is an **alternate dual frame** of the frame if for every  $x \in P(X)$ , we have

$$x = \sum_i \langle x, y_i \rangle x_i.$$

We always have one natural alternate dual frame for any frame.

**Proposition 4.2.** Let  $Z$  be a Banach space with a unconditional basis  $(e_i, e_i^*)$ ,  $X \subset Z$  and  $P$  a bounded linear projection of  $Z$  onto  $X$  so that  $(Pe_i) = (x_i)$  is a frame in  $X$ . Then  $(P^*e_i^*)$  is an alternate dual frame for the frame.

*Proof.* For any  $x \in P(X)$ ,

$$\begin{aligned} x &= \sum_i \langle x, e_i^* \rangle e_i = P(x) = \sum_i \langle Px, e_i^* \rangle x_i = \\ & \sum_i \langle x, P^*e_i^* \rangle x_i. \end{aligned}$$

□

The point of Theorem 4.6 below is that every alternate dual arises in this fashion. In a Hilbert space,  $P$  can be taken to be an orthogonal projection (see Theorem 7.3 below).

In Section 7 we show that any alternate dual pair  $(x_i, y_i)$  on a Hilbert space  $H$  can be dilated to a Riesz basis dual pair  $(u_i, u_i^*)$ . More specifically there is a Hilbert space  $K \supset H$  and a Riesz basis  $(u_i)$  for  $K$  such that  $x_i = Pu_i$  and  $y_i = Pu_i^*$ , where  $P$  is the orthogonal projection from  $K$  onto  $H$  and  $(u_i^*)$  is the (unique) dual Riesz basis for  $(u_i)$ . We will prove in this section that this is also true in the Banach space setting. For this purpose we need to introduce the concept of strongly disjoint frames which was introduced and studied in [22] for the Hilbert space frame case.

**Definition 4.3.** If  $X^*$  and  $Y$  are Banach spaces and  $(x_i)$  in  $X^*$  and  $(y_i)$  in  $Y$ , we say that  $(x_i)$  and  $(y_i)$  are **strongly disjoint** if for every  $x \in X$  we have:

$$\sum_i \langle x, x_i \rangle y_i = 0.$$

Note that if  $(x_i)$  is in  $X$ , by considering  $X \subset X^{**}$ , we have that  $(x_i)$  is strongly disjoint from  $(y_i)$  if for every  $x^* \in X^*$  we have

$$\sum_i \langle x^*, x_i \rangle y_i = 0.$$

**Proposition 4.4.** *If  $((x_i), (y_i))$  is a framing modeled on  $(Z, (e_i))$  and  $((z_i), (w_i))$  is a framing for  $(W, (f_i))$ , and  $(x_i), (w_i)$  and  $(y_i), (z_i)$  are strongly disjoint, then  $x_i \oplus z_i$  and  $y_i \oplus w_i$  are alternate dual frames for  $Z \oplus W$ . Hence, if  $(x_i \oplus z_i)$  is a unconditional basis, then  $(y_i \oplus w_i)$  is a unconditional basis which is the dual basis.*

*Proof.* For any  $(x, y) \in Z \oplus W$  we have:

$$\begin{aligned} \sum_i \langle x \oplus y, y_i \oplus w_i \rangle x_i \oplus z_i &= \sum_i (\langle x, y_i \rangle + \langle y, w_i \rangle) x_i \oplus z_i = \\ &= \sum_i \langle x, y_i \rangle x_i \oplus 0 + 0 \oplus \sum_i \langle y, w_i \rangle z_i = x \oplus y. \end{aligned}$$

Moreover, the above series converge unconditionally, which is all we need.  $\square$

The proof of the classification of the dilation property for alternate dual frames is somewhat technical. To simplify it a little, we first prove a lemma.

**Lemma 4.5.** *Let  $X$  be a Banach space and  $P, Q$  be projections on  $X$ . If  $QP = Q$  (so  $Q|_{PX} : PX \rightarrow PX$  is an onto isomorphism) then there is a  $C$  so that for any  $x \in PX$  and any  $y \in (I - Q)X$  we have:  $\|x + y\| \geq C \max(\|x\|, \|y\|)$ . Finally,  $(I - Q)|_{(I - P)X} : (I - P)X \rightarrow (I - Q)X$  is an onto isomorphism.*

*Proof.* For any  $x \in PX$ ,  $x = Qx + (I - Q)x$ , and since  $Q|_{PX}$  is an isomorphism, there is a constant  $B$  so that  $\|x\| \leq B\|Qx\|$ . Hence, for any  $y \in (I - Q)X$ , if  $\|x + y\| = 1$ , then we have two possibilities:

**Case I.**  $\|Qx\| \geq \frac{1}{4\|I - P\|}$ .

In this case we have

$$\|x + y\| = \|Qx + [(I - Q)x + y]\| \geq D\|Qx\| \geq \frac{D}{4\|I - P\|} = \frac{D}{4\|I - P\|} \|x + y\|.$$

**Case II.**  $\|Qx\| \leq \frac{1}{4\|I - P\|}$ .

In this case,

$$1 = \|x + y\| \geq \|y\| - \|x\| \geq \|y\| - B\|Qx\| \geq \|y\| - \frac{B}{4\|I - P\|}.$$

So,

$$\|y\| \leq 1 + \frac{B}{4\|I - P\|} = \frac{B + 4\|I - P\|}{4\|I - P\|} = A.$$

That is,

$$\|x + y\| \geq \frac{1}{A} \|y\|.$$

Hence,  $\text{span}(PX, (I - Q)X) \cong PX \oplus (I - Q)X$  is a direct sum.

Finally,

$$(I - Q)(I - P) = I - Q - P + QP = I - Q - P + Q = I - P.$$

Hence,  $I - Q$  is an isomorphism of  $(I - P)X$  onto  $(I - Q)X$ .  $\square$

Now we are ready to prove our main result.

**Theorem 4.6.** *Let  $((x_i), (y_i))$  be a framing modeled on  $(Z, (e_i))$  with  $\theta, \Gamma$  as in the definition and let the projection  $P = \theta\Gamma$  be the projection of  $Z$  onto  $\theta X$ . Suppose  $((x_i), (z_i))$  is also a framing. Then there is a Banach space  $Y \cong (I - P)Z$  and an unconditional basis  $(u_i), (u_i^*)$  for  $Z \cong X \oplus Y$  of the form:*

$$u_i = x_i \oplus w_i,$$

and

$$u_i^* = z_i \oplus v_i,$$

with  $w_i \in Y, v_i \in Y^*$ .

*Proof.* We let  $\Gamma, \theta$  be given by the framing model. We proceed in steps.

**Step I.** *The map  $\theta_z : X \rightarrow Z$  given by:  $\theta_z x = \sum_i \langle x, z_i \rangle e_i$  is an into isomorphism taking  $X$  to a complemented subspace of  $Z$ . Moreover,  $Q = \theta_z \Gamma$  is the projection of  $Z$  onto this subspace, and so  $Q\theta = \theta_z$ . Hence,  $QP = Q$ . Finally,  $(z_i)$  is strongly disjoint from  $((I - Q)e_i)$ .*

Since  $(x_i), (z_i)$  is a framing we have for every  $x \in X$ ,

$$x = \sum_i \langle x, z_i \rangle x_i,$$

and this series converges unconditionally. Therefore, the map  $\theta_z$  is an into isomorphism and  $\Gamma\theta_z = I$  since for all  $x \in X$ ,

$$\Gamma\theta_z x = \Gamma \sum_i \langle x, z_i \rangle e_i = \sum_i \langle x, z_i \rangle x_i = x.$$

Hence,  $Q = \theta_z \Gamma$  is a projection of  $Z$  onto  $\theta_z X$ . Now,  $Q\theta = \theta_z \Gamma\theta = \theta_z I$ . It follows that that

$$QP = \theta_z \Gamma\theta\Gamma = \theta_z I\Gamma = Q.$$

Finally, for any  $x \in X$  we have

$$\sum_i \langle x, z_i \rangle (I - Q)e_i = (I - Q) \left( \sum_i \langle x, z_i \rangle e_i \right) = (I - Q)\theta_z x (I - Q)Q\theta_z x = 0.$$

So  $(z_i)$  is strongly disjoint from  $((I - Q)e_i)$ .

**Step II.**  $(I-Q)|_{(I-P)Z} : (I-P)Z \rightarrow (I-Q)Z$  is an onto isomorphism. Therefore, letting  $A = ((I-Q)|_{(I-P)Z})^{-1}$ , we have that  $A$  is an isomorphism of  $(I-Q)Z$  onto  $(I-P)Z$  and  $(I-Q)A = I|_{(I-Q)Z}$ .

By Step I,  $Q|_{PZ} : PZ \rightarrow QZ$  satisfies  $Q\theta = \theta_z$  (Since  $PZ = \theta X$ ), and is therefore an isomorphism onto. We get Step II by applying Lemma 4.5.

**Step III.**  $(x_i)$  is strongly disjoint from  $(A^*(I-P)^*e_i^*)$ .

For all  $x \in X$  we have,

$$\begin{aligned} \sum_i \langle x, A^*(I-P)^*e_i^* \rangle x_i &= \sum_i \langle Ax, (I-P)^*e_i^* \rangle x_i = \\ \Gamma\left(\sum_i \langle (I-P)Ax, e_i^* \rangle e_i\right) &= \Gamma((I-P)Ax) = 0, \end{aligned}$$

since

$$\Gamma(I-P) = \Gamma(I - \theta\Gamma) = \Gamma - \Gamma\theta\Gamma = \Gamma - \Gamma = 0.$$

**Step IV.**  $((I-Q)e_i)$  and  $(A^*(I-P)^*e_i^*)$  are alternate dual frames.

For all  $x \in (I-Q)Z$  we have,

$$\sum_i \langle x, A^*(I-P)^*e_i^* \rangle (I-Q)e_i = (I-Q) \sum_i \langle Ax, (I-P)^*e_i^* \rangle e_i = (I-Q)Ax = x,$$

since  $((I-P)^*e_i^*)$  is an alternate dual frame to  $((I-P)e_i)$ .

It follows from Steps I-IV and Proposition 4.4 that  $((u_i), (u_i^*))$  is a frame model.

**Step V.** The sequence  $(u_i)$  is an unconditional basis.

By lemma 4.5, we know that  $(u_i)$  spans  $X \oplus Y$ . So we only need to show that this sequence is  $\omega$ -independent. Suppose then that

$$\sum_i a_i u_i = \sum_i a_i x_i \oplus \sum_i a_i (I-Q)e_i = 0.$$

Then,

$$\sum_i a_i x_i = 0 = \sum_i a_i (I-Q)e_i.$$

Hence,

$$\Gamma\left(\sum_i a_i e_i\right) = \sum_i a_i x_i = 0.$$

Thus,

$$\theta\Gamma\left(\sum_i a_i e_i\right) = \sum_i a_i P e_i = 0.$$

Now, applying Step I,

$$0 = Q \sum_i a_i P e_i = \sum_i a_i Q P e_i = \sum_i a_i Q \theta x_i = \sum_i a_i \theta_z x_i = \sum_i a_i Q e_i.$$

Finally, this yields,

$$\sum_i a_i e_i = \sum_i a_i Q e_i + \sum_i a_i (I - Q) e_i = 0.$$

Since  $(e_i)$  is an unconditional basis, we have that  $a_i = 0$ , for all  $n$ . Therefore,  $(u_i)$  is  $\omega$ -independent.

The proof of Theorem 4.6 is now complete.  $\square$

The following special case of Theorem 4.6 generalizes our dilation property for alternate duals of Hilbert space frames given in Theorem 7.3.

**Corollary 4.7.** *Let  $(y_i)$  be an alternate dual for a frame  $(x_i)$  on a Banach space  $X$ . Then there is a Banach space  $Y$  and an unconditional basis  $(u_i)$  for  $X \oplus Y$  such that  $x_i = P u_i$  and  $y_i = P^* u_i^*$ , where  $P$  is the projection from  $X \oplus Y$  onto  $X$  and  $(u_i^*)$  is the dual unconditional basis of  $(u_i)$ .*

**Remark 4.8.** *An examination of the proof of Theorem 4.6 shows that a **bounded frame** is an inner direct summand of a bounded unconditional basis for a larger Banach space.*

## 5. FRAMES AND THE APPROXIMATION PROPERTY

We conclude our study of Banach space frames by considering the relationship between frames and atomic decompositions and the various forms of the approximation properties. This will also help define which Banach spaces can have these various frame notions. First we need two more definitions.

**Definition 5.1.** *A Banach space  $X$  is said to have the **approximation property** (A.P. for short) if, for every compact set  $K$  in  $X$  and every  $\epsilon > 0$ , there is an operator  $T : X \rightarrow X$  of finite rank so that  $\|Tx - x\| \leq \epsilon$ , for every  $x \in K$ .*

It is clear that a complemented subspace of a Banach space with the A.P. also has the A.P.

**Definition 5.2.** *A sequence  $(E_i)_{n=1}^\infty$  of finite dimensional subspaces of a Banach space  $X$  is called a **finite dimensional decomposition** (FDD) for  $X$  if for each  $x \in X$  there is a unique sequence of vectors  $x_i \in E_i$  so that*

$$(5.1) \quad x = \sum_i x_i.$$

In this case we write:

$$(5.2) \quad X = \sum_{n=1}^{\infty} \oplus E_i.$$

If the convergence in (5.1) is unconditional, then we call (5.2) an **unconditional finite dimensional decomposition (UFDD)** of  $X$ .

We refer the reader to [3,5] for a detailed treatment of FDD's for Banach spaces. So a basis is a FDD for  $X$  with  $\dim E_i = 1$ , for all  $i$ , and an unconditional basis for  $X$  is a UFDD of this type. If  $\sum \oplus E_i$  is a finite dimensional decomposition for  $X$ , we have a natural sequence  $P_n$  of commuting finite rank projections on  $X$  given by:

$$P_n \left( \sum_{k=1}^{\infty} x_k \right) = \sum_{k=1}^n x_k.$$

We call  $K = \sup_n \|P_n\|$  the **FDD constant** of  $X$ , and the projections  $Q_n = P_n - P_{n-1}$  the **coordinate projections**. The projections  $P_n$  show that a space with a finite dimensional decomposition also has the bounded approximation property. Also, a space with the bounded approximation property has the approximation property.

**Definition 5.3.** A Banach space  $X$  is said to have *GL-Lust* (For Gordon-Lewis Local Unconditional Structure [21]) if there is a constant  $K > 0$  so that for every finite dimensional subspace  $E \subset X$  there is a Banach space  $F$  with a 1-unconditional basis and operators  $A : E \rightarrow F$ ,  $B : F \rightarrow X$  with  $BAx = x$ , for all  $x \in E$ , and  $\|A\| \|B\| \leq K$ .

There are several different forms of local unconditional structure for Banach spaces. We refer the reader to [25] for an analysis of their interrelationships. If a Banach space has an unconditional basis then it has GL-Lust. In fact, if a Banach space  $X$  is complemented in a Banach space with an unconditional basis then  $X$  has GL-Lust. The converse fails since  $L^1[0,1]$  has GL-Lust but fails to have an unconditional basis.

We will now answer the obvious questions concerning which Banach spaces have frames or atomic decompositions.

**Remark 5.4.** *There are Banach spaces - even Banach lattices - which do not have frames or atomic decompositions defined on them.*

It is known [36] that there are Banach spaces (even Banach lattices) which fail the approximation property. These spaces cannot have frames or atomic decompositions since they cannot be complemented in a space with a basis - or even the A.P.

**Remark 5.5.** *There are Banach spaces which have atomic decompositions, but no frames.*

The simplest examples for this are  $L^1[0, 1]$  or  $C[0, 1]$ . Both these spaces have bases and hence atomic decompositions, but neither of them embeds as a complemented subspace of a Banach space with a unconditional basis [30,31].

**Remark 5.6.** *There are non-Hilbert spaces for which every subspace has an atomic decomposition. It is a famous open problem in Banach space theory whether there are non-Hilbert spaces for which every subspace has a frame.*

It is a result of Johnson (see [6]) that convexified Tsirelson's space  $T^2$  is a Banach space with the property that every subspace of every quotient space has a basis - and hence an atomic decomposition. However, there are subspaces of  $T^2$  which fail GL-Lust [30] and hence do not embed complementably into any Banach space with a unconditional basis. So these subspaces of  $T^2$  cannot have frames. It is an old question in Banach space theory whether a Banach space for which every subspace has a unconditional basis (or just GL-Lust) must be a Hilbert space. The general consensus at this time is that the answer to this question is "yes". If every subspace of  $X$  has a frame, then every subspace of  $X$  would have GL-Lust. So it is likely that every non-Hilbert space has a subspace without a frame.

**Remark 5.7.** *There is a Banach space  $X$  with an atomic decomposition so that  $X^*$  is separable and  $X^*$  fails to have an atomic decomposition. There is a Banach space  $X$  with a frame so that  $X^*$  fails to have an atomic decomposition. But, if a Banach space  $X$  has a frame and  $X^*$  is separable, then  $X^*$  has a frame. If  $X^*$  has an atomic decomposition, then so does  $X$ . There is a Banach space  $X$  so that  $X^*$  has a frame, but  $X$  does not have a frame.*

It is known (see [31], Theorem 1.e.7) that there is a Banach space  $X$  with a basis whose dual is separable but fails the approximation property. This is then a space with an atomic decomposition and a separable dual which fails to have any atomic decomposition. The space  $\ell_1$  has a unconditional basis (hence a frame) while its dual is non-separable and hence has no atomic decompositions. It is known (see [25]) that if  $X^*$  has BAP then so does  $X$ . Hence, if  $X^*$  has an atomic decomposition, then so does  $X$ . If  $X$  has a frame and  $X^*$  is separable, we can show directly that  $X^*$  has a frame. Let  $Z$  be a Banach space with an unconditional 1-dimensional expansion of the identity  $(A_i)$  and let  $P$  be a projection from  $Z$  onto  $X$ . Now  $(PA_i)$  is a frame for  $X$  and  $((PA_i)^*)$  is a frame for  $X^*$  (This is essentially the Orlicz-Pettis Theorem). Any pre-dual of  $\ell_1$  which is complemented in a space with an unconditional basis must be isomorphic to  $c_0$  (see [27], Theorem 4.3). So if  $X$  is a pre-dual of  $\ell_1$  not isomorphic to  $c_0$ , then  $X$  is a Banach space without any frames whose dual has a unconditional basis (hence a frame).

**Remark 5.8.** *There are Banach spaces with atomic decompositions which do not have bases. It is an open question in Banach space theory whether there are Banach spaces with frames which do not have unconditional bases.*

The first part follows from a result of Szarek [37] that there are Banach spaces with finite dimensional decompositions which do not have bases. The second part is a reformulation of the famous open problem: If  $X$  is a complemented subspace of a Banach space with a unconditional basis, must  $X$  have a unconditional basis? This is the “atomic lattice” form of the open problem: Is every complemented subspace of a Banach lattice isomorphic to a Banach lattice? However, there is one case where the answer to this problem is known. Kalton and Wood ([29], Corollary 5.5) proved a fundamental result about rank 1 Hermitian operators which implies (See [16,34] for a detailed analysis of this construction) that a norm one complemented subspace of a *complex* Banach space  $Z$  with a 1-unconditional basis, must have a 1-unconditional basis. In the language of frames this becomes:

**Remark 5.9.** *Let  $X$  is a complex Banach space. Then  $X$  has a normalized tight frame if and only if  $X$  has a 1-unconditional basis. There is a real Banach space with a normalized tight frame which does not have a 1-unconditional basis.*

Benyamini, Flinn and Lewis [22] have shown that there is a real Banach space  $X$  with a norm one complemented subspace  $Y$  while  $Y$  fails to have a 1-unconditional basis. So this is a situation which differs from the Hilbert space case. That is, there exists a real Banach space with a 1-unconditional basis and a norm one projection defined on it which takes the 1-unconditional basis to a normalized tight frame, but the range of the projection does not have any 1-unconditional bases on it.

**Remark 5.10.** *It is an open question in Banach space theory whether a frame for a complex Banach space is equivalent to  $s$  normalized tight frame.*

There is an obvious approach for trying to generalize the Kalton, Wood theorem to arbitrary projections on a complex Banach space. That is, if  $X$  is a complemented subspace of a complex Banach space with a unconditional basis, is  $X$  norm one complemented in some other Banach space with a 1-unconditional basis? This is an open question at this time, and is equivalent to the question in Remark 5.10.

**Remark 5.11.** *There are Banach spaces (even complex ones) with frames but which fail to have normalized tight frames.*

The beginning construction of the construction of Gowers and Maurey [20] consists in producing, for each  $K > 0$ , a Banach space  $X$  which has a unconditional basis but  $X$  contains no sequences which are  $K$ -unconditional bases for their span. By standard “blocking techniques” [29], if  $X$  is an infinite dimensional subspace of a Banach space with a  $K$ -unconditional basis, then for every  $\epsilon > 0$ ,  $X$  contains

a sequence which is a  $(K + \epsilon)$ -unconditional basis for its span. In particular, the spaces of Gowers and Maurey have frames but do not embed (and so do not embed complementably) into any Banach space with a 1-unconditional basis, and hence do not have any tight frames.

**Remark 5.12.** *The Kalton-Peck space  $KP$  [29] is a Banach space which is a unconditional (even symmetric) sum of two dimensional spaces, but fails to have a frame. This space does have a basis and hence an atomic decomposition.*

The space  $KP$  is a Banach space which embeds into a Banach space with a unconditional basis but never embeds complementably into a space with a unconditional basis.

**Remark 5.13.** *There are Banach spaces with atomic decompositions for which no subspace has a frame.*

A Banach space  $X$  is **hereditarily indecomposable** (called an H.I. space) if no subspace  $Z \subset X$  can be written as  $Z \cong E \oplus F$  where both  $E$  and  $F$  are infinite dimensional Banach spaces. Gowers and Maurey [21] constructed the first H.I. spaces, and even so that they have bases. It is immediate that every subspace of a H.I. space is also an H.I. space. It is also immediate that any Banach space with an unconditional basis is not an H.I. space. Since every (complemented) subspace of a space with a unconditional basis must contain further subspaces with unconditional bases, it follows that the H.I. spaces (and hence all of their subspaces) fail to have frames.

## 6. ALTERNATE DUAL FRAMES FOR HILBERT SPACE

Recall that a frame  $(y_i)$  for a Hilbert space  $H$  is called an **alternate dual** frame for a frame  $(x_i)$  for  $H$  if we have the **reconstruction formula**

$$(6.1) \quad x = \sum_n \langle x, y_n \rangle x_n, \quad \text{for all } x \in H.$$

There is a simple method for finding an alternate dual frame for  $(x_n)$ . Choose any  $J \subset N$  for which  $(x_n)_{n \in J}$  is still a frame for  $H$  with frame operator say  $S_J$ . Then for any  $x \in H$ ,

$$x = \sum_{n \in J} \langle x, S_J x_n \rangle x_n.$$

So letting  $y_n = 0$ , for all  $n \in J^c$ , and  $y_n = S_J x_n$ , for all  $n \in J$ , we see that  $(y_n)$  is an alternate dual frame for  $(x_n)$ .

**Proposition 6.1.** *If  $(x_n), (y_n)$  are Hilbertian sequences in a Hilbert space  $H$  satisfying for every  $x \in H$ ,*

$$(6.2) \quad x = \sum_n \langle x, y_n \rangle x_n,$$

*then  $(x_n), (y_n)$  are alternate dual frames.*

*Proof.* It is clear that these are “alternate duals” in the sense of the reconstruction formula, We just need to show that they are actually frames. Since  $(y_n)$  is Hilbertian it follows that for any  $x \in H$ ,  $(\langle x, y_n \rangle) \in \ell_2$ . Hence, by (6.2), the operator  $A : \ell_2 \rightarrow H$  given by  $Ae_n = x_n$  is an onto operator where  $(e_n)$  is an orthonormal basis for  $\ell_2$ . Therefore,  $(x_n)$  is a frame. By symmetry,  $(y_n)$  is a frame.  $\square$

We can generalize Proposition 6.1 to a classification of those frames which are just equivalent to alternate dual frames for a given frame.

**Proposition 6.2.** *Let  $(x_n), (y_n)$  be frames for a Hilbert space  $H$ . Then  $(y_n)$  is equivalent to an alternate dual for  $(x_n)$  if and only if the operator  $S : H \rightarrow H$  given by*

$$Sx = \sum_n \langle x, x_n \rangle y_n$$

*is an invertible operator.*

*Proof.*  $\Rightarrow$  Choose an invertible operator  $T$  on  $H$  so that  $(Ty_n)$  is an alternate dual for  $(x_n)$ . Then for all  $x \in H$ ,

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle Ty_n.$$

Hence,  $S = T^{-1}$  is an invertible operator on  $H$ .

$\Leftarrow$  Given the invertible operator  $S$ , for all  $x \in H$ ,

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle S^{-1}y_n.$$

Hence,  $(S^{-1}y_n)$  is an alternate dual for  $(x_n)$ .  $\square$

The next observation will allow us to work only with normalized tight frames when characterizing the alternate duals for a frame.

**Lemma 6.3.** *If  $(x_n)$  is a frame with alternate dual frame  $(y_n)$ , and  $S$  is an invertible operator  $H$ , then  $(Sx_n)$  is a frame for  $H$  with  $((S^{-1})^*y_n)$  an alternate dual frame.*

*Proof.* Since  $(Sy_n)$  is a frame, we just check for all  $x \in H$ ,

$$\sum_{n=1}^{\infty} \langle x, (S^{-1})^*y_n \rangle Sx_n = S \left( \sum_{n=1}^{\infty} \langle S^{-1}x, y_n \rangle x_n \right) = S(S^{-1}x) = x.$$

□

We say that two frames  $(x_n)$  and  $(y_n)$  are **equivalent** if  $Tx_n = y_n$  extends to a invertible operator on  $H$ . This means that if  $A_1, A_2$  are preframe operators on  $H$ , then the induced frames are equivalent if and only if there is an invertible operator  $T$  on  $H$  so that  $TA_1 = A_2$ . But, this is equivalent to  $A_1^*T^* = A_2^*$ . That is, our two frames are equivalent if and only if  $\text{Rng } A_1^* = \text{Rng } A_2^*$ . We can now give the corresponding classification of alternate dual frames for a given frame.

**Proposition 6.4.** *Let  $(x_n)$  be a normalized tight frame for a Hilbert space  $H$  with frame transform  $\theta$ , and let  $(y_n)$  be a frame with frame transform  $\theta'$ . Let  $P_\theta$  be the orthogonal projection of  $\ell_2$  onto the range of  $\theta H$ . Then  $(y_n)$  is an alternate dual frame for  $(x_n)$  if and only if  $P_\theta\theta' = \theta$ .*

*Proof.* Assume that  $(y_n)$  is an alternate dual frame for  $(x_n)$ . Then, for every  $x \in H$ ,

$$\begin{aligned} \langle \theta x, P_\theta\theta'x \rangle &= \langle \theta x, \theta'x \rangle = \sum_n \langle x, x_n \overline{\langle x, y_n \rangle} \rangle = \\ &= \left\langle \sum_n \langle x, x_n \rangle y_n, x \right\rangle = \langle x, x \rangle = \|x\|^2 = \|\theta x\|^2 = \langle \theta x, \theta x \rangle. \end{aligned}$$

Hence,  $\theta = P_\theta\theta'$ .

Conversely, if  $\theta = P_\theta\theta'$ , then for every  $x, y \in H$ ,

$$\begin{aligned} \langle x, y \rangle &= \langle \theta x, \theta y \rangle = \langle \theta x, P_\theta\theta'y \rangle = \langle \theta x, \theta'y \rangle = \\ &= \sum_n \langle x, x_n \rangle \overline{\langle y, y_n \rangle} = \left\langle \sum_n \langle x, x_n \rangle y_n, y \right\rangle. \end{aligned}$$

Hence,  $x = \sum_n \langle x, x_n \rangle y_n$ . □

By considering frames as just bounded, linear, onto maps from  $\ell_2$  to  $H$ , it is natural to classify alternate dual frames in terms of operators.

**Proposition 6.5.** *Let  $(x_n)$  be a normalized tight frame for a Hilbert space  $H$  with frame transform  $\theta$ , and let  $K = (\theta H)^\perp$ . Then the manifold  $\theta + B(H, K)$  in  $B(H)$  represents the family of all alternate dual frames for  $(x_n)$ .*

*Proof.* First choose  $T \in B(H, K)$ . Then, since the ranges of  $T$  and  $\theta$  are orthogonal subspaces of  $\ell_2$ , for all  $x \in H$  we have

$$\|(\theta + T)x\|^2 = \|\theta x + Tx\|^2 = \|\theta x\|^2 + \|Tx\|^2.$$

Since  $\theta$  is a co-isometry, it follows that  $\theta + T$  is an isomorphism. Hence,  $R = (\theta + T)^*$  defines a frame on  $H$ . To see that this is an alternate dual frame for  $(x_n)$ , we just observe that the frame transform for  $R$  is  $\theta' = R^* = \theta + T$ , and  $\theta H \perp TH$  yields that  $P_\theta\theta' = \theta$ .

Conversely, let  $\theta'$  be the frame transform for an alternate dual frame for  $(x_n)$ . It follows from Proposition 6.3 that  $P_\theta\theta' = \theta$ . Hence,  $P_\theta(\theta' - \theta) = 0$ . i.e.  $\theta' - \theta \in B(H, K)$ , and  $\theta' = \theta + (\theta' - \theta) \in \theta + B(H, K)$ .  $\square$

With the above, we can quickly recover a result from [22].

**Corollary 6.6.** *A frame  $(x_n)$  has a unique alternate dual frame if and only if it is a Riesz basis.*

*Proof.* With the notation as in Proposition 6.5,  $(x_n)$  has a unique alternate dual if and only if  $B(H, K) = 0$ . But  $K = (\theta H)^\perp = 0$  if and only if  $\theta$  is an isomorphism of  $H$  onto  $\ell_2$ . That is,  $\theta^*e_n = x_n$  is an isomorphism taking the unit vector basis of  $\ell_2$  onto  $(x_n)$ .  $\square$

As another consequence of Proposition 6.5, we will prove a new inequality concerning the frame operator for alternate duals of a normalized tight frame.

**Corollary 6.7.** *If  $(x_n)$  is a normalized tight frame and  $(y_n)$  is any alternate dual frame with frame operator  $S$ , then for every  $x \in H$ ,  $\|S^{1/2}x\| \geq \|x\|$ .*

*Proof.* Let  $\theta$  be the frame transform for  $(x_n)$  let and  $\theta + T$  be an alternate dual frame for  $(x_n)$  given by Proposition 6.5. Then  $\text{Rng } T = (\theta H)^\perp = \ker \theta^*$ . Hence,  $T\theta^* = 0 = \theta^*T$ . Now,  $S = (\theta + T)(\theta^* + T^*) = \theta\theta^* + TT^* \geq \theta\theta^*$ . Hence,

$$\begin{aligned} \|S^{1/2}x\|^2 &= \langle S^{1/2}x, S^{1/2}x \rangle = \langle Sx, x \rangle \geq \langle \theta\theta^*x, x \rangle = \\ &= \langle \theta^*x, \theta^*x \rangle = \|\theta^*x\|^2 = \|x\|^2. \end{aligned}$$

$\square$

The next result shows that the alternate dual frames for Hilbert spaces are just given by the projections of  $\ell_2$  onto the range of the frame transform.

**Proposition 6.8.** *Let  $(x_n)$  be a frame for a Hilbert space  $H$  with frame transform  $\theta$ . Then the family of alternate dual frames for  $(x_n)$  are all frames of the form  $(\theta^{-1}Qe_n)$ , where  $Q$  is any projection of  $\ell_2$  onto  $\theta H$ .*

*Proof.* Let  $\theta : H \rightarrow \ell_2$  be the frame transform  $\theta x = \sum_n \langle x, x_n \rangle e_n$ . Now let  $Q$  be a projection of  $\ell_2$  onto  $\theta(H)$  and let  $z_n = \theta^{-1}Qe_n$ , for all  $n = 1, 2, 3, \dots$ . Then for any  $x \in H$  we have

$$\begin{aligned} \sum_n \langle x, x_n \rangle z_n &= \sum_n \langle x, x_n \rangle \theta^{-1}Qe_n = \theta^{-1}Q\left(\sum_n \langle x, x_n \rangle e_n\right) \\ &= \theta^{-1}\sum_n \langle x, x_n \rangle e_n = \theta^{-1}\theta(x) = x. \end{aligned}$$

It follows that  $(z_n)$  is an alternate dual frame for  $(x_n)$ .

Conversely, assume that  $(z_n)$  is an alternate dual frame for  $(x_n)$  and define  $Q : \ell_2 \rightarrow \ell_2$  by  $Qe_n = \theta z_n$ . Since  $(e_n)$  is an orthonormal basis for  $\ell_2$  and  $(z_n)$  (and hence also  $(\theta z_n)$ ) are frames for their spans, it follows that  $Q$  is a bounded linear operator on  $\ell_2$ . Now, for all natural numbers  $m$  we have,

$$\begin{aligned} QQe_m &= Q\theta z_m = Q \sum_n \langle z_m, x_n \rangle e_n = \sum_n \langle z_m, x_n \rangle Qe_n \\ &= \sum_n \langle z_m, x_n \rangle \theta z_n = \theta \sum_n \langle z_m, x_n \rangle z_n = \theta z_m = Qe_m. \end{aligned}$$

Therefore,  $Q = Q^2$  is a bounded linear projection on  $\ell_2$  with  $z_n = \theta^{-1}Qe_n$ .  $\square$

Let us now consider a typical example illustrating Proposition 6.8.

**Example 6.9.** *We consider the normalized tight frame:*

$$\left\{ \frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \dots \right\}.$$

If  $\theta$  is the frame transform and  $P$  is the projection of  $\ell_2$  onto the image of  $\theta$ , then we can check that:

$$P(e_{2n-1}) = P(e_{2n}) = \frac{e_{2n-1} + e_{2n}}{2}.$$

If we define a new projection  $Q$  of  $\ell_2$  onto the range of the frame transform by:  $Qe_{2n-1} = e_{2n-1} + e_{2n}$  and  $Qe_{2n} = 0$ , we get that  $(\theta^{-1}Qe_n)$  is the tight frame:

$$\{\sqrt{2}e_1, 0, \sqrt{2}e_2, 0, \dots\}.$$

A Riesz basis which is not orthogonal, is an example of a frame which has no alternate dual frame which is tight. That is, a (non-orthogonal) Riesz basis has only one alternate dual - and this alternate dual is a non-orthogonal Riesz basis and hence not tight.

In general it is quite difficult to identify when a frame has an alternate dual which is a tight frame. The easiest cases are when there are subsequences of the frame which are well behaved. For example, if a frame contains a Riesz basis, then it has an alternate dual frame which is a Riesz basis (possibly with a sequence of zeroes inserted into it.) If a frame has a subsequence which is a normalized tight frame then it has an alternate dual frame which is a normalized tight frame. Finally, if a frame has a subsequence which is a tight frame, then it has an alternate dual frame which is a tight frame. These results can be easily checked by hand. Also they follow immediately from our next proposition. This proposition is precisely the dualization of Proposition 2.5.

**Proposition 6.10.** *Let  $(x_n)$  is a frame for a Hilbert space  $H$  given by the quotient map  $\theta^* : \ell_2 \rightarrow H$ , where  $\theta^*e_n = x_n$ . Then  $(z_n)$  is a normalized tight frame and an alternate dual frame for  $(x_n)$  if and only if there is a subspace  $E$  of  $\ell_2$  so that  $\theta^*|_E$  is a co-isometry and if  $P$  is the orthogonal projection onto  $E$ , then  $\theta^*Pe_n = z_n$ .*

The general case now looks like: A frame  $(z_n)$  is an alternate dual of  $(x_n)$  if and only if there is an orthogonal projection  $P$  on  $\ell_2$  so that  $\theta^*P$  is an onto map producing a frame  $(\theta^*Pe_n)$  with frame operator  $S$  and  $z_n = S^{-1/2}\theta^*Pe_n$ . We summarize this in the next corollary.

**Corollary 6.11.** *A frame  $(x_n = \theta^*Pe_n)$  has an alternate dual frame which is a tight frame if and only if there is a subspace  $E$  of  $\ell_2$  so that  $\theta^*|_E$  is a constant times a co-isometry. (or equivalently,  $\theta^*Pe_n$  is a tight frame).*

## 7. THE DILATION PROPERTY FOR ALTERNATE DUAL FRAMES

Suppose that  $\{u_n\}$  is a Riesz basis for a Hilbert space  $K \supset H$  with its unique dual  $\{u_n^*\}$ . If  $P$  is the orthogonal projection from  $K$  onto  $H$  then  $\{Pu_n\}$  is a frame for  $H$  with an alternate dual  $\{Pu_n^*\}$ . In general  $\{Pu_n^*\}$  is not the canonical dual for  $\{Pu_n\}$  unless  $P$  commutes with the frame operator of  $\{u_n\}$  (see Proposition 1.15, [22]). So the natural question arising here is to determine when a given frame  $\{x_n\}$  and one of its alternate duals  $\{y_n\}$  can be dilated to a Riesz basis  $\{u_n\}$  for some larger Hilbert space  $K$  so that  $x_n = Pu_n$  and  $y_n = Pu_n^*$ . For a more general setting, let  $\mathcal{F}$  be a family of frames for  $H$ . The general question then is: When can we dilate  $\mathcal{F}$  to a family of Riesz bases for the same larger Hilbert space  $K$ ? In this section we will first prove the dilation theorem for alternate duals which generalizes the dilation result for the single frame case, and then generalize this to families of frames.

The concepts of disjoint and strongly disjoint frames were introduced in [22] (also independently by R. Balan in [1]). (Recall that we generalized these notions to abstract Banach spaces in Definition 4.3). Strongly disjoint n-tuples of frames are also called **super-frames** (see [22]). Let  $\{x_n\}$  and  $\{y_n\}$  be two frames for Hilbert spaces  $H$  and  $K$  with frame transforms  $\theta_1$  and  $\theta_2$ . We say that  $(\{x_n\}, \{y_n\})$  is a **strongly disjoint pair (resp. disjoint pair)** if  $rng(\theta_1)$  and  $rng(\theta_2)$  are orthogonal (resp.  $rng(\theta_1) + rng(\theta_2)$  is a Banach direct sum). Note that  $(\{x_n\}, \{y_n\})$  is a disjoint pair if and only if  $\{x_n \oplus y_n\}$  is a frame for the direct sum space  $H \oplus K$ , while it is a strongly disjoint pair if and only if  $\{x_n \oplus y_n\}$  is a frame for  $H \oplus K$  which is similar to (with similarity given by a diagonal invertible operator  $S \oplus T$ ) a normalized tight frame  $\{u_n \oplus v_n\}$  (see Chapter 2 in [22]). To prove the dilation result, we need the following:

**Lemma 7.1.** *If  $(x_n), (y_n)$  are alternate dual frames for a Hilbert space  $H$  and  $(z_n), (w_n)$  are alternate dual frames for a Hilbert space  $M$  with  $(x_n), (w_n)$  and  $(y_n), (z_n)$  strongly disjoint, then  $x_n \oplus z_n$  and  $y_n \oplus w_n$  are alternate dual frames for  $H \oplus M$ . Hence, if  $x_n \oplus z_n$  is a Riesz basis, then  $y_n \oplus w_n$  is a Riesz basis which is the dual basis.*

*Proof.* Since our sequences are clearly Hilbertian, we only need to check that the reconstruction formula of Proposition 2.1 holds. For any  $(x, y) \in H \oplus M$  we have

$$\begin{aligned} \sum_n \langle x \oplus y, y_n \oplus w_n \rangle x_n \oplus z_n &= \sum_n (\langle x, y_n \rangle + \langle y, w_n \rangle) x_n \oplus z_n = \\ &= \sum_n \langle x, y_n \rangle x_n \oplus 0 + 0 \oplus \sum_n \langle y, w_n \rangle z_n = x \oplus y. \end{aligned}$$

□

We also will need some specific properties of the frame transform and its interaction with the respective frame operators as well as the orthogonal projections onto the range of the frame transform.

**Proposition 7.2.** *Let  $(x_n), (y_n)$  be alternate dual frames for  $H$  with frame operators  $S_x, S_y$  and frame transforms  $\theta_x, \theta_y$  respectively. Let  $P, Q$  be the respective orthogonal projections of  $\ell_2$  onto the range of these frame transforms. Then the following hold:*

- (1)  $Pe_n = \theta_x S_x x_n$ , for all  $n = 1, 2, \dots$ ,
- (2)  $P\theta_y = \theta_x S_x$ . Hence,  $P_{Q\ell_2}$  is an isomorphism of  $Q\ell_2$  onto  $P\ell_2$  and similarly for  $P^\perp$ . Therefore,  $\text{span} \{(P\ell_2), (Q^\perp\ell_2)\} = \ell_2$ .

*Proof.*

- (1) For any  $x \in H$ , recalling that by definition  $S_x^{-1} = \theta_x^* \theta_x$ , we have

$$\begin{aligned} \langle \theta_x x, \theta_x S_x x_n \rangle &= \langle x, \theta_x^* \theta_x S_x x_n \rangle = \langle x, S_x^{-1} S_x x_n \rangle = \\ &= \langle x, x_n \rangle = \langle \theta_x x, e_n \rangle = \langle \theta_x x, Pe_n \rangle. \end{aligned}$$

- (2) For all  $x \in \ell_2$ ,

$$\begin{aligned} P\theta_y x &= P \left( \sum_n \langle x, y_n \rangle e_n \right) = \sum_n \langle x, y_n \rangle Pe_n = \\ &= \sum_n \langle x, y_n \rangle \theta_x S_x x_n = \theta_x S_x \sum_n \langle x, y_n \rangle x_n = \theta_x S_x x. \end{aligned}$$

where we applied (1) in the third equality above. □

Now we are ready for the first dilation result. Although this result was done for general Banach spaces in Theorem 4.6 and Corollary 4.7, the proof there is quite technical and so we feel that it is worthwhile to include a simple proof for the Hilbert space case.

**Theorem 7.3.** *Suppose that  $(x_n), (y_n)$  are alternate dual frames in a Hilbert space  $H$ . Then there is a Hilbert space  $H \subset M$  and a Riesz basis  $(u_n), (u_n^*)$  for  $M$  with  $P_H u_n = x_n$  and  $P_H u_n^* = y_n$ .*

*Proof.* The main idea is to reduce this to the case of Lemma 7.1. To this end, let us use the notation of Proposition 7.2. Our space  $M$  will be  $H \oplus Q^\perp \ell_2$ . We will do this in steps.

**Step I.**  $B =: Q^\perp|_{P^\perp \ell_2}$  is an isomorphism of  $P^\perp \ell_2$  onto  $Q^\perp \ell_2$  with inverse say  $A$ . Therefore,  $Q^\perp A = I_{Q^\perp \ell_2}$ .

By Proposition 7.2 (2),  $Q : P \ell_2 \rightarrow Q \ell_2$  is an isomorphism onto. Hence,  $Q^\perp : P^\perp \ell_2 \rightarrow Q^\perp \ell_2$  is an isomorphism onto.

**Step II.**  $(y_n)$  and  $(Q^\perp e_n)$  are a strongly disjoint frame pair.

For all  $x \in H$

$$\theta_y x = \sum_n \langle x, y_n \rangle e_n = \sum_n \langle x, y_n \rangle Q e_n.$$

Hence,  $\sum_n \langle x, y_n \rangle Q^\perp e_n = 0$ .

**Step III.**  $(x_n)$  and  $(A^* P^\perp e_n)$  are a strongly disjoint frame pair.

For any  $x \in H$ ,

$$\sum_n \langle x, x_n \rangle A^* P^\perp e_n = A^* \sum_n \langle x, x_n \rangle P^\perp e_n = A^* 0 = 0.$$

**Step IV.**  $(Q^\perp e_n)$  and  $(A^* P^\perp e_n)$  are alternate dual frames.

For any  $y \in Q^\perp \ell_2$ ,

$$\sum_n \langle y, A^* P^\perp e_n \rangle Q^\perp e_n = Q^\perp \left( \sum_n \langle A y, P^\perp e_n \rangle e_n \right) = Q^\perp A y = y.$$

By steps I-IV and Lemma 7.1, we have that  $(x_n \oplus Q^\perp e_n)$  and  $(y_n \oplus A^* P^\perp e_n)$  are alternate dual frames for  $H \oplus Q^\perp \ell_2$ . Finally, Proposition 7.2 (2) implies that the frame transform for the frame  $(x_n \oplus Q^\perp e_n)$  is onto  $\ell_2$  and hence this sequence is a Riesz basis.  $\square$

**Proposition 7.4.** *Let  $\{x_n\}$  and  $\{y_n\}$  be frames for  $H$ . If one of the following holds, then there exists a Hilbert space  $K \supset H$ , and two Riesz bases  $\{u_n\}$  and  $\{v_n\}$  for  $K$  such that  $x_n = P_H u_n$  and  $y_n = P_H v_n$ :*

- (1)  $\{x_n\}$  and  $\{y_n\}$  are similar,
- (2)  $\{x_n\}$  and  $\{y_n\}$  are disjoint

(3)  $\{y_n\}$  is similar to an alternate dual of  $\{x_n\}$ .

*Proof.* (1) Let  $y_n = Tx_n$  for some invertible operator in  $B(H)$  and let  $\{u_n\}$  be a Riesz basis for a Hilbert space  $K \supset H$  such that  $P_H u_n = x_n$ , for all  $n \in \mathbb{J}$ . Then  $y_n = P_H v_n$ , where  $v_n = (T \oplus P_H^\perp)u_n$ , which is also a Riesz basis.

(2) Since  $\{x_n\}$  and  $\{y_n\}$  are disjoint, it follows that both  $\{x_n \oplus y_n\}$  and  $\{y_n \oplus x_n\}$  are frames for  $H \oplus H$ . Clearly  $\{x_n \oplus y_n\}$  and  $\{y_n \oplus x_n\}$  are similar. So from (1), there is a Hilbert space  $K \supset H \oplus H$ , and two Riesz bases  $\{u_n\}$  and  $\{v_n\}$  for  $K$  such that  $P_{H \oplus H} u_n = x_n \oplus y_n$  and  $P_{H \oplus H} v_n = y_n \oplus x_n$ . Let  $P$  be the projection from  $K$  onto  $H \oplus 0 \oplus 0$ . Then  $Pu_n = x_n$  and  $Pv_n = y_n$ , as expected.

(3) Let  $\{z_n\}$  be an alternate dual of  $\{x_n\}$  which is similar (by  $T$ , say) to  $\{y_n\}$ . Then, by Theorem 7.3, there is a Hilbert space  $K \supset H$ , and two Riesz bases  $\{u_n\}$  and  $\{v_n\}$  for  $K$  such that  $P_H u_n = x_n$  and  $P_H v_n = z_n$ . Let  $w_n = T \oplus P_H^\perp v_n$ . Then  $\{w_n\}$  is also a Riesz basis for  $K$  and  $P_H w_n = y_n$ .  $\square$

Now we turn to the general dilation property for a family of frames. Recall that if  $X$  is a closed subspace of a Hilbert space  $H$  (or even a Banach space) then all of the closed complements of  $X$  have the same dimension.

Let  $\{x_n\}$  be a frame for  $H$  and  $\theta$  be its frame transform. Recall that we defined the **deficiency** of  $\{x_n\}$  to be the dimension of the complement of  $\theta(H)$  in  $l^2(\mathbb{J})$ , which is  $\dim \theta(H)^\perp$ . This is also commonly called in the literature the **deficiency** of the frame. Clearly similar frames have the same excess since their frame transforms have the same range.

We say that a family of frames  $\mathcal{F}$  has the *dilation property* if there is a larger Hilbert space  $K \supset H$  such that each frame of  $\mathcal{F}$  is the compression of a Riesz basis for  $K$ .

**Theorem 7.6.** *Let  $\mathcal{F}$  be a family of frames for  $H$ . Then  $\mathcal{F}$  has the dilation property if and only if all the members in  $\mathcal{F}$  have the same excess.*

*Proof.* First assume that  $\mathcal{F}$  has the dilation property. Let  $\{x_n\}$  and  $\{y_n\}$  be two frames in  $\mathcal{F}$  and let  $\theta_x$  and  $\theta_y$  be their respective frame transforms. Suppose that  $\{u_n\}$  and  $\{v_n\}$  are two Riesz bases for  $K$  such that  $P_H u_n = x_n$  and  $P_H v_n = y_n$  for all  $n \in \mathbb{J}$ . Write  $K = H \oplus M$ , where  $M = P_H^\perp K$ . Then both  $\{P_M u_n\}$  and  $\{P_M v_n\}$  are frames for  $M$ . Let  $\theta_1$  and  $\theta_2$  be the frame transforms for these two frames. Then  $\dim M = \dim \theta_1(M) = \dim \theta_2(M)$ . However, since  $\{x_n \oplus P_M u_n\}$  and  $\{y_n \oplus P_M v_n\}$  are Riesz bases for  $H \oplus M$ , it follows that  $\theta_1(M)$  (resp.  $\theta_2(M)$ ) is a Banach complementary of  $\theta_x(H)$  (resp.  $\theta_y(H)$ ) (see Theorem 2.9 (iii) in [22]). Thus  $\{x_n\}$  and  $\{y_n\}$  have the same deficiency ( $\dim M$ ).

Now assume that all the frames in  $\mathcal{F}$  have the same deficiency. Fix a frame  $\{x_n\}$  in  $\mathcal{F}$  with its frame transform  $\theta_x$ . Let  $K = H \oplus P^\perp l^2(\mathbb{J})$ , where  $P$  is the orthogonal projection from  $l^2(\mathbb{J})$  onto the range of  $\theta_x$ . Then  $\{x_n \oplus P^\perp e_n\}$  is a Riesz basis for

$K$  ( $\{e_n\}$  is as usual the standard orthonormal basis for  $l^2(\mathbb{J})$ ). Let  $\{y_n\}$  be any frame in  $\mathcal{F}$  with frame transform  $\theta_y$  and let  $Q$  be the orthogonal projection from  $l^2(\mathbb{J})$  onto the range of  $\theta_y$ . Then, by our assumption,  $P^\perp(l^2(\mathbb{J}))$  and  $Q^\perp(l^2(\mathbb{J}))$  have the same dimension. Let  $W$  be a unitary operator from  $Q^\perp(l^2(\mathbb{J}))$  onto  $P^\perp(l^2(\mathbb{J}))$ . Then  $\{y_n \oplus WQ^\perp e_n\}$  is a Riesz basis for  $K$ , since  $\{y_n \oplus Q^\perp e_n\}$  is a Riesz basis for  $H \oplus Q^\perp(l^2(\mathbb{Z}))$ . Let  $v_n = y_n \oplus Q^\perp e_n$ . Then clearly  $P_H v_n = y_n$ . Thus we have shown that there is a Hilbert space  $K$  which contains  $H$  and for every frame in  $\mathcal{F}$  there is a Riesz basis for  $K$  such that the frame is the compression of this Riesz basis to  $H$ . Therefore  $\mathcal{F}$  has the dilation property.  $\square$

From Proposition 7.4 and Theorem 7.6, we have

**Corollary 7.7.** *Two frames for  $H$  have the same deficiency if either they are alternate duals or they are disjoint.*

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