Fusion frames and Unbiased Basic Sequences

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Abstract The construction of Parseval frames with special, rigid geometric properties has left many open problems even after decades of efforts. The construction of similar types of fusion frames is even less developed. We construct a large family of equi-isoclinic Parseval fusion frames by taking the Naimark complement of the union of orthonormal bases. If these bases are chosen to be mutually unbiased, then the resulting fusion frame subspaces are spanned by mutually unbiased basic sequences. By giving an explicit representation for Naimark complements, we are able to construct concrete fusion frames in their respective Hilbert spaces.

1 Introduction

Hilbert space frame theory is a powerful tool for robust and stable representation of signals [19, 20]. It has found a broad range of applications to problems in wireless communication, time-frequency analysis, coding, sampling theory and much more [13]. A generalization of frames, called *fusion frames*, was developed in [10] (see also [28, 30, 31]) and quickly found application to problems in engineering including sensor networks, distributed processing, hierarchial data processing, packet encoding and more [2, 6, 12]. In frame theory, we measure the amplitudes of the projections of a signal onto the frame vectors. A fusion frame is a collection of subspaces of a Hilbert space (see Section 3 for definitions) where the signal is represented by the projections onto the fusion frame subspaces. This provides a platform

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for performing hierarchial data processing. Fusion frames provide multi-level processing capabilities with resilience to noise and erasures [2, 11, 23, 24] due to their redundancy. For applications, we generally need additional properties on the fusion frame, the most desired being the *Parseval* property. Apart from the use of group representations [20], and although a recent breakthrough called *spectral tetris* has allowed for large classes of Parseval fusion frames to be constructed [7, 9], it has proven to be extremely difficult to construct Parseval fusion frames.

One of the main drawbacks of fusion frame theory is that until now, results were geared towards constructing tight fusion frames. In [26], there is a nearly complete classification of dimensions for the spaces, dimensions for the subspaces, number of subspaces, and weights for which tight fusion frames exist. In applications however, it is possible that the fusion frame is given - such as the case of sensor networks - and we do not get to construct a tight fusion frame. Recently, in [4], the notion of *non-orthogonal fusion frames* was introduced to deal with this case. By using non-orthogonal projections, the authors are able to show that most fusion frames can be turned into tight fusion frames if we use non-orthogonal projections.

Apart from tightness, additional desirable properties in frame design are of geometric nature: frames whose vectors have *equal norm* or are *equiangular*. Among equal-norm Parseval frames there is a special class obtained by forming the union of orthonormal bases and scaling the vectors with a common factor. Within this class, using mutually unbiased bases provides a near match to equiangular Parseval frames which is desirable for applications [1], in particular in quantum state tomography [33]. The construction of large frames made of mutually unbiased bases is possible in prime power dimensions [32], but is otherwise similarly challenging as the construction of highly or minimally redundant equiangular Parseval frames.

We wish to consider the fusion frame analogue of these desired geometric properties in frame design. One could consider the fusion frame version of equiangular Parseval frames either to be Parseval fusion frames whose subspaces have equal chordal distances [14] or as the stricter requirement that the subspaces are equi-isoclinic [25]. In [23, 24], it is shown that a tight fusion frame (see Section 3 for definitions) has maximal robustness against one subspace erasure when all subspaces have equal dimensions. They further show that a tight fusion frame consisting of equi-dimensional subspaces with equal chordal distances (called *equi-distance tight fusion frames*) gives the best robustness with respect to two and more subspace erasures when the performance is measured in terms of the mean-square error. Equi-distance Parseval fusion frames can be constructed, for example, from Hadamard matrices [22]. When the worst-case error is optimized instead, then *equi-isoclinic* fusion frames, if they exist, are found to be optimal for two erasures. Optimality for larger numbers of erasures requires a more detailed analysis [2]. Lemmens and Seidel first introduced the notion of equi-isoclinic subspaces [25], which was further studied by Hoggar [21] and others [15, 16, 17, 18].

In dimensions where equi-distance or equi-isoclinic Parseval fusion frames are not realizable, one still has a notion of optimality in the more general type of Grassmannian frames. In [5], the authors study the problem of packing equi-dimensional subspaces of $\mathbb{R}^M$ so they are as far apart as possible (called *optimal packings*). They give packings so that either all the principal angles are the same or have at most two different values.

In this paper we will use the notion of Naimark complements to construct a large class of equi-isoclinic Parseval fusion frames. In addition, we construct the special type of equi-isoclinic Parseval fusion frame for which the subspaces are spanned by mutually unbiased basic sequences.

2 Preliminary Results

In this section we will introduce definitions and the main ideas studied in this paper. We begin with Hilbert space frames.

Definition 1. A family of vectors $\{f_i\}_{i\in I}$ is a frame for a Hilbert space $\mathscr{H}$ if there are constants $0 < A \le B < \infty$, called a lower and upper frame bound, respectively, so that for all $f \in \mathscr{H}$ we have

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

If $A = B$, this is called an *A-tight* frame, and if $A = B = 1$, it is a *Parseval frame*. The *analysis operator* of the frame is $T : \mathscr{H} \to \ell_2(I)$ given by

$$Tf = \sum_{i\in I} \langle f, f_i\rangle e_i,$$

where $\{e_i\}_{i\in I}$ is the canonical orthonormal basis of $\ell_2(I)$. The *synthesis operator* of the frame is $T^* : \ell_2(I) \to \mathscr{H}$ given by

$$T^*\left(\sum_{i\in I} a_i e_i\right) = \sum_{i\in I} a_i f_i.$$

The *frame operator* is the positive, self-adjoint and invertible operator $S = T^*T : \mathscr{H} \to \mathscr{H}$ given by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i, \text{ for all } f \in \mathscr{H}.$$

It follows that $AI \le S \le BI$ and the frame is Parseval if and only if $S = I$.

Definition 2. Given two subspaces W_1, W_2 of a Hilbert space $\mathscr{H}$ with dim $W_1 = k \le$ dim $W_2 = \ell$, the *principal angles* $(\theta_1, \theta_2, \dots \theta_k)$ between the subspaces are defined as follows: The first principal angle is

$$\theta_1 = \min\{\arccos|\langle f, g\rangle| : f \in S_{W_1}, g \in S_{W_2}\}$$

where $S_{W_i} = \{f \in W_i : \|f\| = 1\}$. Two vectors f_1, g_1 are called *principal vectors* if they give the minimum above.

The other principal angles and vectors are then defined recursively via

$$\theta_i = min\{\arccos|\langle f,g\rangle| : f \in S_{W_1}, g \in S_{W_2}, \text{ and } f \perp f_j, g \perp g_j, 1 \le j \le i-1\}.$$

Conway, Hardin and Sloane introduced the *chordal distance* between subspaces of a Hilbert space [14].

Definition 3. If W_1, W_2 are subspaces of $\mathscr{H}$ of dimension k, the *chordal distance* $d_c(W_1, W_2)$ between the subspaces is given by:

$$d_c^2(W_1,W_2) = k - tr[P_1P_2] = k - \sum_{j=1}^{k} \cos^2\theta_j,$$

where P_i is the orthogonal projection onto W_i and $\{\theta_j\}_{j=1}^k$ are the principal angles for W_1, W_2.

If P_1, P_2 are the orthogonal projections onto the k-dimensional subspaces W_1, W_2, the non-zero singular values of P_1P_2 are the squares of the cosines of the principal angles $\{\theta_j\}_{j=1}^k$ between W_1 and W_2. Hence,

$$tr[P_1P_2] = \sum_{j=1}^{k} \cos^2\theta_j = k - d_c^2(W_1,W_2).$$

Definition 4. Two k-dimensional subspaces W_1, W_2 of a Hilbert space are isoclinic with parameter λ, if the angle θ between any $f \in W_1$ and its orthogonal projection Pf in W_2 is unique with $\cos^2\theta = \lambda$.

Multiple subspaces are equi-isoclinic if they are pairwise isoclinic with the same parameter λ.

An alternative definition is given in [16] where two subspaces are called isoclinic if the stationary values of the angles of two lines, one in each subspace, are equal. The geometric characterization given by Lemmens and Seidel is that when a sphere in one subspace is projected onto the other subspace, then it remains a sphere, although the radius may change [25]. This is all equivalent to the principal angles between the subspaces being identical.

Much work has been done on finding the maximum number of equi-isoclinic subspaces given the dimensions of the overall space and the subspaces (and often the parameter λ). Specifically, Seidel and Lemmens [25] give an upper bound on the number of real equi-isoclinic subspaces and Hoggar [21] generalizes this to vector spaces over $\mathbb{R}$ and $\mathbb{C}$.

We will show that equi-isoclinic subspaces can be constructed with the help of the Naimark complement from a simple type of Parseval frame formed by taking the union of orthonormal bases and scaling the vectors by a common factor. A special case occurs when we form the union of *mutually unbiased bases*.

Definition 5. A family of orthonormal sequences $\{g_{ij}\}_{i=1}^N$, $j = 1,2,\dots,L$ in an M-dimensional Hilbert space $\mathscr{H}_M$ is called mutually unbiased if there is a constant

$0 < c$ so that

$$|\langle g_{ij}, g_{k\ell}\rangle| = c, \text{ for all } j \neq \ell.$$

If $N = M$, then we say the family consists of mutually unbiased bases. In this case, c must equal $\sqrt{\frac{1}{N}}$.

3 Fusion Frames and Their Complements

Fusion frames were developed in [10] and quickly generated much literature (see www.fusionframe.org) because of their application to problems in engineering.

Definition 6. Given a Hilbert space $\mathscr{H}$ and a family of closed subspaces $\{W_i\}_{i\in I}$ with associated positive weights v_i, $i \in I$, a collection of weighted subspaces $\{(\mathscr{W}_i, v_i)\}_{i\in I}$ is a *fusion frame* for $\mathscr{H}$ if there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|P_i f\|^2 \leq B\|f\|^2 \qquad \text{for any } f \in \mathscr{H},$$

where P_i is the orthogonal projection onto $\mathscr{W}_i$.

The constants A and B are called *fusion frame bounds*. A fusion frame is called *tight* if A and B can be chosen to be equal, *Parseval* if $A = B = 1$, and *orthonormal* if

$$\mathscr{H} = \oplus_{i\in I} W_i.$$

The fusion frame operator is the positive, self-adjoint and invertible operator $S_W : \mathscr{H} \to \mathscr{H}$ given by

$$S_W f = \sum_{i\in I} v_i^2 P_i f, \text{ for all } f \in \mathscr{H}.$$

It is known [12] that $\{W_i, v_i\}_{i\in I}$ is a fusion frame with fusion frame bounds A, B if and only if $AI \leq S_W \leq BI$. Any signal $f \in \mathscr{H}$ can be reconstructed [12] from its fusion frame measurements $\{v_i P_i f\}_{i\in I}$ by performing

$$f = \sum_{i\in I} v_i S^{-1}(v_i P_i f).$$

A frame $\{f_i\}_{i\in I}$ can be thought of as a fusion frame of one dimensional subspaces where $W_i = span\ \{f_i\}$ for all $i \in I$. The fusion frame is then $\{W_i, \|f_i\|\}$. A difference between frames and fusion frames is that for frames, an input signal $f \in \mathscr{H}$ is represented by a collection of scalar coefficients $\{\langle f, f_i\rangle\}_{i\in I}$ that measure the projection of the signal onto each frame vector f_i, while for fusion frames, an input signal $f \in \mathscr{H}$ is represented by a collection of *vector coefficients* $\{\Pi_{W_i}(f)\}_{i\in I}$ corresponding to projections onto each subspace W_i.

There is an important connection between fusion frame bounds and bounds from frames taken from each of the fusion frame's subspaces [10].

Theorem 1. *For each $i \in I$, let $v_i > 0$ and W_i be a closed subspace of $\mathscr{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i with frame bounds A_i, B_i. Assume that $0 < A = \inf_{i\in I} A_i \le \sup_{i\in I} B_i = B < \infty$. Then the following conditions hold:*

1. *$\{W_i, v_i\}_{i\in I}$ is a fusion frame for $\mathscr{H}$.*
2. *$\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathscr{H}$.*

In particular, if $\{W_i, v_i\}_{j\in J_i}\}_{i\in I}$ is a fusion frame for $\mathscr{H}$ with fusion frame bounds C, D, then $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathscr{H}$ with frame bounds AC, BD. Also, if $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathscr{H}$ with frame bounds C, D, then $\{W_i, v_i, \}_{j\in J_i}\}_{i\in I}$ is a fusion frame for $\mathscr{H}$ with fusion frame bounds $\frac{C}{B}, \frac{D}{A}$.

Corollary 1. *For each $i \in I$, let $v_i > 0$ and W_i be a closed subspace of $\mathscr{H}$. The following are equivalent:*

1. *$\{W_i, v_i\}_{i\in I}$ is a fusion frame for $\mathscr{H}$ with fusion frame bounds A, B.*
2. *For every orthonormal basis $\{e_{ij}\}_{j\in K_i}$ for W_i, the family $\{v_i e_{ij}\}_{i\in I, j\in K_i}$ is a frame for $\mathscr{H}$ with frame bounds A, B.*
3. *For every Parseval frame $\{f_{ij}\}_{i\in I, j\in J_i}$ for W_i, the family $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathscr{H}$ with frame bounds A, B.*

Corollary 2. *For each $i \in I$, let $v_i > 0$ and W_i be a closed subspace of $\mathscr{H}$. The following are equivalent:*

1. *$\{W_i, v_i\}_{i\in I}$ is a Parseval fusion frame for $\mathscr{H}$.*
2. *For every orthonormal basis $\{e_{ij}\}_{j\in K_i}$ for W_i, the family $\{v_i e_{ij}\}_{i\in I, j\in K_i}$ is a Parseval frame for $\mathscr{H}$.*
3. *For every Parseval frame $\{f_{ij}\}_{i\in I, j\in J_i}$ for W_i, the family $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a Parseval frame for $\mathscr{H}$.*

We now turn our attention to complements of fusion frames.

3.1 Naimark Complements

A fundamental result in frame theory is *Naimark's Theorem* (see e.g. [13, 20] and [8] for a generalization).

Theorem 2. *A family $\{f_i\}_{i=1}^N$ is a Parseval frame for a Hilbert space $\mathscr{H}$ if and only if there is a Hilbert space $\ell_2(N)$ with orthonormal basis $\{e_i\}_{i=1}^N$, $\mathscr{H} \subset \ell_2(N)$, and the orthogonal projection P of $\ell_2(N)$ onto $\mathscr{H}$ satisfies $Pe_i = f_i$, for all $i = 1, 2, \ldots, N$.*

Definition 7. If $\{Pe_i\}_{i=1}^N$ as in Naimark's Theorem is a Parseval frame for $\mathscr{H}_M$, then $\{(I-P)e_i\}_{i=1}^N$ is a Parseval frame for the subspace $\mathscr{H}_{N-M}$ of $\ell_2(N)$ called the Naimark complement of $\{Pe_i\}_{i=1}^N$.

For a Parseval fusion frame $\{W_i,v_i\}_{i=1}^M$ we also have a Naimark complement. For this, we choose orthonormal bases $\{g_{ij}\}_{i=1}^{n_j}$ for $W_j,\ j=1,2,\dots,M$. Then $\{\sqrt{v_j}g_{ij}\}_{i=1,j=1}^{n_j\ ,\ M}$ is a Parseval frame which we can denote as $\{Pe_{ij}\}_{i=1,j=1}^{n_j\ ,\ M}$. Since

$$\langle Pe_{ij},Pe_{k\ell}\rangle = -\langle (I-P)e_{ij},(I-P)e_{k\ell}\rangle, \qquad j\neq l,$$

it follows that $\{(I-P)e_{ij}\}_{i=1}^{n_j}$ is an orthogonal set. Letting

$$W_j' = span\ \{(I-P)e_{ij}\}_{i=1}^{n_j},$$

it can be shown that $\{W_j',\sqrt{1-v_j^2}\}_{j=1}^M$ is a Parseval fusion frame called the *Naimark complement fusion frame*. We note that the Naimark complement fusion frame actually depends on the choice of orthonormal bases for the fusion frame, but we will now check that all possible Naimark complements are unitarily equivalent.

Lemma 1. *Let $\mathscr{H}_1,\ \mathscr{H}_3$ be M-dimensional Hilbert spaces, $\mathscr{H}_2,\ \mathscr{H}_4$ - $(N-M)$-dimensional Hilbert spaces, and let $U:\mathscr{H}_1\to\mathscr{H}_3$ be a unitary operator. If $\{f_i\oplus g_i\}_{i=1}^N\subseteq\mathscr{H}_1\oplus\mathscr{H}_2$ and $\{Uf_i\oplus h_i\}_{i=1}^N\subseteq\mathscr{H}_3\oplus\mathscr{H}_4$ are orthonormal sets, then $Vg_i=h_i$ is a unitary operator.*

Proof. For all families of scalars $\{a_i\}_{i=1}^N$ we have

$$\|\sum_{i=1}^N a_i(f_i\oplus g_i)\|^2=\sum_{i=1}^N |a_i|^2=\|\sum_{i=1}^N a_i(Uf_i\oplus h_i)\|^2.$$

Thus

$$\|\sum_{i=1}^N a_if_i\|^2+\|\sum_{i=1}^N a_ig_i\|^2=\|\sum_{i=1}^N a_i(f_i\oplus g_i)\|^2=\|\sum_{i=1}^N a_i(Uf_i\oplus h_i)\|^2=$$
$$\|\sum_{i=1}^N a_iUf_i\|^2+\|\sum_{i=1}^N a_ih_i\|^2=\|\sum_{i=1}^N a_if_i\|^2+\|\sum_{i=1}^N a_ih_i\|^2.$$

Hence, $\|\sum_{i=1}^N a_ig_i\|^2=\|\sum_{i=1}^N a_ih_i\|^2$. $\square$

One consequence of the above lemma is that Naimark complements for Parseval fusion frames are independent of the choice of orthonormal bases for the subspaces up to the application of a unitary operator. For our constructions, this will allow us to explicitly identify our equi-isoclinic fusion frames in the Naimark complement of our original fusion frame.

Corollary 3. *If $\{f_i\}_{i=1}^N$ is a Parseval frame for $\mathscr{H}_M$ and $\{g_i\}_{i=1}^N$ is a family of vectors in $\mathscr{H}_{N-M}$ so that $\{f_i\oplus g_i\}_{i=1}^N$ is an orthonormal basis for $\mathscr{H}_N$, then up to a unitary operator, $\{g_i\}_{i=1}^N$ is the Naimark complement of $\{f_i\}_{i=1}^N$.*

3.2 Identifying the Naimark Complement

Now we will give an explicit representation of the Naimark complements for the fusion frames we will be working with.

Proposition 1. *Let $A_1, \ldots, A_M$ be $N \times N$ unitary matrices, and let*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix}$$

be a unitary matrix. Then

$$\mathscr{A} = \begin{bmatrix} a_{11}A_1 & a_{12}A_2 & \cdots & a_{1M}A_M \\ a_{21}A_1 & a_{22}A_2 & \cdots & a_{2M}A_M \\ \vdots & \vdots & \dots & \vdots \\ a_{M1}A_1 & a_{M2}A_2 & \cdots & a_{MM}A_M \end{bmatrix}$$

is a unitary matrix.

Proof. A direct calculation shows that $A^*A = Id$. □

Corollary 4. *Given orthonormal bases $\{g_{ij}\}_{i=1}^N$, $j = 1, 2, \ldots, M$ for $\mathscr{H}_N$, consider the $N \times MN$ matrix*

$$[A_1\ A_2\ \cdots\ A_M] = \begin{bmatrix} | & & | & | & & | & & | & & | \\ g_{11} & \cdots & g_{N1} & g_{12} & \cdots & g_{N2} & \cdots & g_{1M} & \cdots & g_{NM} \\ | & & | & | & & | & & | & & | \end{bmatrix}.$$

This is the synthesis operator for the M-tight frame $\{g_{ij}\}_{i=1,j=1}^{N,\ M}$. Choose a unitary matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix}.$$

Then the column vectors of the matrix

$$\mathscr{A}' = \sqrt{\frac{1}{M}} \begin{bmatrix} a_{21}A_1 & a_{22}A_2 & \cdots & a_{2M}A_M \\ \vdots & \vdots & \dots & \vdots \\ a_{M1}A_1 & a_{M2}A_2 & \cdots & a_{MM}A_M \end{bmatrix}$$

are the Naimark complement for the Parseval frame $\{\sqrt{\frac{1}{M}}g_{ij}\}_{i=1,j=1}^{N,\ M}$.

Proof. This is immediate from Corollary 3 and Proposition 1. □

Corollary 5. *Letting W_j be the span of the column vectors of $a_{1j}A_j$, then $\{W_j, \sqrt{a_{1j}}\}_{j=1}^{M}$ is a Parseval fusion frame for $\mathscr{H}_N$. Letting W_j' be the span of the column vectors of*

$$\begin{bmatrix} a_{2j}A_j \\ a_{3j}A_j \\ \vdots \\ a_{Mj}A_j \end{bmatrix}$$

then $\{W_j', \sqrt{\sum_{i=2}^{M} a_{ij}}\}_{j=1}^{M}$ is a Parseval fusion frame which is the Naimark complement of the Parseval fusion frame $\{W_j, \sqrt{a_{1j}}\}_{j=1}^{M}$.

Now that we have identified our Parseval fusion frames, we find the principal angles and the chordal distance between our fusion frame subspaces by recalling results from [8]:

Theorem 3. *Let $\{W_i, v_i\}_{i=1}^{K}$ be a Parseval fusion frame for $\mathscr{H}_N$ with Naimark complement Parseval fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^{K}$. Let $\{\theta_{\ell,ij}\}_{\ell=1}^{k}$ be the principal angles for the subspaces W_i, W_j. Then the principal angles $\{\theta'_{l,ij}\}$ for the subspaces W_i', W_j' are:*

$$\left\{ \arccos\left[\frac{v_i}{\sqrt{1-v_i^2}} \frac{v_j}{\sqrt{1-v_j^2}} \cos(\theta_{\ell,ij}) \right] \right\}_{\ell=1}^{k}.$$

Theorem 4. *Let $\{W_i, v_i\}_{i=1}^{K}$ be a Parseval fusion frame for $\mathscr{H}_N$. Then the Naimark complement Parseval fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^{K}$ satisfies*

$$d_c^2(W_i', W_j') = \left[1 - \frac{v_i^2}{1-v_i^2} \frac{v_j^2}{1-v_j^2} \right] k + \left[\frac{v_i^2}{1-v_i^2} \frac{v_j^2}{1-v_j^2} \right] d_c^2(W_i, W_j).$$

This construction creates equi-isoclinic fusion frame subspaces in our Naimark complement which we shall further discuss in Section 4. Now, however, we briefly leave our discussion of Naimark complements to consider the *spatial complement* of a fusion frame.

3.3 Spatial Complements

Definition 8. Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathscr{H}$. If the family $\{(W_i^{\perp}, v_i)\}_{i\in I}$, where $W_i^{\perp}$ is the orthogonal complement of W_i, is also a fusion frame, then we call $\{(W_i^{\perp}, v_i)\}_{i\in I}$ the orthogonal fusion frame to $\{(W_i, v_i)\}_{i\in I}$.

Proposition 2. *Let $\{W_i, v_i\}_{i\in I}$ be a Parseval fusion frame for $\mathscr{H}$. Then*

1. $\sum_{i\in I} v_i^2 \geq 1$

2. $\sum_{i\in I} v_i^2 = 1$ *if and only if* $W_i = \mathscr{H}$ *for all* $i \in I$

Proof. To prove the first statement, let P_i be the orthogonal projection of $\mathscr{H}$ onto W_i. Since the fusion frame is Parseval, for all $f \in \mathscr{H}$,

$$f = \sum_{i\in I} v_i^2 P_i(f).$$

Thus

$$\begin{aligned} \|f\| &= \|\sum_{i\in I} v_i^2 P_i(f)\| \\ &\leq \sum_{i\in I} \|v_i^2 P_i(f)\| \\ &= \sum_{i\in I} v_i^2 \|P_i(f)\| \\ &\leq \left(\sum_{i\in I} v_i^2\right) \|f\| \end{aligned} \tag{1}$$

Then the second statement follows from considering when equality holds in inequality (1) for all $f \in \mathscr{H}$:

$$\begin{aligned} \sum_{i\in I} v_i^2 = 1 &\Leftrightarrow \sum_{i\in I} v_i^2 \|P_i(f)\| = \|f\| \\ &\Leftrightarrow P_i(f) = f \text{ for all } i \in I \\ &\Leftrightarrow W_i = \mathscr{H} \text{ for all } i \in I. \end{aligned}$$

□

We need a result from [6].

Theorem 5. *Let* $\{(W_i, v_i)\}_{i\in I}$ *be a fusion frame for* $\mathscr{H}$ *with optimal fusion frame bounds* $0 < A \leq B < \infty$ *such that* $\sum_{i\in I} v_i^2 < \infty$. *Then the following conditions are equivalent:*

1. $\bigcap_{i\in I} W_i = \{0\}$.

2. $B < \sum_{i\in I} v_i^2$.

3. The family $\{(W_i^{\perp}, v_i)\}_{i\in I}$ *is a fusion frame for* $\mathscr{H}$ *with optimal fusion frame bounds* $\sum_{i\in I} v_i^2 - B$ *and* $\sum_{i\in I} v_i^2 - A$.

Corollary 6. *Let* $\{(W_i, v_i)\}_{i\in I}$ *be a Parseval fusion frame for* $\mathscr{H}$ *with* $\sum_{i\in I} v_i^2 > 1$. *Then* $\{(W_i^{\perp}, v_i)\}_{i\in I}$ *is a* $\left(\sum_{j\in I} v_j^2 - 1\right)$ *tight fusion frame and*

$$\left\{ W_i^{\perp}, \frac{v_i}{\sqrt{\sum_{j\in I} v_j^2 - 1}} \right\}$$

is a Parseval fusion frame.

Moreover, we have

$$d_c^2(W_i, W_j) = d_c^2(W_i^{\perp}, W_j^{\perp}).$$

Proof. The only part that needs a proof is the moreover part, but this is a well known result from [27] and [29] where the authors show that the non-zero principal angles between two subspaces and the non-zero principal angles between their orthogonal complements are equal. □

4 Equi-Isoclinic Fusion Frames and Mutually Unbiased Basic Sequences

Now we combine our results to find equi-isoclinic fusion frames, and we find a special case where the fusion frame subspaces are spanned by mutually unbiased basic sequences. We will need a result from [3].

Theorem 6. *Let P be the orthogonal projection of $\ell_2(M)$ onto $\mathscr{H}_N$. Given the Parseval frame $\{Pe_i\}_{i=1}^M$ for $\mathscr{H}_N$, and $J \subseteq \{1,2,\ldots,M\}$, the following are equivalent:*

1. $\mathrm{span}\{(I-P)e_i\}_{i\in J} = (I-P)(\ell_2(M))$.
2. *$\{Pe_i\}_{i\in J^c}$ is linearly independent.*

Now we present the main results.

Theorem 7. *Let $\{g_{ij}\}_{i=1,j=1}^{N\ ,\ M}$ be the union of M orthonormal bases for a Hilbert space $\mathscr{H}_N$. Let $\{W_j, \sqrt{\frac{1}{M}}\}_{i=1}^M$ be the Parseval fusion frame for $\mathscr{H}_N$ where $W_j =$ span $\{g_{ij}\}_{i=1}^N = \mathscr{H}_N$. Let $\{W_j', \sqrt{\frac{M-1}{M}}\}_{j=1}^M$ be the Naimark complement, a Parseval fusion frame for $\mathscr{H}_{(M-1)N}$. Using Naimark's theorem, we can write $g_{ij} = \sqrt{\frac{1}{M}}Pe_{ij}$ where P is an orthogonal projection from $\ell_2(MN)$ onto $\mathscr{H}_N$. Then the following hold:*

1. *For any $j \in \{1,2,\ldots,M\}$,*

$$\mathrm{span}\{W_i^{'}\}_{i\neq j} = \mathscr{H}_{(M-1)N}\,.$$

2. *The principal angles for $W_i^{'}, W_j^{'}$ are*

$$\theta_{l,ij} = \arccos\left[\frac{1}{M-1}\right],$$

so this is a equi-isoclinic Parseval fusion frame.

Proof. To prove (1), we use that $\{g_{ij}\}_{i=1}^N$ is, for any fixed j, a linearly independent set. This implies by the preceding theorem that selecting the complement of the index set, that is all $j' \neq j$, provides a spanning subset $\{\sqrt{\frac{1}{M}}(I-P)e_{ij'} : j' \neq j\}$ for the Naimark complement. Since the subspaces W_j' are spanned by $\{\sqrt{\frac{1}{M}}(I-P)e_{ij'}\}$, the result follows.

To prove (2), we note that the principal angles for W_i, W_j are all $\arccos 1 = 0$. The conversion for principal angles gives

$$\arccos\left[\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\cos 0\right] = \arccos\left[\frac{1}{M-1}\right].$$

□

The second item in the theorem has a converse.

Corollary 7. *Let $\{W_i, v_i\}_{i=1}^M$ be a Parseval fusion frame for $\mathscr{H}_{N(M-1)}$ with* $\dim W_i = N$ *for all $i = 1,2,\ldots,M$. Then*

$$\sum_{i=1}^M v_i^2 = M-1.$$

If $v_i = \sqrt{\frac{M-1}{M}}$ for all $i = 1,2,\ldots,M$, then all principal angles between the subspaces W_i are equal to $\arccos\left[\frac{1}{M-1}\right]$.

Proof. Since $\sum_{i=1}^M \dim W_i = NM$, the Naimark complement fusion frame $\{W_i', \sqrt{1-v_i^2}\}_{i=1}^M$ spans $\mathscr{H}_{NM-N(M-1)} = \mathscr{H}_N$. Since $\dim W_i = \dim W_i' = N$, $W_i' = \mathscr{H}_N$ for every i. The result is now immediate from the preceding theorem. □

Additionally, in the special case where our bases are mutually unbiased, the resulting Naimark fusion frame subspaces are spanned by mutually unbiased basic sequences.

Corollary 8. *Let $\{g_{ij}\}_{i=1,j=1}^{N\ ,\ M}$ be a family of mutually unbiased bases for a Hilbert space $\mathscr{H}_N$. Let $\{W_j, \sqrt{\frac{1}{M}}\}_{i=1}^M$ be the Parseval fusion frame for $\mathscr{H}_N$ where $W_j =$ span $\{g_{ij}\}_{i=1}^N$. Let $\{W_j', \sqrt{\frac{M-1}{M}}\}_{j=1}^M$ be the Naimark complement Parseval fusion frame for $\mathscr{H}_{(M-1)N}$. Using Naimark's theorem, we can write $g_{ij} = \sqrt{\frac{1}{M}}Pe_{ij}$ where P is an orthogonal projection from $\ell_2(MN)$ onto $\mathscr{H}_N$. Then the family $\{\sqrt{\frac{M}{M-1}}(I-P)e_{ij}\}_{i=1}^N$, for $j = 1,2,\ldots,M$ consists of mutually unbiased orthonormal sequences, and the subspaces*

$$W_j = \mathrm{span}\{(I-P)e_{ij}\}_{i=1}^N$$

are equi-isoclinic.

Proof. The equi-isoclinic property is a consequence of the preceding theorem. The fact that the basic sequences spanning the subspaces are mutually unbiased comes from the fact that the Gram matrix of the Naimark complement differs on all off-diagonal elements only by an overall change of sign from the Gram matrix of the mutually unbiased bases. $\square$

Maximal mutually unbiased bases are known to exist in all prime power dimensions p^r [32], so by the Naimark complement there are Parseval fusion frames with $M \le p^r + 1$ equi-isoclinic subspaces of dimension p^r, spanned by mutually unbiased basic sequences in a Hilbert space of dimension $N = (M-1)p^r$. The union of these unbiased basic sequences forms a tight frame of redundancy $M/(M-1)$, and thus we can achieve redundancies arbitrarily close to one if p^r is chosen sufficiently large.

Lastly, we now present a concrete example of a special case of our main theorem. In $\mathscr{H}_N$, consider M copies of an orthonormal basis $\{g_i\}_{i=1}^N$ as $\{g_{ij}\}_{i=1,j=1}^{N,M}$. Hence $\{\frac{1}{\sqrt{M}}g_{ij}\}_{i=1,j=1}^{N,M}$ is a Parseval frame. Let $T : \mathscr{H}_N \to \ell_2(MN)$ be the analysis operator of this Parseval frame and let $W_j = \operatorname{span}\{g_{ij}\}_{i=1}^N$. Then $\{W_j, \frac{1}{\sqrt{M}}\}_{j=1}^M$ is a Parseval fusion frame. Let P be the orthogonal projection of $\ell_2(MN)$ onto the range of T. Let $\{e_{ij}\}_{i=1,j=1}^{N,\,M}$ be the orthonormal basis for $\ell_2(MN)$ so that $Pe_{ij} = Tg_{ij}$. Write our Parseval fusion frame in the order $\frac{1}{\sqrt{M}}Pe_{11}, \frac{1}{\sqrt{M}}Pe_{12}, \dots, \frac{1}{\sqrt{M}}Pe_{1M}, \frac{1}{\sqrt{M}}Pe_{21}, \dots, \frac{1}{\sqrt{M}}Pe_{NM}$. Now,

$$T = \begin{pmatrix} Pe_{11} \\ \vdots \\ Pe_{1M} \\ Pe_{21} \\ \vdots \\ Pe_{2M} \\ \vdots \\ Pe_{N1} \\ \vdots \\ Pe_{NM} \end{pmatrix} = \begin{pmatrix} \frac{1}{M} & \cdots & \frac{1}{M} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \frac{1}{M} & \cdots & \frac{1}{M} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{M} & \cdots & \frac{1}{M} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{1}{M} & \cdots & \frac{1}{M} & \cdots & 0 & \cdots & 0 \\ & & & & & & \ddots & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \frac{1}{M} & \cdots & \frac{1}{M} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \frac{1}{M} & \cdots & \frac{1}{M} \end{pmatrix}$$

and

$$\begin{pmatrix} (I-P)e_{11} \\ \vdots \\ (I-P)e_{1M} \\ (I-P)e_{21} \\ \vdots \\ (I-P)e_{2M} \\ \vdots \\ (I-P)e_{N1} \\ \vdots \\ (I-P)e_{NM} \end{pmatrix} =$$

$$\begin{pmatrix}
1-\frac{1}{M} & \cdots & -\frac{1}{M} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
-\frac{1}{M} & \cdots & 1-\frac{1}{M} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1-\frac{1}{M} & \cdots & -\frac{1}{M} & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & -\frac{1}{M} & \cdots & 1-\frac{1}{M} & \cdots & 0 & \cdots & 0 \\
 & & & & & & \ddots & & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{M} & \cdots & -\frac{1}{M} \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & -\frac{1}{M} & \cdots & 1-\frac{1}{M}
\end{pmatrix}$$

In this setting, W_1' is

$$\begin{pmatrix}
1-\frac{1}{M} & \cdots & -\frac{1}{M} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1-\frac{1}{M} & \cdots & -\frac{1}{M} & \cdots & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 & & & & & & \ddots & & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1-\frac{1}{M} & \cdots & -\frac{1}{M} \\
\vdots & & \vdots & \vdots & & \vdots & \cdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0
\end{pmatrix}$$

The above represents an orthonormal basis for W_1'. Similarly we have W_j' for $j = 1, 2, \ldots, M$. This leads to

Proposition 3. *For the projections above we have:*

1. $W_j = \{Pe_{1j}, Pe_{2j}, \ldots, Pe_{Nj}\}$ *implies* $\mathrm{span}\{W_i^{'}\}_{i \neq j} = \mathscr{H}_{(M-1)N}$.
2. *The family* $\{W_i, v_i\}_{i=1}^{M}$, $v_i = \frac{1}{\sqrt{M}}$ *for* $i = 1, 2, \ldots, M$ *is a Parseval fusion frame for* $\mathscr{H}_N$.
3. *The family* $\{W_i^{'}, \sqrt{1 - v_i^2} = \sqrt{\frac{M-1}{M}}\}_{i=1}^{M}$ *is a Parseval fusion frame for* $\mathscr{H}_{(M-1)N}$.
4. *Letting* $U_i = \mathrm{span}\{(I-P)e_{ij}\}_{j=1}^{M}$, *then* $\{U_i, 1\}_{i=1}^{N}$ *is an orthonormal fusion frame with* $\dim U_i = M - 1$.
5. *The principal angles for* $W_i^{'}, W_j^{'}$ *are*

$$\arccos\left[\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\frac{\frac{1}{\sqrt{M}}}{\sqrt{1-(\frac{1}{\sqrt{M}})^2}}\cos 0\right] = \arccos\left[\frac{1}{M-1}\right],$$

and this is an equi-isoclinic Parseval fusion frame.

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