Gabor frames over irregular lattices

Peter G. Casazza*and Ole Christensen

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Abstract

We give necessary and sufficient conditions for $g \in W(L^\infty, \ell^1)$ to generate a Gabor frame over certain irregular lattices.

1 Introduction

A set of vectors $\{f_n\}$ in a Hilbert space H is called a **frame** if there exist constants $A, B > 0$, such that

$$A||f||^2 \leq \sum |\langle f, f_n \rangle|^2 \leq B||f||^2, \ \forall f \in H.$$

The numbers A, B are called the **lower** and **upper** frame bounds respectively. If we have only the right-hand-side inequality above we call $\{f_n\}$ a **Bessel sequence** with **Bessel bound B**.

In applications, **Gabor frames**, that is, frames for $L^2(\mathbb{R})$ of the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ play an important role. By introducing the operators T_a, E_b on $L^2(\mathbb{R})$ given by

$$(T_a f)(x) = f(x-a), \quad (E_b f)(x) = e^{2\pi ibx}g(x)$$

we can write $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}} = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. One usually thinks about $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ as the set of time-frequency shifts of $g \in L^2(\mathbb{R})$ along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in $\mathbb{R}^2$.

The purpose of this paper is to give classification results for frames of the type $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and also for irregular frames of the type $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$, where $\{x_m\}_{m\in\mathbb{Z}}, \{y_n\}_{m\in\mathbb{Z}} \subseteq \mathbb{R}$. That is, we consider Gabor frames, where $a\mathbb{Z} \times b\mathbb{Z}$ is replaced by irregular lattices in $\mathbb{R}^2$.

The motivation for this research is a fundamental result of Feichtinger and Gröchenig [16]:

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Theorem 1.1 (Feichtinger and Gröchenig).
Let $0 \neq g \in L^2(\mathbb{R})$ *satisfy*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\langle E_x T_y g, g\rangle| dx\ dy < \infty. \qquad (1)$$

Then there exists an open set $U \subset \mathbb{R}^2$ *such that* $\{E_{x_n} T_{y_n} g\}_{n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *for every separated set* $\{(x_n, y_n)\}_{n\in\mathbb{Z}} \subset \mathbb{R}^2$ *for which*

$$\bigcup [(x_n, y_n) + U] = \mathbb{R}^2.$$

We refer the reader to [9] for a unified treatment of the Feichtinger and Gröchenig theory and a proof of Theorem 1.1. Theorem 1.1 is very important because it implies there is a box Q in $\mathbb{R}^2$ so that given a tiling $\{Q_n\}_{n\in\mathbb{Z}}$ of $\mathbb{R}^2$ by Q and any choices of separated points $(x_n, y_n) \in Q_n$ (see definition below) we have that $\{E_{x_n} T_{y_n} g\}_{n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$. For applications, it is important to know the "size" of the box Q and much work has gone into this computation. In particular, Gröchenig [19] (Theorems T and S, pages 24-25) gives conditions for identifying Q. For a more concrete result we refer to the recent paper [30] by Sun and Zhou. See also [2, 26] for estimates on the size of Q. In the case where g is compactly supported, Gröchenig ([20], Theorem 3) has readily verifiable conditions for finding Q. Note that the condition (1) means that g belongs to Feichtinger's algebra S_0. It is known [14] that if $f, f^{'}, f^{''} \in L^1(\mathbb{R})$, then inequality (1) holds. In this paper we offer an alternative approach to computing the box Q by requiring that g is in the Wiener space $W(L^\infty, \ell^1)$ (see below for definitions). Although this looks like a strong assumption, it is possible that this conditions is necessary. We cannot prove the necessity at this time, but we will show that $g \in W(L^\infty, \ell^2)$ is necessary. We give readily verifiable conditions on g so that we can find a box $Q = [0, b_0] \times [0, a_0]$ so that whenever $b \in (0, b_0]$ and $y_n \in [na_0, (n+1)a_0]$ then $\{E_{mb} T_{y_n} g\}_{n,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Frames of this type are called **semi-irregular Gabor frames**. In general, it is an exceptionally difficult problem to classify the functions g which give Gabor frames. For example, it is still an open problem to find all $a, b, c > 0$ so that $\{E_{mb} T_{na} \chi_{[0,c]}\}_{m,n\in\mathbb{Z}}$ is a Gabor frame. This is known as the **abc-problem** and a deep study of this problem was done by Janssen [25]. Also, Casazza and Kalton [5] have shown that the problem of classifying just the characteristic functions of measurable subsets $F \subset \mathbb{R}$ so that $\{E_m T_n \chi_F\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ is equivalent to a classical problem of Littlewood concerning certain complex polynomials and when they have roots on the unit circle. The reason we are able to get exact classifications in the present paper is that we are requiring $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ to give a frame for a whole range of values of a, b. It turns out that this is a strong assumption and puts identifiable restrictions on the functions g.

Our main source of inspiration for this paper was [16]. However, there are other papers dealing with irregular Gabor frames we should mention. In [20], Gröchenig gives sufficient conditions for a family $\{E_{x_m} T_{y_n} g\}_{m,n\in\mathbb{Z}}$ to be a frame in the case where g is a band-limited function. Ramanathan and Steger [27] gave a necessary condition for $\{E_{x_n}T_{y_n}g\}_{n\in\mathbb{Z}}$ to a a frame in terms of the Beurling density of $\{(x_n, y_n)\}$. In the special case of the Gaussians, Seip and Wallsten [28] were even able to show that this condition is also sufficient.

In the rest of this introduction we discuss some assumptions (and the relations between them) that will be used throughout the paper. First, a sequence of real numbers $\{y_n\}$ is δ-**separated**, with $\delta > 0$, if $|y_n - y_m| \geq \delta$, for all $n \neq m$. A sequence is **relatively separated** if it is a finite union of δ-separated sequences, for some $\delta > 0$.

The **Wiener space** $W(L^\infty, \ell^p)$ $1 \leq p < \infty$ is defined as the set of functions $g \in L^2(\mathbb{R})$ for which (for a certain value of $c > 0$)

$$||g||_{W,p,c} := \left(\sum_{n\in\mathbb{Z}} ||T_{nc}g \cdot \chi_{[0,c[}||_\infty^p \right)^{1/p} < \infty.$$

$W(L^\infty, \ell^p)$ is a Banach space with respect to the natural norm. The space is independent of the choice of c, in the sense that different choices give the same space with equivalence of the norms. In fact, it is easily checked that for all $g \in L^2(\mathbb{R})$,

$$||g||_{W,p,b} \leq 2||g||_{W,p,a} \leq \frac{4b}{a}||g||_{W,p,b}, \quad \text{for all} \quad a \leq b.$$

Also, $W(L^\infty, \ell^p)$ is dense in $L^2(\mathbb{R})$. We refer to [15] for more information about these and similar spaces. We will primarily work with the Wiener space $W(L^\infty, \ell^1)$ and write $\|\cdot\|_{W,a}$ for $\|\cdot\|_{W,1,a}$.

Lemma 1.2 *Let $g \in W(L^\infty, \ell^1)$, and assume that $\{y_n\}_{n\in\mathbb{Z}}$ is δ-separated. Then*

$$\sum_{n\in\mathbb{Z}} |g(t - y_n)| \leq ||g||_{W,\delta}, \quad a.e. \;\; t.$$

If furthermore $b \in]0, \frac{1}{\delta}[$, then

$$\sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} g(t - y_n)\overline{g(t - y_n - k/b)}| \leq 2\|g\|_{W,\delta}^2, \quad a.e. \;\; t.$$

Proof: For the first part, fix $t \in \mathbb{R}$, and observe that every interval $[k\delta, (k+1)\delta[, k \in \mathbb{Z}$, contains at most one point of the form $t - y_n,\ n \in \mathbb{Z}$. Therefore,

$$\sum_{n\in\mathbb{Z}} |g(t - y_n)| \leq \sum_{k\in\mathbb{Z}} ||g\chi_{[k\delta,(k+1)\delta[}||_\infty = \sum_{k\in\mathbb{Z}} ||gT_{k\delta} \cdot \chi_{(0,\delta[}||_\infty = ||g||_{W,\delta}, \quad \text{a.e. t.}$$

For the second part, we have

$$\sum_{k\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} g(t - y_n)g(t - y_n - k/b)| \leq \sum_{n\in\mathbb{Z}} |g(t - y_n)| \sum_{k\in\mathbb{Z}} |g(t - y_n - k/b)|.$$

By assumption the sequence $\{y_n + \frac{k}{b}\}_{k\in\mathbb{Z}}$ is δ-separated, so the first part gives that $\sum_{k\in\mathbb{Z}} |g(t - y_n - k/b)| \leq ||g||_{W,\delta}$ and the Lemma follows. □

2 The Gabor Frame Identity

In this section we consider the form of the Weyl-Heisenberg frame Identity needed for our work. This identity has a rich history. It was originally due to Daubechies [12] for Gabor frames where it was used to give sufficient conditions for (g, a, b) to generate a Gabor frame. Walnut [31, 22] used Wiener amalgum spaces and the WH-frame identity to give other sufficient conditions to have a Gabor frame and introduced what is now called the Walnut representation of the frame operator. The WH-frame identity was extended to translation invariant systems by Janssen [24], Proposition 1.2.1. The minimal conditions necessary for it to hold are in [6]. An analysis of the convergence properties of the series appearing in the WH-frame Identity appears in [4]. The form we give below is due to Janssen [24].

Theorem 2.1 *Let $g \in L^2(\mathbb{R})$, $b > 0$ and let $\{y_n\}_{n\in\mathbb{Z}}$ be real numbers. Assume that $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence. Then for all bounded and compactly supported functions f we have:*

$$\sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{y_n}g\rangle|^2 =$$

$$\frac{1}{b}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \overline{f(t)}f(t-k/b)\sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-k/b)}\ dt =$$

$$b^{-1}\int_{\mathbb{R}} |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2\ dt +$$

$$b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}} \overline{f(t)}f(t-k/b)\sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-k/b)}\ dt$$

We now have a simple sufficient condition for the existence of WH-frames.

Corollary 2.2 *Let $g \in L^2(\mathbb{R})$, $b > 0$ and let $\{y_n\}_{n\in\mathbb{Z}}$ be real numbers with $\{E_{mb}T_{y_n}\}_{m,n\in\mathbb{Z}}$ a Bessel sequence. Assume that*

$$A \ :\ = essinf_{t\in\mathbb{R}}\left[\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 - \sum_{k\neq 0}|\sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-\frac{k}{b})}|\right] > 0$$

and let

$$B := esssup_{t\in\mathbb{R}}\sum_{k\in\mathbb{Z}}|\sum_{n\in\mathbb{Z}} g(t-y_n)\overline{g(t-y_n-\frac{k}{b})}| < \infty.$$

Then $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds $\frac{A}{b}, \frac{B}{b}$.

Proof: Defining

$$H_k(t) := \sum_{n\in\mathbb{Z}} |T_{y_n}g(t)T_{y_n+k/b}g(t)|$$

it is easy to see that $\sum_{k\neq 0}|T_{-k/b}H_k(t)| = \sum_{k\neq 0}|H_k(t)|$. An application of Cauchy-Schwarz gives that for functions f which are bounded with compact support,

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|f(t)f(t-k/b)|\sum_{n\in\mathbb{Z}}|g(t-y_n)g(t-y_n-k/b)|\;dt$$

$$\leq \int_{\mathbb{R}}|f(t)|^2(\sum_{k\neq 0}|H_k(t)|+H_0(t)|)dt<\infty.$$

By the WH-Frame Identity,

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}}|\langle f,E_{mb}T_{y_n}g\rangle|^2 = \\
&|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\overline{f(t)}f(t-k/b)\sum_{n\in\mathbb{Z}}g(t-y_n)\overline{g(t-y_n-k/b)}\;dt| \\
\geq\;& \frac{1}{b}\int|f(t)|^2\left[\sum_{n\in\mathbb{Z}}|g(t-y_n)|^2-\sum_{k\neq 0}|\sum_{n\in\mathbb{Z}}g(t-y_n)\overline{g(t-y_n-\frac{k}{b})}|\right]dt \\
\geq\;& \frac{A}{b}||f||^2.
\end{aligned}$$

A similar arguement gives the upper estimate. Since those two estimates hold on a dense subset of $L^2(\mathbb{R})$, they hold on $L^2(\mathbb{R})$. $\square$

Note that the expression used to define A in Theorem 2.1 is not a periodic function and therefore the infimum in this theorem has to be over $\mathbb{R}$.

3 Frames of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$

Before we present our results we remind the reader that a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is very sensitive to (even arbitrary small) changes of the parameters a, b. In [17] Feichtinger and Janssen have constructed an example, where

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for all $a=\frac{1}{2m}$, $m\in\mathbb{N}$ and $b\in]0,1[$, and

(ii) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is never a frame when $a=\frac{l}{3^k}$, $k,l\in\mathbb{N}$ and $b\in]0,1[$.

This kind of problem can be avoided by restricting the class of functions g. We need a Lemma, which is proved in the second half of the proof of Theorem 4.1.8 in [22].

Lemma 3.1 *Let $g\in W(L^\infty,\ell^1)$. Fix a natural number N and $0<a\in\mathbb{R}$. Let $g_0=g\cdot\chi_{[-aN,aN]}$ and $g_1=g-g_0$. If $1/b\geq 2aN$ then*

$$\sum_{k\neq 0}||\sum_{n\in\mathbb{Z}}T_{y_n}g\cdot T_{y_n+k/b}\overline{g}||_\infty\leq 8||g_0||_{W,a}||g_1||_{W,a}+4||g_1||^2_{W,a}.$$

Theorem 3.2 *Let $g \in W(L^\infty, \ell^1)$. Then the following are equivalent:*

(a) There exists a box $Q := [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$ and $A > 0$ such that

$$P(x,y) := \sum_{n \in \mathbb{Z}} |g(x - ny)|^2 \geq A \text{ for a.e. } (x,y) \in Q.$$

(b) There exists $a_0 > 0$ so that for all $0 < c_0 < a_0$, there are $b_0, A > 0$ such that for all $a \in [c_0, a_0], b \in (0, b_0]$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound A.

Proof: $(a) \Rightarrow (b)$: Since $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}T_c g\}_{m,n\in\mathbb{Z}}$ are Gabor frames together with the same frame bounds for all $c \in \mathbb{R}$, by replacing g with $T_{a_1}g$ we may assume that Q is of the form $[0, b_1] \times [a_2, b_2]$. Let $a_0 = \min(b_1, b_2 - a_2)$.

Now, let $y \in [0, a_0]$, $x \in]0, a_0]$. Since $(\ell+1)y - \ell y = y \leq a_0$, there is an $\ell \in \mathbb{N}$ so that $a_2 \leq \ell y \leq b_2$. Now for all $0 \leq x \leq a_0 \leq b_1$ we have that $(x, \ell y) \in Q$. Hence,

$$A \leq \sum_{n \in \mathbb{Z}} |g(x - n\ell y)|^2 \leq \sum_{n\in\mathbb{Z}} |g(x - ny)|^2.$$

For y fixed, the function $P(x,y)$ is y-periodic in the variable x, and the above inequality holds for all $0 \leq x \leq a_0$ and all $0 < y \leq a_0$, so it follows that the inequality holds for all $x \in \mathbb{R}$.

Now, fix $0 < c_0 < a_0$ and $\epsilon > 0$ so that

$$16\|g\|_{W,c_0} + 4\epsilon^2 \leq \frac{A}{2}.$$

Next, choose a natural number N so that for every $a \geq c_0$, if $g_0 = g \cdot \chi_{[-aN,aN]}$ and $g_1 = g - g_0$ then $\|g_1\|_{W,c_0} < \epsilon$. Finally, let $b_0^{-1} = 2a_0 N$. Now, fix $c_0 \leq a \leq a_0$. For any $0 < b \leq b_0$ we have that $b^{-1} \geq 2aN$. So by Lemma 3.1, for g_0, g_1 as above,

$$\sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g(t-na)\overline{g(t-na-k/b)}| \leq \sum_{k \neq 0} \|\sum_{n\in\mathbb{Z}} T_{na}g \cdot T_{na+k/b}\overline{g}\|_\infty$$

$$\leq 8\|g_0\|_{W,a}\|g_1\|_{W,a} + 4\|g_1\|_{W,a}^2$$

$$\leq 16\|g\|_{W,c_0}\|g_1\|_{W,c_0} + 4\|g_1\|^2_{W,c_0} \leq 16\|g\|_{W,c_0}\epsilon + 4\epsilon^2 \leq \frac{A}{2}.$$

It follows that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ satisfies the CC-Condition (see [3]) and therefore is a frame with lower frame bound $A/2$ for all $c_0 < a \leq a_0$ and $0 < b \leq b_0$.

$(b) \Rightarrow (a)$: Assume (b). In Theorem 2.1, if we consider $f \in L^2(\mathbb{R}$ with support in $[0,1]$ and $0 < b \leq b_0 \leq 1$, we have that $\overline{f(t)}f(t - k/b) = 0$ a.e., for all $k \neq 0$. Hence, for all $c_0 \leq a \leq a_0$ and $y_n = na$, we have from Theorem 2.1 and our assumption that $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ has lower frame bound A,

$$A = A\|f\|^2 = A\int_0^1 |f(x)|^2\,dx \leq \sum_{n,m\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2$$

$$= b^{-1}\int_{\mathbb{R}}|f(x)|^2\sum_{n\in\mathbb{Z}}|g(x-na)|^2\ dx = b^{-1}\int_0^1\sum_{n\in\mathbb{Z}}|f(x)|^2|g(x-na)|^2\ dx.$$

It follows that

$$A \le b^{-1}\sum_{n\in\mathbb{Z}}|g(x-ny)|^2 \quad \text{a.e.} \quad (x,y)\in[0,1]\times[c_0,a_0].$$

□

An examination of the proof of Theorem 3.2 gives a more explicit result:

Corollary 3.3 *Let $g\in W(L^\infty,\ell^1)$ and assume there is a box $Q=[a_1,b_1]\times[a_2,b_2]$ so that*

$$A\le\sum_{n\in\mathbb{Z}}|g(x-ny)|^2 \quad a.e. \quad (x,y)\in Q.$$

Let $a_0 = min(b_1-a_1, b_2-a_2)$ and $0<c_0<a_0$. Choose $\epsilon>0$ such that $8\epsilon\|g\|_{W,c_0}+4\epsilon^2\le\frac{A}{2}$, and choose a natural number N so that $\sum_{|n|\ge N}\|g\cdot\chi_{[c_0n,c_0(n+1)]}\|_\infty\le\epsilon$. Choose b_0 so that $1/b_0\ge 2a_0N$. Then for all $0<a\le a_0$ and all $0<b\le b_0$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with frame bounds $A/2, B=A/2+\|g\|_{W,c_0}$.

Corollary 3.4 *Part (a) in Theorem 3.2 holds if any of the following holds:*

(i) There exists a point x_0 where g is continuous and non-zero.

(ii) There exists a point $(x_0,y_0)\in\mathbb{R}^2$ where $P(x,y):=\sum_{n\in\mathbb{Z}}|g(x-ny)|^2$ is continuous and non-zero.

(iii) g is bounded below on an interval.

We note that there are functions $g\in L^2(\mathbb{R})$ which cannot be used to give Gabor frames for any values of $a,b>0$.

Example 3.5 *There exists a function $0\ne g\in W(L^\infty,\ell^1)$ so that the family $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$ for any $a,b>0$.*

For this example, we construct a Cantor set E of measure $1/2$ in $[0,1]$ by removing the middle one fourth of each interval (instead of middle thirds) in the usual Cantor set construction. That is, in the first step the interval $[\frac{3}{8},\frac{5}{8}]$ is removed, then the process is repeated on the intervals $[0,\frac{3}{8}],[\frac{5}{8},1]$, etc. Let $g=\chi_E$. Now, for $a>1/2$, $I:=[1/2,\min(a,5/8)]$ is removed in the Cantor set construction and so $G_0(t):=\sum_{n\in\mathbb{Z}}|g(t-na)|^2=0$ for all $t\in I$. Hence by Proposition 4.3 $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ cannot form a Gabor frame for any $b>0$. For $0<a\le 1/2$, let $I_0\subset[0,a]$ be an interval removed in the construction. Similarly, for $i=0,1,2,\ldots,k$, with k the greatest integer less than or equal to $a/2$ we can find an interval I_{i+1} removed in the construction so that $I_{i+1}\subset I_i+a$. Letting $I=\cup_j(I_k+ja)$ we have that G_0 vanishes on I and so $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ cannot form a Gabor frame for any $b>0$.

4 Frames of the form $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$

Frames of the form $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ are often obtained via perturbation of a regular Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, cf. [29]. Another approach is to apply the Fourier transform $\mathcal{F}$, which transforms $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ into the shift-invariant system $\{T_{mb}E_{-y_n}\mathcal{F}g\}_{m,n\in\mathbb{Z}}$; after that, the theory developed by e.g. Janssen [24] can be applied. Here we present a different approach.

We first prove a necessary condition for an irregular Gabor family $\{E_{x_n}T_{y_m}g\}_{m,n\in\mathbb{Z}}$ to be a frame. To prove the result we need to recall a result of Christensen, Deng and Heil [11] which is a generalizatin of a fundamental density result of Ramanthan and Steger [27].

Theorem 4.1 *Let $\Lambda = \{(x_n, y_n)\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}\times\mathbb{R}$, $g \in L^2(\mathbb{R})$ and assume that $\{E_{x_n}T_{y_n}g\}_{n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. Then there is a constant $R > 0$ so that for all $c, d \in \mathbb{R}$ we have*

$$([c, c+R]\times[d, d+R]) \cap \Lambda \neq \emptyset.$$

Recall that a family of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ has **uniform density** $D = D(\{\lambda_n\})$ if there is an $L > 0$ such that for all $n \in \mathbb{Z}$ we have $|\lambda_n - \frac{n}{D}| \leq L$. Jaffard [23] has classified when families of exponentials form a frame for $L^2[0, b]$.

Theorem 4.2 *If a family of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ is relatively separated and has a subset of uniform density $D > (b-a)$, then $\{e^{2\pi i\lambda_n t}\}_{n\in\mathbb{Z}}$ forms a frame for $L^2[a, b]$.*

Proposition 4.3 *Assume that $\{E_{x_m}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B. There is an $a > 0$ so that $\{E_{x_m}\}_{m\in\mathbb{Z}}$ is a frame for $L^2[0, a]$ with frame bounds say A_1, B_1. Furthermore*

$$A/B_1 \leq \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq B/A_1 \ a.e. \tag{1}$$

Proof: Let $\Lambda = \{(x_m, y_n) : m, n \in \mathbb{Z}\}$. By Theorem 4.1 there is a constant $R > 0$ so that for all $c, d \in \mathbb{R}$:

$$([c, c+R]\times[d, d+R]) \cap \Lambda \neq \phi.$$

In particular, if $F = \{x_n : n \in \mathbb{Z}\}$ then for all $c \in \mathbb{R}$, $[c, c+R] \cap E \neq \phi$. This means that $(x_n)_{n\in\mathbb{Z}}$ is a set of uniform density in $\mathbb{R}$. Hence, by Theorem 4.2 there is a constant $a > 0$ so that $\{E_{x_n}\}_{n\in\mathbb{Z}}$ is a frame for $L^2[0, a]$ with frame bounds say A_1, B_1. For any interval $I = [b, b+a] \subset \mathbb{R}$ and any bounded function $f \in L^2(I)$ we have

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\langle f\cdot T_{y_n}\overline{g}, E_{x_m}\rangle|^2 \leq \sum_{n\in\mathbb{Z}} B_1\|f\cdot T_{y_n}\overline{g}\|^2.$$

Similarly,

$$\sum_{n\in\mathbb{Z}} A_1\|f\cdot T_{y_n}\overline{g}\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{x_m}T_{y_n}g\rangle|^2.$$

Also,

$$\sum_{n\in\mathbb{Z}} \|f\cdot T_{y_n}\overline{g}\|^2 = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |f(t)\overline{g(t-y_n)}|^2 \; dt =$$

$$\sum_{n\in\mathbb{Z}} \int_I |f(t)|^2 |g(t-y_n)|^2 \; dt = \int_I |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \; dt.$$

Combining the above we have for all $f \in L^2(I)$:

$$\int_I |f(t)|^2 \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \; dt \leq \frac{B}{A_1}\|f\|^2 = \frac{B}{A_1}\int_I |f(t)|^2 \; dt.$$

It follows that

$$\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq \frac{B}{A_1}.$$

The other inequality is done similarly with the other frame inequality. □

Note that if $x_m = mb$ for all $m \in \mathbb{Z}$, then $\{E_{x_m}\}_{m\in\mathbb{Z}}$ is a frame for $L^2[0,1/b]$ with bounds $A_1 = B_1 = b$, so we obtain the classical result as a special case.

Now we prove a classification theorem for certain irregular Gabor frames.

Theorem 4.4 *Let $g \in W(L^\infty, \ell^1)$. The following are equivalent:*

(a) g is bounded below on an interval in $\mathbb{R}$.

(b) There are numbers $a_0, b_0, A > 0$ so that for all $0 < b \leq b_0$ and all $y_n \in [a_0 n, a_0(n+1)]$, $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with lower frame bound A.

Proof: $(a) \Rightarrow (b)$: Assume there is an interval $[c,d] \subset \mathbb{R}$ and an $A > 0$ so that

$$A \leq |g(t)|^2 \;\; \text{a.e.} \;\; t \in [c,d].$$

Let $a_0 = \frac{d-c}{2}$. Then for every $0 < a \leq a_0$ and for every $y_n \in [an, a(n+1)]$ and every $t \in \mathbb{R}$, there is an $n \in \mathbb{N}$ so that $t - y_n \in [c,d]$. Hence,

$$A \leq \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \;\; \text{a.e.} \;\; t \in \mathbb{R}.$$

By changing g on a zero-set we may assume $\sum_{n\in\mathbb{Z}} \sup_{a_0 n \leq t < a_0(n+1)} |g(t)| = \|g\|_{W,a}$. By Theorem 2.1 we will be done by establishing the following claim:

Claim: There is a $b_0 > 0$ so that for all $0 < b \leq b_0$ and all $y_n \in [an, a(n+1)]$ we have

$$\sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g(t-y_{2n})\overline{g(t-y_{2n}-k/b)}| \leq \frac{A}{4}.$$

and

$$\sum_{k\neq 0} |\sum_{n\in\mathbb{Z}} g(t-y_{2n+1})\overline{g(t-y_{2n+1}-k/b)}| \leq \frac{A}{4}.$$

We will do the first inequality above since the second follows by only notational changes. Choose a natural number N and $\epsilon > 0$ (to be specified later) so that

$$\sum_{|n|\geq N} \|g \cdot \chi_{[a_0 n, a_0(n+1))}\|_\infty \leq \epsilon.$$

Let $g_0 = g \cdot \chi_{[-a_0 N, a_0 N]}$ and $g_1 = g - g_0$. Now

(1) $\|g_1\|_{W,a_0} \leq \epsilon$.

(2) If $\frac{1}{b} \geq 2a_0 N = \frac{1}{b_0}$, then

$$g_0(t - y_n)g(t - y_n - k/b) = 0 \text{ for all t and all } k \neq 0.$$

Now we compute. (Since $\{y_{2n}\}$ is an a_0-separated sequence, we can apply Lemma 1.2 in the last inequality below.)

$$\sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |g(t - y_{2n})||g(t - y_{2n} - k/b)| =$$

$$\sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |(g_0 + g_1)(t - y_{2n})||(g_0 + g_1)(t - y_{2n} - k/b)|$$

$$\leq \sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |g_0(t-y_{2n})||g_0(t-y_{2n}-k/b)| + \sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |g_0(t-y_{2n})||g_1(t-y_{2n}-k/b)| +$$

$$\sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |g_1(t - y_{2n})||g_0(t - y_{2n} - k/b)| + \sum_{k\neq 0}\sum_{n\in\mathbb{Z}} |g_1(t - y_{2n})||g_1(t - y_{2n} - k/b)|$$

$$\leq 0 + 4\|g_0\|_{W,a_0}\|g_1\|_{W,a_0} + 4\|g_0\|_{W,a_0}\|g_1\|_{W,a_0} + 4\|g_1\|^2_{W,a_0} \leq 8\|g_0\|_{W,a_0}\epsilon + 4\epsilon^2.$$

If $\epsilon > 0$ is chosen so that the right hand side of the above inequality is $\leq A/4$, we are finished.

$(b) \Rightarrow (a)$: We will do this in steps.

Step I: If $E, F \subset [a, b]$ with $|E|, |F| > 0$, then there is a $0 \leq x \leq b - a$ so that $|(x + E) \cap F| > 0$.

Note that $\chi_{(-E)} * \chi_F \neq 0$ since

$$\int_{\mathbb{R}} \chi_{(-E)} * \chi_F(x)\ dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(-E)}(x - y)\chi_F(y)\ dy\ dx$$

$$= \int_{\mathbb{R}} \chi_F(y) \int_{\mathbb{R}} \chi_{(-E)}(x - y)\ dx\ dy = \int_{\mathbb{R}} \chi_F(y)|E|\ dy = |E| \cdot |F| > 0.$$

Now, for all x in a set of positive measure,

$$0 \neq (\chi_{(-E)} * \chi_F)(-x) = \int_{\mathbb{R}} \chi_{(-E)}(-x - y)\chi_F(y)\ dy = \int_{\mathbb{R}} \chi_{(x+E)}(y)\chi_F(y)\ dy.$$

It follows that $\chi_{(x+E)} \cdot \chi_F \neq 0$ on a set of positive measure. In particular, $|(x + E) \cap F| > 0$.

Step II: We assume the result fails (i.e. g is not bounded below on any interval) and show that for any $0 < a$ and any $0 < b_0$ and any $\epsilon > 0$, there are $y_n \in [an, a(n+1)]$ so that

$$\text{ess inf}_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq \epsilon.$$

Since $g \in W(L^\infty, \ell^1)$, there is a natural number N so that

$$\sum_{|n|\geq N} \|\chi_{[an,a(n+1)]}g\|_\infty^2 \leq \frac{\epsilon}{2}.$$

We assumed that g is not bounded below on any interval. So there are measurable sets $E_n \subset [a(n+1/3), a(n+1-1/3)]$ for $n = -N+1, -N+2, \ldots, N-1$ with $|E_n| > 0$ and

$$\|\chi_{E_n} \cdot g\|_\infty^2 < \frac{\epsilon}{4N}, \quad \text{for all} \quad |n| \leq N-1.$$

Letting $F_n = E_n - a(n+1/3)$, we see that $F_n \subset [a/3, 2a/3]$.

By iterating Step I, we see that there are numbers $0 \leq x_n \leq a/3$ so that

$$E := \cap_{|n|\leq N-1}(x_n + F_n) > 0$$

has positive measure. Let $y_n = x_n + (n + a/3)$ for $|n| \leq N-1$ and $y_n = an$ otherwise. So $y_n \in [an, a(n+1)]$, for all $n \in \mathbb{Z}$. But, for all $t \in E$

we have:

$$\sum_{|n|\geq N} |g(t+y_n)|^2 \leq \sum_{|n|\geq N} \|\chi_{[an,a(n+1)]}g\|^2 \leq \frac{\epsilon}{2}.$$

Also, for $|n| < N$ and $t \in E$ we have $t + y_n \in E_n$ and so

$$\sum_{|n|<N} |g(t+y_n)|^2 \leq \sum_{n\in\mathbb{Z}} \|\chi_{E_n} g\|_\infty^2 \leq 2N\frac{\epsilon}{4N} = \frac{\epsilon}{2}.$$

This completes Step II.

This completes the proof since now it is clear that we cannot have a universal lower frame bound A for all choices of $y_n \in [an, a(n+1)]$ for any choice of $a > 0$. That is, (b) does not hold. □

We now note that if $g \in W(L^\infty, \ell^1)$, the necessary condition in Proposition 4.3 is sufficient to ensure existence of a frame for small values of b. This is a variation of a result of Heil and Walnut ([22], Theorem 4.1.8) and is an immediate consequence of the proof of Theorem 4.4.

Theorem 4.5 *Assume that $g \in W(L^\infty, \ell^1)$, $\{y_n\}_{n\in\mathbb{Z}}$ is a relatively separated sequence of real numbers and that there exists a constant $A > 0$ such that*

$$A \leq \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \ a.e.$$

Then there exists a $b_0 > 0$ such that $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ for all $b \in (0, b_0]$.

We do not know if the assumption $g \in W(L^\infty, \ell^1)$ is necessary in Theorems 4.4 and 4.5. But the next proposition shows that g at least must be in $W(L^\infty, \ell^2)$.

Proposition 4.6 *Let $g \in L^2(\mathbb{R})$ and assume that there exist $a, b > 0$ such that for all $y_n \in [na, (n+1)a]$, $\{E_{mb}T_{y_n}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B. Then*

$$bA \leq \sum_{n\in\mathbb{Z}} ess\ inf_{t\in[an,a(n+1)]}\ |\chi_{[an,a(n+1)]}g(t)|^2 \leq \sum_{n\in\mathbb{Z}} ||\chi_{[an,a(n+1)]}g||_\infty^2 \leq bB.$$

In particular, $g \in W(L^\infty, \ell^2)$.

Proof: We will prove the upper inequality first since we need it to prove the left hand side inequality. By Proposition 4.3 we have

$$A \leq \frac{1}{b}\sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq B \ \text{ a.e. on } \ [a\ell, a\ell + k/b],$$

and hence this inequality holds on all of $\mathbb{R}$. Fix a natural number N and an $\epsilon > 0$. There is a $\delta > 0$ so that for all $-N \leq n \leq N$ there is a measurable set $E_n \subset [an, a(n+1)]$ satisfying:

(1) $0 < |E_n| \leq \delta$.
(2) $\pm\delta + E_n \subset [an, a(n+1)]$.
(3) For all $t \in E_n$ we have

$$\|\chi_{[a,b]}g\|_\infty \leq |g(t)| + \epsilon.$$

Now we proceed as in the proof of Theorem 4.4. We can find an $F \subset [0, a]$ with $0 < |F|$ and $y_n \in [an, a(n+1)]$ so that $F + y_n \subset E_n$, for all $-N \leq n \leq N$. Hence, if $t \in F$ we have

$$bB \geq \sum_{n=-N}^{N} |g(t-y_n)|^2 \geq \sum_{n=-N}^{N} \left(\|\chi_{[an,a(n+1)]}g\|_\infty^2 + \epsilon\right).$$

Since $\epsilon > 0$ was arbitrary and $|F| > 0$ we have

$$\sum_{n=-N}^{N} \|\chi_{[an,a(n+1)}]g\|_\infty^2 \leq bB.$$

Since N was also arbitrary we have

$$\sum_{n\in\mathbb{Z}} \|\chi_{[an,a(n+1)]}g\|_\infty^2 \leq bB.$$

The lower bound is similar to the above. This time choose $F \subset [0, a]$, $|F| > 0$ and $E_n \subset [an, a(n+1)]$ for all $-N \leq n \leq N$ and $y_n \in [an, a(n+1)]$ so that $y_n + F \subset E_n$. Now, for all $t \in F$ and we have

$$|g(t+y_n)|^2 \leq \text{ess inf}_{t\in[an,a(n+1)]}\ |\chi_{[an,a(n+1)]}g(t)|^2 + \frac{\epsilon}{2^{|n|}}.$$

Hence for all $t \in F$,

$$\sum_{n=-N}^{N} |g(t-y_n)|^2 \leq \sum_{n=-N}^{N} \|\chi_{[an,a(n+1)]}g\|^2 + 3\epsilon.$$

By the upper bound inequality proved above, we can choose N so that

$$\sum_{|n|\geq N} \|\chi_{[an,a(n+1)]}g\|_\infty^2 < \epsilon.$$

Hence, for any choice of $y_n \in [an, a(n+1)]$ for $|n| \geq N$ we have

$$bA \leq \sum_{n\in\mathbb{Z}} |g(t-y_n)|^2 \leq \sum_{n=-N}^{N} |g(t-y_n)|^2 + \sum_{|n|\geq N} \|\chi_{[an,a(n+1)]}g\|_\infty^2$$

$$\leq 3\epsilon + \epsilon + \sum_{n\in\mathbb{Z}} \text{ess inf}_{t\in[an,a(n+1)]}|\chi_{[an,a(n+1)]}g(t)|^2.$$

Since $\epsilon > 0$ was arbitrary we have

$$bA \leq \sum_{n\in\mathbb{Z}} \text{ess inf}_{t\in[an,a(n+1)]}|\chi_{[an,a(n+1)]}g(t)|^2.$$

□

It is clear that if g is bounded below on an interval then $P(x,y) = \sum_{n\in\mathbb{Z}} |g(x-ny)|^2$ is bounded below on a box. We do not know of an example where P is bounded below on a box while g is not bounded below on an interval. However, we can show that there is a $g \in L^2(\mathbb{R})$ and a $y \in \mathbb{R}$ so that $P(x,y)$ is bounded below a.e. x while g is not bounded below on an interval. It is an exercise in a point-set topology course to show that the interval $[0,1]$ can be written as the disjoint union of two sets E, F so that neither E nor F contains an interval and each of them is of measure $1/2$. Now let

$$g = \chi_E + \chi_{(F+1)}.$$

Then $P(x,1) = 1$ a.e. x and g is not bounded below on any interval in $\mathbb{R}$.

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Peter G. Casazza
Department of Mathematics
University of Missouri
Columbia

Mo 65211
USA
E-mail: pete@casazza.math.missouri.edu

Ole Christensen
Technical University of Denmark
Department of Mathematics
Building 303
2800 Lyngby
Denmark
E-mail: Ole.Christensen@mat.dtu.dk