Hilbert Space Frames Containing a Riesz Basis and Banach Spaces Which Have No Subspace Isomorphic to c_0

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We prove that a Hilbert space frame $\{f_i\}_{i \in I}$ contains a Riesz basis if every subfamily $\{f_i\}_{i \in J}$, $J \subseteq I$, is a frame for its closed span. Secondly we give a new characterization of Banach spaces which do not have any subspace isomorphic to c_0. This result immediately leads to an improvement of a recent theorem of Holub concerning frames consisting of a Riesz basis plus finitely many elements

1. INTRODUCTION

Let $\mathscr{H}$ be a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. A family $\{f_i\}_{i \in I}$ of elements of $\mathscr{H}$ is called a *frame* for $\mathscr{H}$ if

$$\exists A, B > 0: A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathscr{H}.$$

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A and B are called *frame bounds*. If $\{f_i\}_{i\in I}$ is a frame, then $Sf := \sum_{i\in I}\langle f, f_i\rangle f_i$ defines a bounded invertible operator on $\mathscr{H}$. This fact leads to the *frame decomposition*: every $f \in \mathscr{H}$ has the representation

$$f = SS^{-1}f = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i.$$

So a frame can be considered as a generalized basis in the sense that every element of $\mathscr{H}$ can be written as a linear combination of the frame elements. Actually one has that

$$\{f_i\}_{i\in I} \text{ is a Riesz basis}$$
$$\Leftrightarrow \left[\{f_i\}_{i\in I} \text{ is a frame and } \sum_{i\in I} c_i f_i = 0 \Rightarrow c_i = 0, \forall i.\right]$$

Recently more authors have been interested in the relation between frames and Riesz bases. Holub [H] concentrates on *near-Riesz bases*, i.e., frames consisting of a Riesz basis plus finitely many elements. He is able to give equivalent characterisations of such frames. Seip [Se] deals only with frames of complex exponentials. Among his very interesting results one finds examples of frames which do not contain a Riesz basis. On the other hand he proves that all frames which have appeared "naturally" until his paper contain a Riesz basis.

Using different techniques, the present authors have constructed a frame not containing a Riesz basis [CC]. Furthermore one of the authors gave the first condition implying that a frame contains a Riesz basis [C1]. The purpose of the first part of the present paper is to show that the conclusion is true under weaker conditions than in [C1]. We also give an example where the new theorem can be used, but where the condition in the version of [C1] is not satisfied.

In the second part of the paper we present a new characterization of Banach spaces which do not have any subspace isomorphic to c_0. This is important in itself, but the reason for combining it with the frame result above is that it immediately leads to an improvement of a recent frame result of Holub [H]. Holub shows that if a frame is norm-bounded below, then it is a near-Riesz basis if and only if it is *unconditional*, which means that if a series $\sum_{i\in I} c_i f_i$ converges, then it converges unconditionally. Actually we are able to prove the same without the condition about norm-boundedness; however, this property follows as a consequence of the situation.

2. FRAMES CONTAINING A RIESZ BASIS

A frame $\{f_i\}_{i\in I}$ is called a *Riesz frame* if very subfamily $\{f_i\}_{i\in J}$ is a frame for its closed span, with a lower bound A common for all those frames. One of the main results in [C1] is

THEOREM 2.1. *Every Riesz frame contains a Riesz basis.*

The main ingredient in the proof is an application of Zorn's lemma. Our goal is to show that the conclusion actually holds without the assumption about a common lower bound. This is important in practice, since one might be in the situation that some theoretical arguments give the frame property, but no knowledge of the bounds. However, the proof of this more general theorem is much more complicated and in fact Theorem 2.1 is part of the results we use in the proof. We need a lemma:

LEMMA 2.2. *Let $\{f_i\}_{i\in I}$ be a frame. Given $\epsilon > 0$ and a finite set $J \subseteq I$, there exists a finite set J' containing J such that*

$$\sum_{i\in I-J'} |\langle f, f_i\rangle|^2 \le \epsilon \cdot \|f\|^2, \qquad \forall f \in \operatorname{span}\{f_i\}_{i\in J}.$$

Proof. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $\operatorname{span}\{f_i\}_{i\in J}$. Given $\epsilon > 0$ we take the index set J' containing I such that

$$\sum_{i\in I-J'} |\langle e_j, f_i\rangle|^2 \le \frac{\epsilon}{n}, \qquad \text{for all } j \in J.$$

Now take $f \in \operatorname{span}\{f_i\}_{i\in J}$. Writing $f = \sum_{j=1}^n \langle f, e_j\rangle e_j$ we get

$$\begin{aligned}\sum_{i\in I-J'} |\langle f, f_i\rangle|^2 &= \sum_{i\in I-J'} \left|\sum_{j=1}^n \langle f, e_j\rangle\langle e_j, f_i\rangle\right|^2 \\ &\le \sum_{i\in I-J'} \sum_{j=1}^n |\langle f, e_j\rangle|^2 \sum_{j=1}^n |\langle e_j, f_i\rangle|^2 \\ &= \sum_{i\in I-J'} \sum_{j=1}^n |\langle e_j, f_i\rangle|^2 \cdot \|f\|^2 \le \epsilon \cdot \|f\|^2.\end{aligned}$$

PROPOSITION 2.3. *Let $\{f_i\}_{i\in I}$ be a frame with the property that every subset of $\{f_i\}$ is a frame for its closed linear span. Then there is an $\epsilon > 0$, and finite subsets $J \subset J' \subset I$, with the property: For every $J'' \subset I - J'$, the family $\{f_i\}_{i\in J\cup J''}$ has lower frame bound $\ge \epsilon$.*

Proof. We assume that proposition fails and construct by induction sequences of finite subsets $J_1, J_2, \dots$ and $J'_1, J'_2, \dots$ with the following properties.

(1) $\bigcup_{j=1}^n J_j \subset J'_n$,

(2) For every $f \in \operatorname{span}\{f_i\}_{i \in \bigcup_{j=1}^n J_j}$, with $\|f\| = 1$,

$$\sum_{i \in I - J'_n} |\langle f, f_i \rangle|^2 \le \frac{1}{n},$$

(3) There is some $f \in \operatorname{span}\{f_i\}_{i \in \bigcup_{j=1}^n J_j}$, with $\|f\| = 1$ and

$$\sum_{i \in \bigcup_{j=1}^n J_j} |\langle f, f_i \rangle|^2 \le \frac{1}{n}.$$

We will quickly check the induction step. Assume $J_1, J_2, \dots, J_n$ and $J'_1, J'_2, \dots, J'_{n-1}$ have been chosen to satisfy (1)–(3) above. By Lemma 2.2, there is a finite set $J'_n \subset I$ with $\bigcup_{j=1}^n \subset J'_n$ satisfying (2) above with the constant $1/(n+1)$. Given $\epsilon = 1/(n+1)$, and $J = \bigcup_{j=1}^n J_j$ and $J \subset J'_n$, our assumption that the proposition fails implies there is a finite set $J_{n+1} \subset I - J'_n$ so that (3) holds for $1/(n+1)$. This completes the induction. We now let $J = \bigcup_{n=1}^\infty J_n$. It is easily seen from (2) and (3) above that $\{f_i\}_{i \in J}$ is not a frame for its closed linear span. That is, for each n, there is a $f \in \operatorname{span}\{f_i\}_{i \in \bigcup_{j=1}^n J_j}$ with $\|f\| = 1$ and satisfying (3). We now have by (2),

$$\sum_{i \in J} |\langle f, f_i \rangle|^2 = \sum_{i \in \bigcup_{j=1}^n J_j} |\langle f, f_i \rangle|^2 + \sum_{i \in \bigcup_{j=n+1}^\infty J_j} |\langle f, f_i \rangle|^2 \le \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

This contradiction completes the proof of the proposition.

THEOREM 2.4. *If every subset of $\{f_i\}_{i \in I}$ is a frame for its closed linear span, then $\{f_i\}_{i \in I}$ contains a Riesz basis.*

Proof. By Proposition 2.3, there exists an $\epsilon > 0$ and finite sets J and J' with $J \subset J'$ so that whenever $J'' \subset I - J'$ the lower frame bound of $\{f_i\}_{i \in J \cup J''}$ is $\ge \epsilon$. Let P denote the orthogonal projection of H onto $\operatorname{span}\{f_i\}_{i \in J}$. Then for all $J'' \subset I - J'$, if $f \in \operatorname{span}\{(I-P)f_i\}_{i \in J''}$, then $f \in \operatorname{span}\{f_i\}_{i \in J \cup J''}$, so

$$\begin{aligned}\epsilon \|f\|^2 &\le \sum_{i \in J \cup J''} |\langle f, f_i \rangle|^2 = \sum_{i \in J \cup J''} |\langle (I-P)f, f_i \rangle|^2 \\ &= \sum_{i \in J''} |\langle (I-P)f, f_i \rangle|^2 = \sum_{i \in J''} |\langle f, (I-P)f_i \rangle|^2.\end{aligned}$$

It follows that for every $J'' \subset I - J'$, the set $\{(I-P)f_i\}_{i \in J''}$ has lower frame bound $\epsilon > 0$. Obviously every frame $\{(I-P)f_i\}_{i \in J'}$ has the same upper bound as $\{f_i\}_{i \in I}$, so Theorem 2.1 applied to $\{(I-P)f_i\}_{i \in I-J'}$ shows that there exists a subset $I' \subseteq I - J'$ such that $\{(I-P)f_i\}_{i \in I'}$ is a Riesz basis for its closed span. Hence, for all sequences $\{c_i\} \in l^2(I')$,

$$\left\| \sum_{i \in I'} c_i f_i \right\| \geq \left\| \sum_{i \in I'} c_i (I-P) f_i \right\| \geq \epsilon \sqrt{\sum_{i \in I'} |c_i|^2},$$

i.e., $\{f_i\}_{i \in I'}$ is a Riesz basis for its closed span. But since $\dim(P\mathscr{H}) < \infty$ it can be extended to a Riesz basis for $\mathscr{H}$ just by adding finitely many elements.

To prove that Theorem 2.4 really is an improvement of Theorem 2.1 one needs an example of a frame, where every subfamily is a frame for its closed span, but where there does not exists a common lower bound for all those frames. We present such an example now:

EXAMPLE. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ and define

$$\{f_i\}_{i \in K} := \left\{ e_i, e_i + \frac{1}{2^i} e_1 \right\}_{i=2}^{\infty}.$$

First we show that every subfamily $\{f_i\}_{i \in I}$, is a frame for its closed span. For convenience, write the subfamily as

$$\{f_i\}_{i \in I} = \{e_i\}_{i \in I} \cup \left\{ e_i + \frac{1}{2^i} e_1 \right\}_{i \in J}.$$

First we assume that $I \cap J = \varnothing$. Then $\{e_i\}_{i \in I \cup J}$ is an orthonormal basis for its closed span. The idea is now to show that $\{f_i\}_{i \in L}$ is a perturbation of this family and thereby conclude that the family itself is a frame. Since

$$\sum_{i \in I \cup J} \|f_i - e_i\|^2 = \sum_{i \in J} \left[\frac{1}{2^i} \right]^2 \leq \sum_{i=2}^{\infty} \left[\frac{1}{2^i} \right]^2 < 1$$

we conclude by [C1, Corollary 2.3(b)] that $\{f_i\}_{i \in L}$ is a Riesz basis for its closed span, as desired.

Now assume that $I \cap J \neq \varnothing$. Write

$$\{f_i\}_{i \in L} = \left\{ e_i, e_i + \frac{1}{2^i} e_1 \right\}_{i \in I \cap J} \cup \{e_i\}_{i \in I - J} \cup \left\{ e_i + \frac{1}{2^i} e_1 \right\}_{i \in J - I}.$$

Clearly $\overline{\text{span}}\{f_i\}_{i\in L} = \overline{\text{span}}\{\{e_1\} \cup \{e_i\}_{i\in I\cup J}\}$. The sequence $\{f_i\}_{i\in L}$ is a Bessel sequence (i.e., the upper frame condition is satisfied) so by [C2, Corollary 4.3] we are done if we can show that the operator

$$T: l^2(L) \to \overline{\text{span}}\{f_i\}_{i\in L}, \qquad T\{e_i\} = \sum_{i\in L} e_i f_i$$

is surjective. Now, let $f \in \overline{\text{span}}f_i\}_{i\in L}$. We want to write f as a linear combination of elements f_i with coefficients from $l^2(L)$. First,

$$\begin{aligned} f &= \sum_{i\in I-J} \langle f, e_i\rangle e_i + \sum_{i\in J-I} \langle f, e_i\rangle e_i + \sum_{i\in I\cap J} \langle f, e_i\rangle e_i + \langle f, e_1\rangle e_1 \\ &= \sum_{i\in I-J} \langle f, e_i\rangle e_i + \sum_{i\in J-I} \langle f, e_i\rangle \left(e_i + \frac{1}{2^i}e_1\right) \\ &\quad - \sum_{i\in J-I} \langle f, e_i\rangle \frac{1}{2^i}e_1 + \langle f, e_1\rangle e_1 + \sum_{i\in I\cap J} \langle f, e_i\rangle e_i. \end{aligned}$$

Choose $n \in I \cap J$. Then

$$\begin{aligned} f &= \sum_{i\in I-J} \langle f, e_i\rangle e_i + \sum_{i\in J-I} \langle f, e_i\rangle \left(e_i + \frac{1}{2^i}e_1\right) \\ &\quad + \left(-2^n \sum_{i\in J-I} \langle f, e_i\rangle \frac{1}{2^i} + 2^n \langle f, e_1\rangle\right)\left(e_n + \frac{1}{2^n}e_1\right) \\ &\quad - \left(-2^n \sum_{i\in J-I} \langle f, e_i\rangle \frac{1}{2^i} + 2^n \langle f, e_1\rangle\right) e_n \\ &\quad + \langle f, e_n\rangle e_n + \sum_{i\in I\cap J-\{n\}} \langle f, e_i\rangle e_i \\ &= \sum_{i\in I\cup J} \langle f, e_i\rangle e_i + \sum_{i\in J-I} \langle f, e_i\rangle \left(e_i + \frac{1}{2^i}e_1\right) \\ &\quad + \left(-2^n \sum_{i\in J-I} \langle f, e_i\rangle \frac{1}{2^i} + 2^n \langle f, e_1\rangle\right) \cdot \left(e_n + \frac{1}{2^n}e_1\right) \\ &\quad + \left(2^n \sum_{i\in J-I} \langle f, e_i\rangle \frac{1}{2^i} - 2^n \langle f, e_1\rangle + \langle f, e_n\rangle\right) e_n + \sum_{i\in I\cap J-\{n\}} \langle f, e_i\rangle e_i. \end{aligned}$$

So every $f \in \overline{\text{span}}\{f_i\}_{i\in L}$ can be written as a linear combination of the elements in $\{f_i\}_{i\in L}$, and obviously the coefficient sequence is in $l^2(L)$. That is, T is surjective, and the proof is complete. Now we show that there is no common lower bound for all subframes. Let $n \in N$ and consider the family $\{e_n, e_n + (1/2^n)e_1\}$, which is a frame for $\mathrm{span}\{e_1, e_n\}$. Since

$$|\langle e_1, e_n\rangle|^2 + \left|\left\langle e_1, e_n + \frac{1}{2^n}e_1\right\rangle\right|^2 = \frac{1}{2^{2n}} \cdot \|e_1\|^2$$

the lower bound for this frame is at most $1/2^{2n}$. Hence there is no common lower bound.

3. BANACH SPACES HAVING A UNCONDITIONAL BASIS AND NEAR-RIESZ BASES IN HILBERT SPACES

In this section we prove the following:

THEOREM 3.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for the Hilbert space $\mathscr{H}$. Then $\{f_i\}_{i=1}^{\infty}$ is unconditional if and only if it is a near-Riesz basis.*

With the additional assumption that the f_i's are norm bounded below the result is proven by Holub [H] using a result of Heil [He, p. 168]. Theorem 3.1 shows that this assumption is superfluous. However, it is a consequence of the situation, since every near-Riesz basis is norm-bounded below. The "if" part above follows from this property and the original result of Holub. The "only if" part is more complicated, but actually we prove a much more general result concerning abstract Banach spaces. This result has independent interest in Banach space theory in that it classifies those Banach spaces which do not have any subspace isomorphic to c_0.

THEOREM 3.2. *Let X be any Banach space. The following are equivalent:*

(1) *No subspace of X is isomorphic to c_0.*

(2) *If $\{y_i\}_{i=1}^{\infty} \subseteq X$ is a sequence, so that whenever $\sum_{i=1}^{\infty} a_i y_i$ converges for some coefficient sequence $\{a_i\}$, the series must converge unconditionally, then there is some $n_0 \in N$ so that $\{y_i\}_{i=n_0}^{\infty}$ is a unconditional basis for its closed span.*

The "only if" part of Theorem 3.1 is a consequence of Theorem 3.2. Actually a Hilbert space satisfies (1) of Theorem 3.2, so if $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis for a Hilbert space $\mathscr{H}$, then there exists a n_0 such that $\{f_i\}_{i=n_0}^{\infty}$ is a unconditional basis for its closed span and a frame for its closed span. Here we used the fact that if one deletes finitely many elements from a frame, then one still has a family which is a frame for its closed span ([C3, Lemma 2] or [CH, Theorem 7] for a more general statement). So by the characterization of Riesz bases in the Introduction, $\{f_i\}_{i=n_0}^{\infty}$ is a Riesz basis for its closed span. This space has finite codimension, so adding finitely many elements we obtain a Riesz basis for $\mathscr{H}$, and Theorem 3.1 follows.

The proof of Theorem 3.2 requires some preliminary results.

LEMMA 3.3. *If X is a Banach space, $\{y_i\}_{i=1}^{\infty} \subseteq X$, and $\{c_i\}_{i=1}^{\infty}$ are scalars so that $\sum_{i=1}^{\infty} c_i y_i$ converges unconditionally in X, then*

$$\lim_{n\to\infty} \sup_{\epsilon_i = \pm 1} \left\| \sum_{i=n}^{\infty} \epsilon_i c_i y_i \right\| = 0.$$

Proof. If the conclusion of the lemma fails, then there is some $\epsilon > 0$ and natural numbers $n_1 < n_2 < \cdots$ and some $\epsilon_i^j = \pm 1$, $j = 1, 2, \ldots$ and $i = n_j, n_j + 1 \ldots$, with $\|\sum_{i=n_j}^{\infty} \epsilon_i^j c_i y_i\| \geq \epsilon$. Since $\sum_{i=1}^{\infty} c_i y_i$ converges unconditionally, $\sum_{i=n_j}^{\infty} \epsilon_i^j c_i y_i$ converges in X, and hence $\lim_{k\to\infty} \|\sum_{i=k}^{\infty} \epsilon_i^j c_i y_i\| = 0$ for all j. Therefore, by switching to a subsequence of n_j (let us call it n_j again), we have for all $j = 1, 2, \ldots,$

$$\left\| \sum_{i=n_j}^{n_{j+1}-1} \epsilon_i^j c_i y_i \right\| \geq \frac{\epsilon}{2}.$$

Letting $d_i := \epsilon_i^j c_i$, for $n_j \leq i \leq n_{j+1} - 1$ we have that

$$\sum_{j=1}^{\infty} \sum_{i=n_j}^{n_{j+1}-1} \epsilon_i^j c_i y_i = \sum_{i=n_1}^{\infty} d_i y_i$$

converges in X, since it is just a change of signs on the unconditionally convergent series $\sum_{i=n_1}^{\infty} c_i y_i$. However,

$$\left\| \sum_{i=n_j}^{n_{j+1}-1} d_i y_i \right\| = \left\| \sum_{i=n_j}^{n_{j+1}-1} \epsilon_i^j c_i y_i \right\| \geq \frac{\epsilon}{2}$$

implies that $\sum_{i=n_1}^{\infty} d_i y_i$ does not converge in X. This contradiction completes the proof of Lemma 3.3.

Next we introduce the notation needed for the proof of Theorem 3.2. If $\{x_i\}_{i=1}^{\infty}$ is a basis for a Banach space X, $n_1 < n_2 < \cdots$ are natural numbers, and $y_j = \sum_{i=n_j}^{n_{j+1}-1} c_i x_i$ are vectors in X, we call $\{y_i\}_{i=1}^{\infty}$ a *block basic sequence* of $\{x_i\}_{i=1}^{\infty}$. If $\{x_i\}_{i=1}^{\infty}$ is a unconditional basis for X, then it is easily seen that a block basic sequence $\{y_i\}_{i=1}^{\infty}$ is a unconditional basis for its closed span with unconditional basis constant less than or equal to the unconditional basis constant for $\{x_i\}_{i=1}^{\infty}$ in X.

A series $\sum_n x_n$ is a Banach space is said to be *weakly unconditionally Cauchy* if given any permutation π of the natural numbers, $(\sum_{k=1}^{n} x_{\pi(k)})$ is a weakly Cauchy sequence; alternatively, $\sum_n x_n$ is weakly unconditionally Cauchy if and only if for each $x^* \in X^*$, $\sum_n |x^*(x_n)| < \infty$ (see [D, Chap. V]). Let $\hat{N}$ denote the family of all finite subsets of the natural numbers. We will need a result which follows immediately from Theorem 6 of [D, p. 44].

PROPOSITION 3.4. *The following statements are equivalent*:

(1) $\Sigma_n x_n$ *is weakly unconditionally Cauchy.*

(2) $\sup_{\Delta \in \hat{N}} \|\Sigma_{n \in \Delta} x_n\| < \infty$.

Also, we will need Theorem 8 from [D, p. 45], which we now state for completeness.

PROPOSITION 3.5. *Let X be a Banach space. Then, in order that each weakly unconditionally Cauchy series in X be unconditionally convergent, it is necessary and sufficient that X contains no copy of* c_0.

Now we are ready to prove Theorem 3.2

(2) ⇒ (1). It suffices to show that c_0 fails property (2). Let $\{e_n\}_{n=1}^{\infty}$ be the unit vector basis of c_0 and define

$$y_{2n} = e_n, \qquad y_{2n+1} = e_n,\ n = 1, 2, \ldots .$$

We will show that $\{y_n\}$ satisfies the hypotheses of (2) but fails the conclusion. So assume that $\Sigma_{n=1}^{\infty} c_n y_n$ converges in c_0. Since $\|y_n\| = 1$, for all n, it follows that $\lim_{n\to\infty} |c_n| = 0$. Given any $\epsilon_n = \pm 1$, $\|\Sigma_{n=m}^{\infty} \epsilon_n c_n y_n\|_\infty \le 2 \sup_{m \le k} |\epsilon_k c_k|$. Hence,

$$\lim_{m\to\infty} \left\| \sum_{n=m}^{\infty} \epsilon_n c_n y_n \right\|_\infty \le 2 \lim_{m\to\infty} \sup_{m \le k} |c_k| = 0.$$

So $\Sigma_{n=1}^{\infty} \epsilon_n c_n y_n$ converges in c_0. That is, whenever $\Sigma_{n=1}^{\infty} c_n y_n$ converges in c_0, then the series converges unconditionally. So the hypotheses of (2) in Theorem 3.2 are satisfied. But clearly the conclusion of (2) fails since any subset of $\{y_n\}$, which contains all but a finite number of the y_n, must contain two equal elements and hence cannot be independent.

(1) ⇒ (2). We proceed by way of contradiction. So assume (1) and the hypotheses of (2) are satisfied, but the conclusion of (2) fails. Alternately applying this assumption and Lemma 3.3, we find natural numbers $n_1, n_2, \ldots,$ and $\epsilon_i^j = \pm 1$, and scalars $\{c_i\}_{i=1}^{\infty}$ so that for all j,

(3) $\|\Sigma_{i=n_j+1}^{n_{j+1}} c_i y_i\| < 1/2^j$,

(4) $1/2 \le (1/2) \sup_{\epsilon_i = \pm 1} \|\Sigma_{i=n_j+1}^{n_{j+1}} \in_i c_i y_i\| \le \|\Sigma_{i=n_j+1}^{n_{j+1}} \epsilon_i^j c_i y_i\| = 1$.

We let $z_j = \Sigma_{i=n_j+1}^{n_{j+1}} \epsilon_i^j c_i y_i$, for $j = 1, 2, \ldots$.

Claim. $\sup_{\Delta \in \hat{N}} \|\Sigma_{i \in \Delta} z_i\| = \infty$.

This claim follows quickly. That is, if this sup were finite, then $\Sigma_i z_i$ would be unconditionally Cauchy by Proposition 3.4. But then since we assumed c_0 does not embed into our space, by Proposition 3.5, it would follow that this series is unconditionally convergent. But this is ridiculous since $\|z_i\| = 1$, for all $i = 1, 2, \ldots$. This completes the proof of the Claim.

By applying our claim and choosing successive subsets $\Delta \in \hat{N}$, and reindexing we have the following. There are natural numbers $n_1, n_2, \ldots,$

natural numbers $0 = m_0 < m_1 < m_2 < \cdots$, a sequence of scalars $\{c_i\}$, and choices of signs $\epsilon_i^j = \pm 1$, so that

(5) $\|\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\| < 1/2^j$,

(6) $1/2 \le (1/2)\sup_{\epsilon_i=\pm 1}\|\sum_{i=n_j+1}^{n_{j+1}} \epsilon_i c_i y_i\| \le \|\sum_{i=n_j+1}^{n_{j+1}} \epsilon_i^j c_i y_i\| = 1$,

(7) $\|\sum_{j=m_k+1}^{m_{k+1}}(\sum_{i=n_j+1}^{n_{j+1}} \epsilon_i^j c_i y_i)\| = K_k \ge k$.

We will now show that the series

(8) $\sum_{k=1}^{\infty}(1/K_k)[\sum_{j=m_k+1}^{m_{k+1}}(\sum_{n_j+1}^{n_{j+1}} c_i y_i)]$

converges in X as a series in $c_i y_i$, but the series does not converge unconditionally.

That the series does not converge unconditionally can be proven easily. For any $k = 1, 2, \ldots,$

$$\sup_{\epsilon_i=\pm 1}\frac{1}{k_k}\left\|\sum_{j=m_k+1}^{m_{k+1}}\left(\sum_{i=n_j+1}^{n_{j+1}} \epsilon_i c_i y_i\right)\right\| \ge \frac{1}{K_k}\left\|\sum_{j=m_k+1}^{m_{k+1}}\left(\sum_{i=n_j+1}^{n_{j+1}} \epsilon_i^j c_i y_i\right)\right\| = 1.$$

That is, the series in (8) fails Lemma 3.3 and hence is not unconditionally convergent. To prove that the series in (8) converges, we must check that the "tail end" of the series converges to 0 in norm. So consider $\sum_{i=s}^{\infty} c_i y_i$, and fix k with $m_k + 1 \le l \le m_{k+1}$ and $n_l + 1 \le s \le n_{l+1}$. Then

$$\left\|\frac{1}{K_k}\sum_{i=s}^{n_{l+1}} c_{iy_i} + \frac{1}{K_k}\sum_{j=l+1}^{m_{k+1}}\left(\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\right) + \sum_{t=k+1}^{\infty}\frac{1}{K_t}\left[\sum_{j=m_t+1}^{m_{t+1}}\left(\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\right)\right]\right\|$$

$$\le \frac{1}{K_k}\left\|\sum_{i=s}^{n_{l+1}} c_i y_i\right\| + \frac{1}{K_k}\sum_{j=l+1}^{m_{k+1}}\left\|\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\right\|$$

$$+ \sum_{t=k+1}^{\infty}\frac{1}{K_t}\left[\sum_{j=m_t+1}^{m_{t+1}}\left\|\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\right\|\right]$$

$$\le \frac{2}{K_k}\sup_{\epsilon_i=\pm 1}\left\|\sum_{i=n_l+1}^{n_{l+1}} \epsilon_i c_i y_i\right\| + \sum_{j=l+1}^{\infty}\left\|\sum_{i=n_j+1}^{n_{j+1}} c_i y_i\right\|$$

$$\le \frac{2}{K_k} + \sum_{i=l}^{\infty}\frac{1}{2^i} \le \frac{2}{k} + \frac{1}{2^{l-1}}.$$

Hence, our series (11) converges. This contradiction completes the proof of Theorem 3.2.

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$c_i y_i\| = 1,$

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proven easily.

$\epsilon_i^j c_i y_i \Big) \Big\| = 1.$

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Then

$\Big(\sum_{i=n_j+1}^{n_{j+1}} c_i y_i \Big) \Big] \Big\|$

letes the proof

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