

A BRIEF INTRODUCTION TO HILBERT SPACE FRAME THEORY AND ITS APPLICATIONS

PETER G. CASAZZA AND JANET C. TREMAIN

ABSTRACT. This is a short introduction to Hilbert space frame theory and its applications for those outside the area who want an introduction to the subject. We will increase this over time. There are incomplete sections at this time. If anyone wants to add a section or fill in an incomplete section on "their applications" contact Pete Casazza.

1. BASIC DEFINITIONS

For a more complete treatment of frame theory we recommend the book of Christensen [18], and the tutorials of Casazza [5, 6] and the Memoir of Han and Larson [23]. For a complete treatment of frame theory in time-frequency analysis we recommend the excellent book of Gröchenig [22]. For an excellent introduction to frame theory and filter banks plus applications we recommend Kovačević and Chebira [29]. Hilbert space frames were introduced by Duffin and Schaeffer in 1952 [20] to address some deep questions in non-harmonic Fourier series. The idea was to weaken Parseval's identity.

Parseval's Identity 1.1. *If $\{f_i\}_{i \in I}$ is an orthonormal sequence in a Hilbert space \mathbb{H} , then for all $f \in \mathbb{H}$ we have*

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 = \|f\|^2.$$

We do not need to have an orthonormal sequence to have equality in Parseval's identity. For example, if $\{e_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are orthonormal bases for a Hilbert space \mathbb{H} then $\{\frac{1}{\sqrt{2}}e_i, \frac{1}{\sqrt{2}}g_i\}_{i \in I}$ satisfies Parseval's identity.

In 1952, Duffin and Schaeffer [20] were working on some deep problems in non-harmonic Fourier series and weakened Parseval's identity to produce what they called a *frame*.

Definition 1.2. *A family of vectors $\{f_i\}_{i \in I}$ is a **frame** for a Hilbert space \mathbb{H} if there are constants $0 < A \leq B < \infty$ so that for all $f \in \mathbb{H}$*

$$(1.1) \quad A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

The authors were supported by NSF DMS 0704216.

A, B are called the **lower** (respectively, **upper**) **frame bounds** for the frame.

If $A = B$ this is an **A-tight frame** and if $A = B = 1$ this is a **Parseval frame**. If $\|f_i\| = \|f_j\|$ for all $i, j \in I$, this is an **equal norm** frame and if $\|f_i\| = 1$ for all $i \in I$ this is a **unit norm** frame. If we have just the right hand inequality in inequality (9.1) we call $\{f_i\}_{i \in I}$ a **B-Bessel** sequence. The frame is **bounded** if $\inf_{i \in I} \|f_i\| > 0$.

We note that there are no restrictions put on the frame vectors. For example, if $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for a Hilbert space \mathbb{H} , then $\{e_1, 0, e_2, 0, e_3, 0, \dots\}$ is a Parseval frame for \mathbb{H} . Also,

$$\left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots\right\},$$

is a Parseval frame for \mathbb{H} .

Definition 1.3. A family of vectors $\{f_i\}_{i \in I}$ in a Hilbert space \mathbb{H} is a **Riesz basic sequence** if there are constants $A, B > 0$ so that for all families of scalars $\{a_i\}_{i \in I}$ we have

$$A \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i f_i \right\|^2 \leq B \sum_{i \in I} |a_i|^2.$$

This family is a **Riesz basis** for \mathbb{H} if it also spans \mathbb{H} . So a Riesz basis for \mathbb{H} is a frame for \mathbb{H} .

If $\{f_i\}_{i \in I}$ is a frame for \mathbb{H} with frame bounds A, B we define the **analysis operator** $T : \mathbb{H} \rightarrow \ell_2(I)$ to be

$$T(f) = \sum_{i \in I} \langle f, f_i \rangle e_i, \quad \text{for all } f \in \mathbb{H},$$

where $\{e_i\}_{i \in I}$ is the natural orthonormal basis of $\ell_2(I)$. The adjoint of the analysis operator is the **synthesis operator** which is given by

$$T^*(e_i) = f_i.$$

It follows that T is a bounded operator if and only if $\{f_i\}_{i \in I}$ is a Bessel sequence. Also,

$$\|T(f)\|^2 = \sum_{i \in I} |\langle f, f_i \rangle|^2.$$

We then have the following theorem.

Theorem 1.4. Let $\{f_i\}_{i \in I}$ be a family of vectors in a Hilbert space \mathbb{H} . The following are equivalent:

- (1) $\{f_i\}_{i \in I}$ is a frame for \mathbb{H} .
- (2) The operator T^* is bounded, linear and onto.
- (3) The operator T is a (possibly into) isomorphism.

Moreover, $\{f_i\}_{i \in I}$ is a Parseval frame if and only if the synthesis operator is a quotient map (i.e. a partial isometry) if and only if $S = I$ if and only if T is a partial isometry.

The **frame operator** for the frame is $S = T^*T : \mathbb{H} \rightarrow \mathbb{H}$ given by

$$Sf = T^*Tf = T^* \left(\sum_{i \in I} \langle f, f_i \rangle e_i \right) = \sum_{i \in I} \langle f, f_i \rangle T^* e_i = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

A direct calculation now yields

$$\langle Sf, f \rangle = \sum_{i \in I} |\langle f, f_i \rangle|^2.$$

So the frame operator is a **positive, self-adjoint, and invertible operator** on \mathbb{H} . Moreover,

$$A \cdot I \leq S \leq B \cdot I.$$

We can **reconstruct** vectors in the space by

$$\begin{aligned} f &= SS^{-1}f \\ &= \sum_{i \in I} \langle S^{-1}f, f_i \rangle f_i \\ &= \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i \\ &= \sum_{i \in I} \langle f, S^{-1/2}f_i \rangle S^{-1/2}f_i. \end{aligned}$$

We call $\{\langle S^{-1}f, f_i \rangle\}_{i \in I}$ the **frame coefficients** of the vector $f \in \mathbb{H}$. Also, since S is invertible, the family $\{S^{-1}f_i\}_{i \in I}$ is also a frame for \mathbb{H} called the **canonical dual frame**. Recall that two sets of vectors $\{f_i\}_{i \in I}$ and $\{g_i\}_{i \in I}$ are **equivalent** if the operator $L(f_i) = g_i$ is a bounded invertible operator.

Theorem 1.5. *Every frame $\{f_i\}_{i \in I}$ (with frame operator S) is equivalent to the Parseval frame $\{S^{-1/2}f_i\}_{i \in I}$.*

The main property of frames which makes them so useful in applied problems is their **redundancy**. That is, each vector in the space has infinitely many representations with respect to the frame but it also has one natural representation given by the frame coefficients. The role played by redundancy varies with specific applications. One important role is its **robustness**. That is, by spreading our information over a wider range of vectors, we are better able to sustain **losses** (called **erasures** in this setting) and still have accurate reconstruction. This shows up in internet coding (for transmission losses), distributed processing (where “sensors” are constantly fading out), modeling the brain (where memory cells are constantly dying out) and a host of other

applications. Another advantage of spreading our information over a wider range is to mitigate the effects of noise in our signal or to make it prominent enough so it can be removed as in signal/image processing. Another advantage of redundancy is in areas such as quantum tomography where we need classes of orthonormal bases which have “constant” interactions with one another or we need vectors to form a Parseval frame but have the absolute values of their inner products with all other vectors the same. In speech recognition, we need a vector to be determined by the absolute value of its frame coefficients. This is a very natural frame theory problem since this is impossible for a linearly independent set to achieve. Redundancy is a fundamental issue in this setting.

Our next proposition shows the relationship between the frame elements and the frame bounds.

Proposition 1.6. *Let $\{f_i\}_{i \in I}$ be a frame for H with frame bounds A, B . We have for all $i \in I$ that $\|f_i\|^2 \leq B$ and $\|f_i\|^2 = B$ implies $f_i \perp \text{span}_{j \neq i} f_j$. If $\|f_i\|^2 < A$, then $f_i \in \overline{\text{span}}(f_j)_{j \neq i}$.*

Proof. If we replace f in the frame definition by f_i we see that

$$A\|f_i\|^2 \leq \|f_i\|^4 + \sum_{j \neq i} |\langle f_i, f_j \rangle|^2 \leq B\|f_i\|^2.$$

The first part of the result is now immediate. For the second part, assume to the contrary that $E = \text{span}(f_j)_{j \neq i}$ is a proper subspace of H . Replacing f_i in the above inequality by $P_{E^\perp} f_i$ and using the left hand side of the inequality yields an immediate contradiction. \square

As a particular case of Proposition 1.6 we have for a Parseval frame $\{f_i\}_{i \in I}$ that $\|f_i\|^2 \leq 1$ and $\|f_i\| = 1$ if and only if $f_i \perp \text{span}_{j \neq i} f_j$. We call $\{f_i\}_{i \in I}$ an **exact frame** if it ceases to be a frame when any one of its vectors is removed. If $\{f_i\}_{i \in I}$ is an exact frame then $\langle S^{-1} f_i, f_j \rangle = \langle S^{-1/2} f_i, S^{-1/2} f_j \rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker delta) since $\{S^{-1/2} f_i\}_{i \in I}$ is now an orthonormal basis for H . That is, $\{S^{-1} f_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ form a biorthogonal system. Also, it follows that $\{e_i\}_{i \in I}$ is an orthonormal basis for \mathbb{H} if and only if it is an exact, Parseval frame. Another consequence of Proposition 1.6 is

Proposition 1.7. *The removal of a vector from a frame leaves either a frame or an incomplete set.*

Proof. By Theorem 1.5, we may assume that $\{f_i\}_{i \in I}$ is a Parseval frame. Now, by Proposition 1.6, for any i , either $\|f_i\| = 1$ and $f_i \perp \text{span}_{j \neq i} f_j$, or $\|f_i\| < 1$ and $f_i \in \text{span}_{j \neq i} f_j$. \square

Since a frame is not ω -independent (unless it is a Riesz basis) a vector in the space may have many representations relative to the frame besides the natural

one given by the frame coefficients. However, the natural representation of a vector is the unique representation of minimal ℓ_2 -norm as the following result of Duffin and Schaeffer [20] shows.

Theorem 1.8. *Let $\{f_i\}_{i \in I}$ be a frame for a Hilbert space H and $f \in H$. If $\{b_i\}_{i \in I}$ is any sequence of scalars such that*

$$f = \sum_{i \in I} b_i f_i,$$

then

$$(1.2) \quad \sum_{i \in I} |b_i|^2 = \sum_{i \in I} |\langle S^{-1}f, f_i \rangle|^2 + \sum_{i \in I} |\langle S^{-1}f, f_i \rangle - b_i|^2.$$

Proof. We have by assumption

$$\sum_{i \in I} \langle S^{-1}f, f_i \rangle f_i = \sum_{i \in I} b_i f_i.$$

Now, taking the inner product of both sides with $S^{-1}f$ we get

$$\sum_{i \in I} |\langle S^{-1}f, f_i \rangle|^2 = \sum_{i \in I} \overline{\langle S^{-1}f, f_i \rangle} b_i,$$

and (1.2) follows easily. \square

A major advantage of frames over wavelets is that orthogonal projections take frames to frames but do not map wavelets to wavelets.

Proposition 1.9. *Let $\{f_i\}_{i \in I}$ be a frame for H with frame bounds A, B , and let P be an orthogonal projection on H . Then $\{Pf_i\}_{i \in I}$ is a frame for $P(H)$ with frame bounds A, B .*

Proof. For any $f \in P(H)$ we have

$$\sum_{i \in I} |\langle f, Pf_i \rangle|^2 = \sum_{i \in I} |\langle Pf, f_i \rangle|^2 = \sum_{i \in I} |\langle f, f_i \rangle|^2.$$

The result is now immediate. \square

It follows that an orthogonal projection P applied to an orthonormal basis $\{e_i\}_{i \in I}$ (or just a Parseval frame) yields a Parseval frame $\{Pe_i\}_{i \in I}$ for $P(H)$. The converse of this is also true and is a result of Naimark (see [10] and Han and Larson [23]).

Theorem 1.10. *A sequence $\{f_i\}_{i \in I}$ is a Parseval frame for a Hilbert space H if and only if there is a larger Hilbert space $\mathbf{K} \supset \mathbb{H}$ and an orthonormal basis $\{e_i\}_{i \in I}$ for K so that the orthogonal projection P_H of K onto H satisfies $Pe_i = f_i$ for all $i \in I$.*

Proof. The “only if” part follows from Proposition 1.9. For the “if” part, if $\{f_i\}_{i \in I}$ is a Parseval for H then the synthesis operator $T^* : \ell_2(I) \rightarrow H$ is a partial isometry. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for $\ell_2(I)$ for which $T^*(e_i) = f_i$ is our frame. Since T is an into isometry we can associate H with $T(H)$. Now let $K = \ell_2(I)$ and let P be the orthogonal projection of K onto $T(H)$. Then for all $i \in I$ and all $g = Tf \in T(H)$ we have

$$\langle Tf, Pe_i \rangle = \langle Tf, e_i \rangle = \langle f, T^*e_i \rangle = \langle f, f_i \rangle = \langle Tf, Tf_i \rangle.$$

It follows that $Pe_i = Tf_i$, and by our association of H with $T(H)$, $\{Tf_i\}_{i \in I}$ is our frame. \square

2. FINITE FRAME THEORY

A good introduction to finite frame theory is [18, 10]. Let $\{f_i\}_{i=1}^M$ be a frame for an N -dimensional Hilbert space \mathbb{H}_N . The frame operator S for this frame is a positive, self-adjoint, invertible operator on \mathbb{H}_N . Let $\{\lambda_n\}_{n=1}^N$ be the eigenvalues of S with respective eigenvectors $\{g_n\}_{n=1}^N$.

General Frame: The sum of the eigenvalues of S equals the sum of the squares of the lengths of the frame vectors:

$$\sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|f_m\|^2.$$

Equal Norm Frame: For an equal norm frame with $\|f_m\| = c$, for $m = 1, 2, \dots, M$ we have

$$\sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|f_m\|^2 = M \cdot c^2.$$

Tight Frame: Since tightness means $A = B$, we have

$$\sum_{m=1}^M |\langle f, f_m \rangle|^2 = A \|f\|^2, \quad \text{for all } f \in \mathbb{H}_N.$$

Moreover, we have that

$$S = A \cdot I_N.$$

Then, the sum of the eigenvalues becomes:

$$N \cdot A = \sum_{m=1}^M \lambda_n = \sum_{m=1}^M \|f_m\|^2.$$

Parseval Frame: If the frame is Parseval, then $A = B = 1$ and so

$$\sum_{m=1}^M |\langle f, f_m \rangle|^2 = \|f\|^2, \quad \text{for all } f \in \mathbb{H}_N.$$

It follows that $S = I$ and

$$N = \sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|f_m\|^2.$$

Equal Norm Tight Frame: For an equal norm A -tight frame we have

$$N \cdot A = \sum_{n=1}^N \lambda_n = \sum_{m=1}^M \|f_m\|^2 = M \cdot c^2.$$

Hence,

$$\sum_{m=1}^M |\langle f, f_i \rangle|^2 = \frac{M}{N} c^2 \|f\|^2, \quad \text{for all } f \in \mathbb{H}_N.$$

Equal Norm Parseval Frame: For an equal norm Parseval frame we have

$$N = \sum_{n=1}^M \lambda_n = \sum_{m=1}^M \|f_i\|^2 = c^2 M.$$

Every finite frame for \mathbb{H}_N can be turned into a tight frame with the addition of at most $N - 1$ -vectors.

Proposition 2.1. *If $\{f_m\}_{m=1}^M$ is a frame for \mathbb{H}_N , then there are vectors $\{h_n\}_{n=2}^N$ so that $\{f_m\}_{m=1}^M \cup \{h_n\}_{n=2}^N$ is a tight frame.*

Proof. Let S be the frame operator for the frame with eigenvectors $\{g_n\}_{n=1}^N$ and respective eigenvalues $\{\lambda_n\}_{n=1}^N$ and satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. We define h_n for $n = 2, 3, \dots, N$ by

$$h_n = \sqrt{\lambda_1 - \lambda_n} g_n.$$

This family works to give a λ_1 -tight frame. □

There is a unique way to get Parseval frames. Fix $N \leq M$ and let $\{g_m\}_{m=1}^M$ be an orthonormal basis for in \mathbb{H}_M . Represent this family with respect to an orthonormal basis $\{e_m\}_{m=1}^M$ for \mathbb{H}_M . Then the family $\{f_m\}_{m=1}^M$ is a Parseval frame for \mathbb{H}_N where

$$f_m = \sum_{n=1}^N \langle g_m, e_n \rangle e_n.$$

Another important property of frames is given in the next proposition.

Proposition 2.2. *Let $\{f_m\}_{m=1}^M$ be vectors in \mathbb{H}_N , $\{e_n\}_{n=1}^N$ be an orthonormal basis for \mathbb{H}_N and let $\{\lambda_n\}_{n=1}^N$ be positive real numbers. The following are equivalent:*

(1) *$\{f_m\}_{m=1}^M$ is a frame for \mathbb{H}_N with frame operator S having eigenvectors $\{e_n\}_{n=1}^N$ and respective eigenvalues $\{\lambda_n\}_{n=1}^N$.*

(2) *The following hold:*

(a) *The column vectors formed by $\{f_m\}_{m=1}^M$ with respect to $\{e_n\}_{n=1}^N$ are orthogonal. That is, if*

$$\varphi_n = (\langle f_1, e_n \rangle, \langle f_2, e_n \rangle, \dots, \langle f_m, e_n \rangle),$$

for all $n = 1, 2, \dots, N$ then for all $n \neq m$ we have

$$\langle \varphi_n, \varphi_m \rangle = 0.$$

(b) *For all $n = 1, 2, \dots, N$ we have $\|\varphi_n\|^2 = \lambda_n$.*

Proof. If $\{f_m\}_{m=1}^M$ is a frame for \mathbb{H}_N with frame operator S having eigenvectors $\{e_n\}_{n=1}^N$ and respective eigenvalues $\{\lambda_n\}_{n=1}^N$, then for all $n = 1, 2, \dots, N$ we have

$$\sum_{m=1}^M \langle e_n, f_m \rangle f_m = \lambda_n e_n.$$

Hence, for $n \neq k = 1, 2, \dots, N$ we have

$$\sum_{m=1}^M \langle e_n, f_m \rangle \langle f_m, e_k \rangle = \sum_{m=1}^M \overline{\langle f_m, e_n \rangle} \langle f_m, e_k \rangle = 0.$$

That is,

$$\langle \varphi_n, \varphi_k \rangle = 0.$$

Similarly,

$$\sum_{m=1}^M \langle e_n, f_m \rangle \langle f_m, e_n \rangle = \sum_{m=1}^M |\langle f_m, e_n \rangle|^2 = \lambda_n.$$

□

3. CONSTRUCTING FINITE FRAMES

For applications, we need to construct finite frames with extra properties such as:

(1) Prescribing in advance the norms of the frame vectors. (See for example [7, 14, 15]).

(2) Constructing equiangular frames. i.e. Frames $\{f_m\}_{m=1}^M$ for which there is a constant $c > 0$ and

$$|\langle f_m, f_n \rangle| = c, \quad \text{for all } n \neq m.$$

(See for example [16, 26, 34, 35]).

(3) Frames for which the operator

$$\pm f \mapsto \{|\langle f, f_m \rangle|\}_{m=1}^M, \text{ is one-to-one.}$$

(See for example [1, 3]).

For a good introduction to constructive methods for frames see [7]. One of the main constructive methods for frames is due to Casazza and Leon [14, 15]. They gave a construction for the important results of Benedetto and Fickus [2] and Casazza, Fickus, Kovačević, Leon, Tremain [9].

Theorem 3.1. [9] *Fix $N \leq M$ and $a_1 \geq a_2 \geq \dots \geq a_M > 0$. The following are equivalent:*

(1) *There is a tight frame $\{f_m\}_{m=1}^M$ for \mathbb{H}_N satisfying $\|f_m\| = a_m$, for all $m = 1, 2, \dots, M$.*

(2) *For all $1 \leq n < N$ we have*

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_m^2}{N - n}.$$

(3) *We have*

$$\sum_{m=1}^M a_m^2 \geq N a_1^2.$$

(4) *If*

$$\lambda = \sqrt{\frac{N}{\sum_{m=1}^M a_m^2}},$$

then

$$\lambda a_m \leq 1, \text{ for all } m = 1, 2, \dots, M.$$

This result was generalized by Casazza and Leon [14] to:

Theorem 3.2. *Let S be a positive self-adjoint operator on H_N and let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0$ be the eigenvalues of S . Fix $M \geq N$ and real numbers $a_1 \geq a_2 \geq \dots \geq a_M > 0$. The following are equivalent:*

(1) *There is a frame $\{f_i\}_{i=1}^M$ for H_N with frame operator S and satisfying $\|f_i\| = a_i$ for all $i = 1, 2, \dots, M$.*

(2) *For every $1 \leq k \leq N$ we have*

$$\sum_{j=1}^k a_j^2 \leq \sum_{j=1}^k \lambda_j,$$

and

$$\sum_{i=1}^M a_i^2 = \sum_{j=1}^N \lambda_j.$$

The next result follows easily from the construction methods above but somehow never made it to the literature.

Corollary 3.3. *Let S be a positive self-adjoint operator on a N -dimensional Hilbert space H_N . For any $M \geq N$ there is an equal norm sequence $\{f_m\}_{m=1}^M$ in H_N which has S as its frame operator.*

Proof. Let $\lambda_1 \geq \lambda_2 \geq \dots \lambda_N \geq 0$ be the eigenvalues of S . Let

$$(3.1) \quad a^2 = \frac{1}{M} \sum_{i=1}^N \lambda_i.$$

Now we check the conditions of Theorem 3.2 to see that there is a sequence $\{f_m\}_{m=1}^M$ in H_N with $\|f_m\| = a$ for all $m = 1, 2, \dots, M$. We are letting $a_1 = a_2 = \dots a_M = a$. For the second equality in Theorem 3.1, by Equation 3.1,

$$(3.2) \quad \sum_{m=1}^M \|f_m\|^2 = \sum_{m=1}^M a_m^2 = Ma^2 = \sum_{i=1}^N \lambda_i.$$

For the first inequality in Theorem 3.1, we note that by Equation 3.1 we have that

$$a_1^2 = a^2 = \frac{1}{M} \sum_{i=1}^N \lambda_i \leq \frac{1}{N} \sum_{i=1}^N \lambda_i \leq \lambda_1.$$

So our inequality holds for $m = 1$. Suppose there is an $1 < m \leq N$ for which this inequality fails and m is the first time this fails. So,

$$\sum_{i=1}^{m-1} a_i^2 = (m-1)a^2 \leq \sum_{i=1}^{m-1} \lambda_i,$$

while

$$\sum_{i=1}^m a_i^2 = ma^2 > \sum_{i=1}^m \lambda_i.$$

It follows that

$$a_m^2 = a^2 > \lambda_m \geq \lambda_{m+1} \geq \lambda_N.$$

Hence,

$$\begin{aligned}
Ma^2 = \sum_{m=1}^M a_m^2 &\geq \sum_{i=1}^m a_i^2 + \sum_{i=m+1}^N a_i^2 \\
&> \sum_{i=1}^m \lambda_i + \sum_{i=m+1}^N a_i^2 \\
&\geq \sum_{i=1}^m \lambda_i + \sum_{i=m+1}^N \lambda_i \\
&= \sum_{i=1}^N \lambda_i.
\end{aligned}$$

But this contradicts Equation 3.2. \square

4. THE GRAMMIAN

If $\{f_i\}_{i=1}^M$ is a frame for \mathbb{H}_N , the **Grammian operator** is defined as

$$G = TT^* = [\langle f_n, f_m \rangle]_{m,n=1}^M.$$

Proposition 4.1. *If $T : \mathbb{H} \rightarrow \mathbb{H}_1$ is a bounded linear operator, then T^*T and TT^* have the same non-zero eigenvalues.*

Proof. If $T^*Tx = \lambda x$, and $0 \neq \lambda$ and $x \neq 0$ then $Tx \neq 0$ and

$$TT^*(Tx) = T(T^*Tx) = T(\lambda x) = \lambda Tx.$$

\square

Corollary 4.2. *Let $\{f_i\}_{i=1}^M$ be a frame for \mathbb{H}_N with frame bounds A, B and frame operator S . The Grammian operator has the same non-zero eigenvalues as S . That is, the largest eigenvalue of G is B and the smallest non-zero eigenvalue is A .*

Corollary 4.3. *Let $\{f_m\}_{m=1}^M$ be vectors in \mathbb{H}_N . The grammian of this family is invertible if and only if $\{f_m\}_{m=1}^M$ is a Riesz basis.*

Proof. If $G = TT^*$ is invertible, by Proposition 4.1, T^*T is invertible. Hence, $\{f_m\}_{m=1}^M$ is a frame for \mathbb{H}_N . Also, we have that T^* is one-to-one. But T^* is bounded, linear and onto. Hence, it is an isomorphism.

If $\{f_m\}_{m=1}^M$ is a Riesz basis then T^* is an isomorphism and we have that T^* is invertible and so $G = TT^*$ is invertible. \square

Proposition 4.4. *Let $T = (a_{mn})_{m,n=1}^M$ be a positive, self-adjoint matrix (operator) on \mathbb{H}_N with $\dim \ker S = M - N$. Then $\{T^{1/2}e_m\}_{m=1}^M$ spans an N -dimensional space and*

$$\langle T^{1/2}e_m, T^{1/2}e_n \rangle = \langle Te_m, e_n \rangle = a_{mn}.$$

Hence, T is the Gramian matrix for the vectors $\{T^{1/2}e_m\}_{m=1}^M$. Moreover,

$$\|T^{1/2}e_m\|^2 = a_{mm},$$

and so if $a_{mm} = 1$ for all $m = 1, 2, \dots, M$ then $\{T^{1/2}e_m\}_{m=1}^M$ is a unit norm family.

5. GABOR FRAMES

This section needs someone to fill it in

Gabor frames form the basis for time-frequency analysis which is the mathematics behind signal processing. This is a huge subject which cannot be covered here except for a few remarks. We recommend the excellent book of Gröchenig [22] for a comprehensive coverage of this subject.

Definition 5.1. *Fix $a, b > 0$. For a function $f \in L^2(\mathbb{R})$ we define*

$$\text{Translation by } a: T_a f(x) = f(x - a),$$

$$\text{Modulation by } b: M_b f(x) = e^{2\pi i b x} f(x).$$

In 1946, Gabor [21] formulated a fundamental approach to signal decomposition in terms of elementary signals. Gabor's approach quickly became a paradigm for the spectral analysis associated with time-frequency methods, such as the short-time Fourier transform and the Wigner transform. For Gabor's method, we need to fix a **window function** $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}^+$. If the family

$$\{M_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ we call this a **Gabor frame**. It is a very deep question which values of a, b, g give Gabor frames. There are some necessary requirements however.

Theorem 5.2. *If the family given by a, b, g yields a Gabor frame then:*

- (1) $ab \leq 1$.
- (2) *If $ab = 1$ then this family is a frame if and only if it is a Riesz basis.*

Also, the Balian-Low Theorem puts some restrictions on the function $g \in L^2(\mathbb{R})$ for the case $ab = 1$.

Theorem 5.3 (Balian-Low Theorem). *If $g \in L^2(\mathbb{R})$, $ab = 1$ and (g, a, b) generates a Gabor frame, then either $xg(x) \notin L^2(\mathbb{R})$ or $g' \notin L^2(\mathbb{R})$.*

The Balian-Low Theorem implies that Gaussian functions e^{-ax^2} cannot yield Gabor frames for $ab = 1$.

6. $\sigma - \Delta$ -QUANTIZATION

7. FRAMES AND THE KADISON-SINGER PROBLEM

For nearly 50 years the Kadison-Singer problem [28] has defied the best efforts of some of the most talented mathematicians of our time.

Kadison-Singer Problem 7.1. *Does every pure state on the (abelian) von Neumann algebra \mathbb{D} of bounded diagonal operators on ℓ_2 have a unique extension to a (pure) state on $B(\ell_2)$, the von Neumann algebra of all bounded linear operators on the Hilbert space ℓ_2 ?*

A **state** of a von Neumann algebra \mathcal{R} is a linear functional f on \mathcal{R} for which $f(I) = 1$ and $f(T) \geq 0$ whenever $T \geq 0$ (i.e. whenever T is a positive operator). The set of states of \mathcal{R} is a convex subset of the dual space of \mathcal{R} which is compact in the w^* -topology. By the Krein-Milman theorem, this convex set is the closed convex hull of its extreme points. The extremal elements in the space of states are called the **pure states** (of \mathcal{R}).

Casazza and Tremain [17] (See also [?] and the references) showed the equivalence between the Kadison-Singer Problem and a problem in frame theory.

Feichtinger Conjecture 7.2. *Is every bounded frame a finite union of Riesz basic sequences?*

There is a huge body of literature on the Kadison-Singer and frame theory and now has its own website. We recommend visiting the KS website for the latest developments on frames and the Kadison-Singer Problem:

www.aimath.org

You can then check under **problems 2006** or the Kadison-Singer Workshop 2006 for information.

8. FUSION FRAMES

A number of new applications have emerged which cannot be modeled naturally by one single frame system. Generally they share a common property that requires distributed processing. Furthermore, we are often overwhelmed by a deluge of data assigned to one single frame system, which becomes simply too large to be handled numerically. In these cases it would be highly beneficial to split a large frame system into a set of (overlapping) much smaller systems, and to process locally within each sub-system effectively.

A distributed frame theory for a set of local frame systems is therefore in demand. A variety of applications require distributed processing. Among them there are, for instance, wireless sensor networks [27], geophones in geophysics measurements and studies [19], and the physiological structure of visual and hearing systems [33]. To understand the nature, the constraints, and related problems of these applications, let us elaborate a bit further on the example of wireless sensor networks.

In wireless sensor networks, sensors of limited capacity and power are spread in an area sometimes as large as an entire forest to measure the temperature, sound, vibration, pressure, motion, and/or pollutants. In some applications, wireless sensors are placed in a geographical area to detect and characterize chemical, biological, radiological, and nuclear material. Such a sensor system is typically redundant, and there is no orthogonality among sensors, therefore each sensor functions as a frame element in the system. Due to practical and cost reasons, most sensors employed in such applications have severe constraints in their processing power and transmission bandwidth. They often have strictly metered power supply as well. Consequently, a typical large sensor network necessarily divides the network into redundant sub-networks – forming a set of subspaces. The primary goal is to have local measurements transmitted to a local sub-station within a subspace for a subspace combining. An entire sensor system in such applications could have a number of such local processing centers. They function as relay stations, and have the gathered information further submitted to a central processing station for final assembly.

In such applications, distributed/local processing is built in the problem formulation. A staged processing structure is prescribed. We will have to be able to process the information stage by stage from local information and to eventually fuse them together at the central station. We see therefore that a mechanism of coherently collecting sub-station/subspace information is required.

Also, due to the often unpredictable nature of geographical factors, certain local sensor systems are less reliable than others. While facing the task of combining local subspace information coherently, one has also to consider weighting the more reliable sets of substation information more than suspected less reliable ones. Consequently, the coherent combination mechanism we just saw as necessary often requires a weighted structure as well.

Definition 8.1. *Let I be a countable index set, let $\{W_i\}_{i \in I}$ be a family of closed subspaces in \mathcal{H} , and let $\{v_i\}_{i \in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i \in I}$ is a fusion frame, if there exist constants $0 < C \leq D < \infty$*

such that

$$(8.1) \quad C\|f\|^2 \leq \sum_{i \in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \quad \text{for all } f \in \mathbb{H},$$

where π_{W_i} is the orthogonal projection onto the subspace W_i . We call C and D the fusion frame bounds. The family $\{(W_i, v_i)\}_{i \in I}$ is called a C -tight fusion frame, if in (8.1) the constants C and D can be chosen so that $C = D$, a Parseval fusion frame provided that $C = D = 1$, and an orthonormal fusion basis if $\mathbb{H} = \bigoplus_{i \in I} W_i$. If $\{(W_i, v_i)\}_{i \in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it a Bessel fusion sequence with Bessel fusion bound D .

Often it will become essential to consider a fusion frame together with a set of local frames for its subspaces. In this case we speak of a fusion frame system.

Definition 8.2. Let $\{(W_i, v_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} , and let $\{f_{ij}\}_{j \in J_i, i \in I}$ be a frame for W_i for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ a fusion frame system for \mathcal{H} . C and D are the associated fusion frame bounds if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i \in I}$, and A and B are the local frame bounds if these are the common frame bounds for the local frames $\{f_{ij}\}_{j \in J_i}$ for each $i \in I$. A collection of dual frames $\{\tilde{f}_{ij}\}_{j \in J_i, i \in I}$ associated with the local frames will be called local dual frames.

To provide a quick inside-look at some intriguing relations between properties of the associated fusion frame and the sequence consisting of all local frame vectors, we present the following theorem from [11] that provides a link between local and global properties.

Theorem 8.3. [11, Thm. 3.2] For each $i \in I$, let $v_i > 0$, let W_i be a closed subspace of \mathcal{H} , and let $\{f_{ij}\}_{j \in J_i}$ be a frame for W_i with frame bounds A_i and B_i . Suppose that $0 < A = \inf_{i \in I} A_i \leq \sup_{i \in I} B_i = B < \infty$. Then the following conditions are equivalent.

- (1) $\{(W_i, v_i)\}_{i \in I}$ is a fusion frame for \mathcal{H} .
- (2) $\{v_i f_{ij}\}_{j \in J_i, i \in I}$ is a frame for \mathcal{H} .

In particular, if $\{(W_i, v_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ is a fusion frame system for \mathcal{H} with fusion frame bounds C and D , then $\{v_i f_{ij}\}_{j \in J_i, i \in I}$ is a frame for \mathcal{H} with frame bounds AC and BD . Also if $\{v_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for \mathcal{H} with frame bounds C and D , then $\{(W_i, v_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ is a fusion frame system for \mathcal{H} with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Tight frames play a vital role in frame theory due to the fact that they provide easy reconstruction formulas. Tight fusion frames will turn out to be particularly useful for distributed reconstruction as well. Notice, that the

previous theorem also implies that $\{(W_i, v_i)\}_{i \in I}$ is a C -tight fusion frame for \mathcal{H} if and only if $\{v_i f_{ij}\}_{j \in J_i, i \in I}$ is a C -tight frame for \mathcal{H} .

The following result from [12] proves that the fusion frame bound C of a C -tight fusion frame can be interpreted as the *redundancy* of this fusion frame.

Proposition 8.4. *Let $\{(W_i, v_i)\}_{i=1}^n$ be a C -tight fusion frame for \mathcal{H} with $\dim \mathcal{H} < \infty$. Then we have*

$$C = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}}.$$

Let $\mathcal{W} = \{(W_i, v_i)\}_{i \in I}$ be a fusion frame for \mathbb{H} . In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}} : \mathbb{H} \rightarrow \left(\sum_{i \in I} \oplus W_i \right)_{\ell_2} \quad \text{with} \quad T_{\mathcal{W}}(f) = \{v_i \pi_{W_i}(f)\}_{i \in I}.$$

It can easily be shown that the *synthesis operator* $T_{\mathcal{W}}^*$, which is defined to be the adjoint operator, is given by

$$T_{\mathcal{W}}^* : \left(\sum_{i \in I} \oplus W_i \right)_{\ell_2} \rightarrow \mathbb{H} \quad \text{with} \quad T_{\mathcal{W}}^*(f) = \sum_{i \in I} v_i f_i, \quad f = \{f_i\}_{i \in I} \in \left(\sum_{i \in I} \oplus W_i \right)_{\ell_2}.$$

The *fusion frame operator* $S_{\mathcal{W}}$ for \mathcal{W} is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}}^* T_{\mathcal{W}}(f) = \sum_{i \in I} v_i^2 \pi_{W_i}(f).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i \in I}$ is a fusion frame for \mathcal{H} with fusion frame bounds C and D , then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on \mathbb{H} , and

$$(8.2) \quad C Id \leq S_{\mathcal{W}} \leq D Id.$$

We refer the reader to [11, Prop. 3.16] for details.

This topic now has its own website and we recommend visiting it for the latest developments on Fusion frames and distributed processing/ sensor networks

www.fusionframe.org

9. EQUIANGULAR FRAMES

The Real Case:

Good references for real equiangular frames are [16, 26, 34, 35]. A unit norm frame with the property that there is a constant c so that

$$|\langle f_i, f_j \rangle| = c, \quad \text{for all } i \neq j,$$

is called an **equiangular frame** at angle c . Equiangular tight frames first appeared in discrete geometry [32] but today (especially the complex case) have applications in signal processing, communications, coding theory and more [24, 34]. A detailed study of this class of frames was initiated by Strohmer and Heath [34] and Holmes and Paulsen [26]. Holmes and Paulsen [26] show that equiangular tight frames give error correction codes that are robust against two erasures. Bodmann and Paulsen [4] analyze arbitrary numbers of erasures for equiangular tight frames. Recently, Bodmann, Casazza, Edidin and Balan [3] show that equiangular tight frames are useful for signal reconstruction when all phase information is lost. Recently, Sustik, Tropp, Dhillon and Heath [35] made an important advance on this subject (and on the complex version). Other applications include the construction of capacity achieving signature sequences for multiuser communication systems in wireless communication theory [37]. The tightness condition allows equiangular tight frames to achieve the capacity of a Gaussian channel and their equiangularity allows them to satisfy an interference invariance property. Equiangular tight frames potentially have many more practical and theoretical applications. Unfortunately, we know very few of them and so their usefulness is largely untapped.

Over the years, many authors have made conjectures which were supposed to classify equiangular tight frames. All of these were doomed to failure because we cannot construct equiangular tight frames unless we first have the requisite number of equiangular lines. The following theorem, which summarizes the results of [16, 26, 34, 35] is probably the best that can be said about the classification question since it assumes the required number of equiangular lines exist.

Theorem 9.1. *The following are equivalent:*

- (I) *The space \mathbf{R}^N has an equiangular tight frame with M elements at angle $1/\alpha$.*
- (II) *We have*

$$M = \frac{(\alpha^2 - 1)N}{\alpha^2 - N},$$

and there exist M equiangular lines in \mathbf{R}^N at angle $1/\alpha$.

Moreover, in this case we have:

- (1) $\alpha \leq N \leq \alpha^2 - 2$.
- (2) $N = \alpha$ if and only if $M = N + 1$.
- (3) $N = \alpha^2 - 2$ if and only if $M = \frac{N(N+1)}{2}$.
- (4) $M = 2N$ if and only if

$$\alpha^2 = 2N - 1 = a^2 + b^2, \quad a, b \text{ integers.}$$

If $M \neq N + 1, 2N$ then:

- (5) α is an odd integer.
- (6) M is even.
- (7) α divides $M-1$.
- (8) $\beta = \frac{M-1}{\alpha}$ is the angle for the complementary equiangular tight frame.

We cannot classify the equiangular tight frames unless we first understand how many equiangular lines can be drawn through the origin in \mathbb{R}^N . This, however, is a very deep problem itself which has been addressed for over 60 years. We need a set of M lines passing through the origin in \mathbf{R}^N which are **equiangular** in the sense that if we choose a set of unit length vectors $\{f_m\}_{m=1}^M$, one on each line, then for $1 \leq m \neq n \leq M$, $|\langle f_m, f_n \rangle|$ is a constant. These inner products represent the cosine of the acute angle between the lines. The problem of constructing any number (especially, the maximal number) of equiangular lines in \mathbf{R}^N is one of the most elementary and at the same time one of the most difficult problems in mathematics. After sixty years of research, the maximal number of equiangular lines in \mathbf{R}^N is known only for 35 dimensions. This line of research was started in 1948 by Hanntjes [25] in the setting of elliptic geometry where he identified the maximal number of equiangular lines in \mathbf{R}^N for $n = 2, 3$. Later, Van Lint and Seidel [32] classified the largest number of equiangular lines in \mathbf{R}^N for dimensions $N \leq 7$ and at the same time emphasized the relations to discrete mathematics. In 1973, Lemmens and Seidel [30] made a comprehensive study of real equiangular line sets which is still today a fundamental piece of work. Gerzon [30] gave an upper bound for the maximal number of equiangular lines in \mathbf{R}^N :

Theorem 9.2 (Gerzon). *If we have M equiangular lines in \mathbf{R}^N then*

$$M \leq \frac{N(N+1)}{2}.$$

In most cases there are many fewer lines than this bound gives. Also, P. Neumann [30] produced a fundamental result in the area:

Theorem 9.3 (P. Neumann). *If \mathbf{R}^N has M equiangular lines at angle $1/\alpha$ and $M > 2N$, then α is an odd integer.*

Finally, there is a lower bound on the angle formed by equiangular line sets.

Theorem 9.4. *If $\{f_m\}_{m=1}^M$ is a set of norm one vectors in \mathbf{R}^N , then*

$$(9.1) \quad \max_{m \neq n} |\langle f_m, f_n \rangle| \geq \sqrt{\frac{M-N}{N(M-1)}}.$$

Moreover, we have equality if and only if $\{f_m\}_{m=1}^M$ is an equiangular tight frame and in this case the tight frame bound is $\frac{M}{N}$.

This inequality goes back to Welch [37]. Strohmer and Heath [34] and Holmes and Paulsen [26] give more direct arguments which also yields the "moreover" part. For some reason, in the literature there is a further assumption added to the "moreover" part of Theorem 9.4 that the vectors span \mathbf{R}^N . This assumption is not necessary. That is, equality in inequality 9.1 already implies that the vectors span the space [16].

The status of the equiangular line problem at this point is summarized in the following chart [30, 16, 36] where N is the dimension of the Hilbert space, M is the maximal number of equiangular lines and these will occur at the angle $1/\alpha$.

Table I: Maximal equiangular line sets

$N =$	2	3	4	5	6	7	...	13	14
$M =$	3	6	6	10	16	28	...	28	28 – 30
$\alpha =$	2	$\sqrt{5}$	3	3	3	3	...	3	5

$N =$	15	16	17	18	19	20
$M =$	36	≥ 40	≥ 48	≥ 48	72 – 76	92 – 96
$\alpha =$	5	5	5	5	5	5

$N =$	21	22	23	...	41	42	43
$M =$	126	176	276	...	276	≥ 276	344
$\alpha =$	5	5	5	...	5	5	7

The Complex Case:

REFERENCES

- [1] R. Balan, P.G. Casazza and D. Edidin, *On signal reconstruction without phase*, Appl. Comp. Harmonic Anal. **20** (2006) 345-356.
- [2] J.J. Benedetto and M. Fickus, *Finite normalized tight frames*, Advances in computational Math., special issue on frames (2002).
- [3] B. Bodmann, P.G. Casazza, D. Edidin and R. Balan, *Frames for linear reconstruction without phase*, Preprint.
- [4] B. Bodmann and V. Paulsen, *Frames, graphs and erasures*, Linear Algebra and Applications, **404** (2005) 118-146.
- [5] P.G. Casazza, *The Art of Frame Theory*, Taiwanese Journal of Math. **4**, No. 2 (2000) 1-127.
- [6] P.G. Casazza, *Modern tools for Weyl-Heisenberg (Gabor) frame theory*, Advances in Imaging and Electron Physics **115** (2000) 1-127.
- [7] P.G. Casazza, *Custom building finite frames*, Contemp Math. **345** (2004) 61-86.
- [8] P.G. Casazza, M. Fickus, J.C. Tremain and E. Weber, *The Kadison-Singer Problem in Mathematics and Engineering: A Detailed Account*, Operator Theory, Operator

- Algebras and Applications, Proceedings of the 25th GPOTS Symposium (2005), D. Han, P.E.T. Jorgensen and D.R. Larson Eds., Contemporary Math **414**(2006) 299-356.
- [9] P.G. Casazza, J. Kovačević, M. Fickus, M. Leon and J.C. Tremain, *A physical interpretation for finite tight frames*, Appl. Comp. Harmonic Anal. (2006) 51-78.
- [10] P.G. Casazza and J. Kovačević, *Equal norm tight frames with erasures*, Adv. Comp. Math **18** (2003) 387-430.
- [11] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, in: Wavelets, Frames and Operator Theory (College Park, MD, 2003), Contemp. Math. **345**, Amer. Math. Soc., Providence, RI, 2004, 87-113.
- [12] P.G. Casazza, G. Kutyniok and S. Li, *Fusion frames and distributed processing*, Preprint.
- [13] P. G. Casazza and G. Kutyniok, *A generalization of Gram-Schmidt orthogonalization generating all Parseval frames*, Adv. Comput. Math. **27** (2007), 65-78.
- [14] P.G. Casazza and M. Leon, *Existence and construction of finite tight frames*, Preprint.
- [15] P.G. Casazza and M. Leon, *Existence and construction of finite frames with a given frame operator*, International Journal of Pure and Applied Mathematics, **63** No. 2 (2010) 149-158.
- [16] P.G. Casazza, D. Redmond and J.C. Tremain, *Real equiangular frames*, Preprint.
- [17] P.G. Casazza and J.C. Tremain, *The Kadison-Singer Problem in Mathematics and Engineering*, Proceedings of the National Academy of Sciences, **103** No. 7 (2006) 2032-2039.
- [18] O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, Boston 2003.
- [19] M. S. Craig and R. L. Genter, *Geophone array formation and semblance evaluation*, Geophysics **71** (2006), 1-8.
- [20] R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*. Trans. AMS **72** (1952) 341-366.
- [21] D. Gabor, *Theory of Communications*, J. Inst. Elec. Engrg. **93** (1946) 429-457.
- [22] K.H. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2000.
- [23] D. Han and D.R. Larson, *Frames, bases and group representations*, Memoirs AMS **697** (2000).
- [24] R.W. Heath, T. Strohmer and A. J. Paulraj, *On quasi-orthogonal signatures for CDMA*, IEEE Trans. Information Theory **52** No. 3 (2006) 1217-1226.
- [25] J. Haantjes, *Equilateral point-sets in elliptic two- and three-dimensional spaces*, Nieuw Arch. Wisk. **22** (1948) 355-362.
- [26] R. B. Holmes and V. I. Paulsen, *Optimal frames for erasures*, Linear Alg. and Applications **377** (204) 31-51.
- [27] S. S. Iyengar and R. R. Brooks, eds., *Distributed Sensor Networks*, Chapman & Hall/CRC, Baton Rouge, 2005.
- [28] R. Kadison and I. Singer, *Extensions of pure states*, American Jour. Math. **81** (1959), 383-400.
- [29] J. Kovačević and A. Chebira, *An introduction to frames*, in *Foundations and Trends in Signal Processing* (2008) NOW publishers.
- [30] P. Lemmens and J. Seidel, *Equiangular lines*, Journal of Algebra **24** (1973) 494-512.
- [31] S. Li, *On general frame decompositions*, Numer. Funct. Anal. Optim. **16** (1995), 1181-1191.

- [32] J. H. van Lint and J. J. Seidel, *Equiangular point sets in elliptic geometry*, Proc. Nederl. Akad. Wetensch. Series A **69** (1966) 335-348.
- [33] C. J. Rozell and D. H. Johnson, *Analyzing the robustness of redundant population codes in sensory and feature extraction systems*, Neurocomputing **69** (2006), 1215–1218.
- [34] T. Strohmer and R. W. Heath, *Grassmannian frames with applications to coding and communication*, Appl. Comp. Harmonic Anal. **14** No. 3 (2003) 257-275.
- [35] M. A. Sustik, J. A. Tropp, I. S. Dhillon and R. W. Heath, Jr., *On the existence of equiangular tight frames*, Linear Alg. and Applications **426** No. 2-3 (2007) 619-635.
- [36] J.C. Tremain, *Concrete Constructions of Equiangular Line Sets*, in preparation.
- [37] L. R. Welch, *Lower bounds on the maximum cross-correlation of signals*, IEEE Trans. Inform. Theory **20** (1974) 397-399.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211-4100

E-mail address: `janet,pete@math.missouri.edu`