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An Introduction to Irregular Weyl-Heisenberg Frames

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ABSTRACT We give an introduction to irregular Weyl-Heisenberg frames showing the latest developments and open problems. We provide several new results for semi-irregular WH-frames as well as giving new and more accessible proofs for several results from the literature.

1 Introduction

In 1946, D. Gabor [25] formulated a fundamental approach to signal decomposition into elementary signals. Gabor's approach has now become the paradigm for the spectral analysis associated with time-frequency methods. Gabor (or Weyl-Heisenberg) frames provide the fundamental tool for modern day signal/image processing. Yet ever more applications are being found for these methods in optics, filter banks, signal detection and much more.

Although there is now an extensive literature on Weyl-Heisenberg frames [5, 22, 25, 29], until recently very little was known about irregular Weyl-Heisenberg frames. However, over the last 5 years there has developed an extensive literature on irregular Weyl-Heisenberg frames. Our goal is to put this literature into a framework that allows nonspecialists to enter the area and have a basis for further study.

2 Preliminaries

Now we introduce the definitions and basic results on frames needed for the study of irregular Gabor frames. If H is a (real or complex) Hilbert space, a sequence of elements of H $\{f_n\}_{n=1}^{\infty}$ is called a **frame** for H if there exists constants $A, B > 0$ so that for all $f \in H$ we have:

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2.$$

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A (resp. B) is called the **lower** (resp. **upper**) frame bound of the frame. Frames were introduced by Duffin and Schaeffer [18] while they were working on some deep problems in nonharmonic Fourier series. If $A = B = \lambda$, $\{f_n\}_{n=1}^{\infty}$ is called a λ -**tight-frame** and if $A = B = 1$ a **Parseval frame**. If $\|f_n\| = \|f_m\|$ for all n, m this is an **equal-norm** frame. And if $0 < \inf_n \|f_n\|$ we say the frame is **bounded**. Note that for any frame $\{f_n\}_{n=1}^{\infty}$,

$$\|f_m\|^4 \leq \sum_{n=1}^{\infty} |\langle f_m, f_n \rangle|^2 \leq B \|f_m\|^2.$$

so $\|f_m\|^2 \leq B$ for all m .

Remark: For years there were “two camps” using different notations for frames. A group of us agreed this year to standardize the notation to that used in this paper. Previously, “Parseval frames” were called “normalized tight frames” and “equal-norm frames” were called “uniform frames”.

It will often be useful to work with sequences which have only the upper frame bound condition. We call $\{f_n\}_{n=1}^{\infty}$ a **Bessel sequence** if there is a constant $K > 0$ so that for all sequence of scalars $\{a_n\}_{n=1}^{\infty}$ we have

$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\|^2 \leq K \sum_{n=1}^{\infty} |a_n|^2.$$

We call K the **Bessel constant** of $\{f_n\}_{n=1}^{\infty}$. Given $f_n \in H$, define the **synthesis operator** $T : \ell_2 \rightarrow H$ by $T e_n = f_n$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for ℓ_2 . Also, for $f \in H$,

$$\langle T^* f, e_n \rangle = \langle f, T e_n \rangle = \langle f, f_n \rangle.$$

Hence,

$$T^* f = \sum_{n=1}^{\infty} \langle f, f_n \rangle e_n$$

and so

$$\|T^* f\|^2 = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2.$$

It follows that T is a bounded operator if and only if $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence. By standard operator theoretic considerations and the frame definition we have

Theorem 2.1. *Let $f_n \in H$ with synthesis operator T . The following are equivalent:*

- (1) $\{f_n\}_{n=1}^{\infty}$ is a frame for H .
- (2) T is a bounded, linear onto operator.
- (3) T^* is a (possibly into) isomorphism.

It follows that the **frame operator** $S = TT^* : H \rightarrow H$ is an invertible operator and for $f \in H$ and we have

$$\begin{aligned} Sf &= TT^*f = T \sum_{n=1}^{\infty} \langle f, f_n \rangle e_n \\ &= \sum_{n=1}^{\infty} \langle f, f_n \rangle T e_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n. \end{aligned}$$

Hence, $\langle Sf, f \rangle = \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2$, and so S is a positive, self-adjoint invertible operator on H . Also,

$$\langle Af, f \rangle = A\|f\|^2 \leq \langle Sf, f \rangle \leq B\|f\|^2 = \langle Bf, f \rangle.$$

So $AI \leq S \leq BI$ and $\{f_n\}_{n=1}^{\infty}$ is a Parseval frame if and only if $S = I$. We say that two frames $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ for H are **equivalent** if there is an invertible operator $L : H \rightarrow H$ so that $Lf_n = g_n$. If $\{f_n\}_{n=1}^{\infty}$ is a frame with frame operator S then for $f \in H$,

$$f = S^{-\frac{1}{2}}SS^{-\frac{1}{2}}f = S^{-\frac{1}{2}} \sum_{n=1}^{\infty} \langle S^{-\frac{1}{2}}f, f_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, S^{-\frac{1}{2}}f_n \rangle S^{-\frac{1}{2}}f_n.$$

Hence, $\{S^{-\frac{1}{2}}f_n\}_{n=1}^{\infty}$ is a Parseval frame for H which is equivalent to $\{f_n\}_{n=1}^{\infty}$. Finally, $\{S^{-1}f_n\}_{n=1}^{\infty}$ is a frame for H called the **canonical dual frame** and satisfies for $f \in H$ the **reconstruction** formula

$$f = \sum_{n=1}^{\infty} \langle S^{-1}f, f_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle S^{-1}f_n.$$

For $1 \leq p < \infty$, let $L^p(\mathbb{R})$ denote the Banach space of complex valued, Lebesgue integrable functions f on \mathbb{R} satisfying

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty.$$

We define the unitary operators $T_a, E_a, a \in \mathbb{R}$, on $L^p(\mathbb{R})$ by:

Translation. $T_a f(t) = f(t - a)$

Modulation. $E_a f(t) = e^{2\pi i a t} f(t)$.

If $g \in H$, $0 < a, b$ and $(g, a, b) = \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call (g, a, b) a **Gabor frame** or a **Weyl-Heisenberg frame** for $L^2(\mathbb{R})$. It is a very deep question to classify the g, a, b so that (g, a, b) is a Gabor frame. For an introduction to Weyl-Heisenberg frames we recommend the excellent books of Gröchenig [26], Feichtinger and Strohmer [22] and the paper of Heil and Walnut [29]. The first modern fundamental work on Gabor frames goes back to

Daubechies, Grossman and Meyer [16, 17]. A deep result of Rieffel [38] implies if (g, a, b) is a Weyl-Heisenberg frame then $a, b \leq 1$.

It is also known (See [29]) that when $ab = 1$, a Weyl-Heisenberg frame becomes a **Riesz basis**. i.e. a sequence $\{f_n\}_{n=1}^{\infty}$ so that there is a constant $K > 0$ satisfying for all sequences of scalars $\{a_n\}_{n=1}^{\infty}$,

$$\frac{1}{K} \sum_{u=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|^2 \leq K \sum_{n=1}^{\infty} |a_n|^2.$$

Our interest is in irregular Weyl-Heisenberg frames. If $(x_n, y_n) \in \mathbb{R}^2$, $g \in L^2(\mathbb{R})$ and $\{E_{x_n} T_{y_n} g\}_{n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, we call it an **irregular Weyl-Heisenberg frame**. A special case of this involves irregular lattices. If $x_m, y_n \in \mathbb{R}$, $\{E_{x_m} T_{y_n} g\}_{n, m \in \mathbb{Z}}$ is an irregular Weyl-Heisenberg system. The **semi-irregular Weyl-Heisenberg frames** are the Weyl-Heisenberg frames of the form $\{E_{mb} T_{y_n} g\}_{n, m \in \mathbb{Z}}$. Most of the standard results on Weyl-Heisenberg frames hold for semi-irregular Weyl-Heisenberg frames so it is this class we will concentrate on first.

We note that it is a very deep question to classify, even for the regular case, those (g, a, b) which generate a WH-frame. Even a special case, $(\chi_{[0, c]}, a, b)$ is open. This is known as the **abc-problem** and a deep analysis of this was done by Janssen [32]. Also, Casazza and Kalton [10] have shown that even for the case $a = b = 1$, the abc-problem is equivalent to a classical problem of Littlewood concerning certain complex polynomials and when they have roots on the unit circle.

3 Density

It is now known that density plays a fundamental role in the theory of Weyl-Heisenberg frames. This notion originated in the study of nonharmonic Fourier series. We say that a family of real numbers $\{\lambda_n\}_{n \in \mathbb{Z}}$ has **uniform density** $D = D(\{\lambda_n\})$ if there is an $L > 0$ such that for all $n \in \mathbb{Z}$ we have $|\lambda_n - \frac{n}{D}| \leq L$. We say that $\{\lambda_n\}$ is **separated** if there is a $\delta > 0$ so that $|\lambda_n - \lambda_m| \geq \delta$, whenever $n \neq m$. It is **relatively separated** if it is a finite union of separated sequences. Duffin and Schaeffer [18] showed that if $\{\lambda_n\}$ is uniformly separated and has uniform density $D > (a - b)$ then the complex exponentials $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbb{Z}}$ form a frame for $L^2[a, b]$. Jaffard [30] gives a sufficient condition for a sequence of exponentials to form a frame for $L^2[a, b]$.

Theorem 3.1. *If a family of real numbers $\{\lambda_n\}_{n \in \mathbb{Z}}$ is relatively separated and has a subset of uniform density $D > (a - b)$ then $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbb{Z}}$ forms a frame for $L^2[a, b]$.*

In a deep paper, Ortega-Cerdá and Seip [41] then finished the classification (started by Duffin and Schaeffer [18] in 1952) of the families of exponentials which give frames for $L^2[a, b]$.

Our first proposition is important in that it explains why we need to assume in all our results that $\{(x_n, y_n)\}_{n \in \mathbb{Z}}$ is a relatively separated sequence. If $f_n \in H$, we say that $\{f_n\}_{n=1}^\infty$ is **a-separated** if there is a constant $a > 0$ so that $\|f_n - f_m\| \geq a$ for all $n \neq m$. We say $\{f_n\}_{n=1}^\infty$ is **relatively separated** if it is a finite union of separated sequences.

Proposition 3.1. *If $\{f_n\}_{n=1}^\infty$ is a Bessel sequence in a Hilbert space H and there is a constant $a > 0$ so that $\|f_n\| \geq a$, for all $n = 1, 2, 3, \dots$, then $\{f_n\}_{n=1}^\infty$ is relatively separated.*

Proof. We proceed by way of contradiction. Assume $0 < a \leq \|f_n\|$ for all n and $\{f_n\}_{n=1}^\infty$ is not relatively separated. Fix a natural number N . Choose $\epsilon > 0$ so that $\epsilon \leq \frac{1}{2}$. We assume that $\{f_n\}_{n=1}^\infty$ is not a union of N , ϵ -separated sets. Choose any maximal set $I_1 \subset \mathbb{N}$ so that $\{f_n\}_{n \in I_1}$ is ϵa -separated. Continuing, we choose, for $1 \leq j \leq N$, a maximal ϵa -separated set $\{f_n\}_{n \in I_j}$ where $I_j \subseteq [\cup_{i=1}^{j-1} I_i]^c$. Pick any $n_j \in I_j$. Then there is a $n_{j-1} \in I_{j-1}$ with $\|f_{n_j} - f_{n_{j-1}}\| \leq \epsilon a$. Continuing, we find $n_i \in I_i$, $1 \leq i < N$, so that $\|f_{n_{i+1}} - f_{n_i}\| \leq \epsilon a$ for all $1 \leq i \leq N$. It follows that $\|f_{n_1} - f_{n_i}\| \leq (i-1)\epsilon a$, for all $1 \leq i \leq N$. Now,

$$\begin{aligned} & \sqrt{\sum_{j=1}^N |\langle f_{n_1}, f_{n_j} \rangle|^2} \geq \\ & \sqrt{\sum_{j=1}^N |\langle f_{n_1}, f_{n_1} \rangle|^2} - \sqrt{\sum_{j=1}^N |\langle f_{n_1}, f_{n_1} - f_{n_j} \rangle|^2} \\ & \geq N^{\frac{1}{2}} \|f_{n_1}\|^2 - \|f_{n_1}\| \sqrt{\sum_{j=1}^N \|f_{n_1} - f_{n_j}\|^2} \\ & \geq N^{\frac{1}{2}} \|f_{n_1}\|^2 - \|f_{n_1}\| N^{\frac{1}{2}} \epsilon a \\ & \geq N^{\frac{1}{2}} \|f_{n_1}\|^2 (1 - \epsilon) \geq \frac{1}{2} N^{\frac{1}{2}} \|f_{n_1}\|^2. \end{aligned}$$

Since N was arbitrary, it follows that $\{f_n\}_{n=1}^\infty$ is not a Bessel sequence which is a contradiction. \square

Corollary 3.1. *If $(x_n, y_n) \in \mathbb{R}^2$, $g \in L^2(\mathbb{R})$ and $\{E_{x_n} T_{y_n} g\}_{n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ then $\{(x_n, y_n)\}_{n=1}^\infty$ is relatively separated in \mathbb{R}^2 .*

Ramanathan and Steger [37] strikingly showed the role that density plays in Gabor frame theory. They proved a fundamental result now known as the **homogeneous approximation property**. Let $g_1, g_2, \dots, g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, \Lambda_2, \dots, \Lambda_r \subset \mathbb{R}^{2d}$ be such that $\cup_{k=1}^r \{E_x T_y g_k\}_{(x,y) \in \Lambda_k}$ is a frame for $L^2(\mathbb{R}^d)$ with frame operator S . Given $h > 0$ and $(x, y) \in \mathbb{R}^{2d}$, let

$$W(h, x, y) = \text{span}\{S^{-1}(E_x T_y g_k) : (x, y) \in Q_h(x, y) \cap \Lambda_k, k = 1, 2, \dots, r\},$$

where $Q_h(x, y)$ is the cube centered at (x, y) with side lengths h .

Theorem 3.2 (Homogeneous Approximation Property). *Let $\{g_k\}_{k=1}^r$ be elements of $L^2(\mathbb{R}^d)$ and $\{\Lambda_k\}_{k=1}^r$ be subsets of \mathbb{R}^d such that $\cup_{k=1}^r \{E_x, T_y g_k\}_{(x,y) \in \Lambda_k}$ is a frame for $L^2(\mathbb{R}^d)$. Then for each $f \in L^2(\mathbb{R}^d)$, and for every $\epsilon > 0$ and every $R > 0$ and every $(x, y) \in \mathbb{R}^d$ we have*

$$\text{dist}(E_x T_y f, W(R, x, y)) < \epsilon.$$

If Λ is a sequence of points in \mathbb{R}^d , the **lower and upper Beurling densities** of Λ are, respectively,

$$D^-(\Lambda) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{R}^{2d}} \frac{|\Lambda \cap Q(r, z)|}{r^{2d}} \quad \text{and} \quad D^+(\Lambda) = \limsup_{r \rightarrow \infty} \sup_{z \in \mathbb{R}^{2d}} \frac{|\Lambda \cap Q(r, z)|}{r^{2d}},$$

where $|E|$ denotes the cardinality of the set E and $Q(r, z)$ denotes the cube centered at $z \in \mathbb{R}^{2d}$ with side lengths r . The Beurling densities are measuring the average number of elements of Λ lying inside sets of unit measure. The density result of Ramanathan and Steger [37] (as extended by Christensen, Deng and Heil [15]) is as follows:

Theorem 3.3. *If $\{E_x T_y g\}_{(x,y) \in \Lambda}$ is a frame for $L^2(\mathbb{R}^d)$, then $1 \leq D^-(\Lambda) \leq D^+(\Lambda) < \infty$.*

Moreover, if our family is a Riesz basis then $D^-(\Lambda) = D^+(\Lambda) = 1$.

Now we will apply these density results to Weyl-Heisenberg frames. These results come from [6].

Proposition 3.2. *Assume that $\{E_{x_n} T_{y_n} g\}_{n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B . There is an $a > 0$ so that $\{E_{x_n}\}_{n \in \mathbb{Z}}$ is a frame for $L^2[0, a]$ with frame bounds say A_1, B_1 . Furthermore*

$$A/B_1 \leq \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 \leq B/A_1 \text{ a.e.} \quad (3.1)$$

In particular, g is bounded.

Proof. Let $\Lambda = \{(x_n, y_n) : n \in \mathbb{Z}\}$. By Theorem 3.3 there is a constant $R > 0$ so that for all $c, d \in \mathbb{R}$:

$$([c, c + R] \times [d, d + R]) \cap \Lambda \neq \emptyset.$$

In particular, if $E = \{x_n : n \in \mathbb{Z}\}$ then for all $c \in \mathbb{R}$, $[c, c + R] \cap E \neq \emptyset$. This means that $\{x_n\}_{n \in \mathbb{Z}}$ is a set of uniform density in \mathbb{R} . Hence, there is a constant $a > 0$ so that $\{x_n\}_{n \in \mathbb{Z}}$ is a frame for $L^2[0, a]$ with frame bounds say A_1, B_1 . For any interval $I = [b, b + a] \subset \mathbb{R}$ and any bounded function $f \in L^2(I)$ we have

$$\sum_{m, n \in \mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g \rangle|^2 = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\langle f \cdot T_{y_n} \bar{g}, E_{x_m} \rangle|^2 \leq \sum_{n \in \mathbb{Z}} B_1 \|f \cdot T_{y_n} \bar{g}\|^2.$$

Similarly,

$$\sum_{n \in \mathbb{Z}} A_1 \|f \cdot T_{y_n} \bar{g}\|^2 \leq \sum_{m, n \in \mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g \rangle|^2.$$

Also,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \|f \cdot T_{y_n} \bar{g}\|^2 &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f(t) \overline{g(t - y_n)}|^2 dt = \\ &= \sum_{n \in \mathbb{Z}} \int_I |f(t)|^2 |g(t - y_n)|^2 dt = \int_I |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 dt. \end{aligned}$$

Combining the above we have for all $f \in L^2(I)$:

$$\int_I |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 dt \leq \frac{B}{A_1} \|f\|^2 = \frac{B}{A_1} \int_I |f(t)|^2 dt.$$

It follows that

$$\sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 \leq \frac{B}{A_1}.$$

The other inequality is done similarly with the other frame inequality. \square

Corollary 3.2. *Let $\{x_m\}_{m \in \mathbb{Z}}$ be a set of uniform density in \mathbb{R} (and so there is an $a > 0$ so that $\{E_{x_m}\}_{m \in \mathbb{Z}}$ is a frame for $L^2[-a, a]$). Let $g \in L^2(\mathbb{R})$ be bounded with support in $[-a, a]$ and $\{y_n\}_{n \in \mathbb{Z}}$ a separated sequence in \mathbb{R} with*

$$0 < A \leq \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 \leq B < \infty \quad \text{a.e.}$$

Then $\{E_{x_m} T_{y_n} g\}_{m, n \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$.

Proof. Let A_1, B_1 be the frame bounds for $\{E_{x_n}\}_{n \in \mathbb{Z}}$ on $L^2[c - a, c + a]$ for $c \in \mathbb{R}$. For $f \in L^2[\mathbb{R}]$,

$$\begin{aligned} &\sum_{m, n \in \mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g \rangle|^2 \\ &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\langle f T_{y_n} \bar{g}, E_{x_m} \rangle|^2 \\ &\leq B_1 \sum_{n \in \mathbb{Z}} \|f \cdot T_{y_n} \bar{g}\|^2 \\ &= B_1 \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} |f(t)|^2 |g(t - y_n)|^2 dt \\ &= B_1 \int_{\mathbb{R}} |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 dt \\ &\leq B_1 B \int_{\mathbb{R}} |f(t)|^2 dt = B_1 B \|f\|^2. \end{aligned}$$

The lower frame bound $A_1 A$ is done similarly. \square

Finally, Balan, Casazza, Heil and Landau [2, 3] have shown that there is an interesting relationship between the density of $\Omega \subseteq \mathbb{R}^d \oplus \mathbb{R}^d$ and the excess of a Gabor frame $\{E_x T_y g\}_{(x,y) \in \Omega}$. More specifically, they relate density to the existence of an infinite subset of the frame which can be deleted and still leave a frame.

4 Semi-Irregular Weyl-Heisenberg Frames

In this section we will work with frames of the form $\{E_{mb} T_{y_n} g\}_{m,n \in \mathbb{R}}$ where $0 < b, y_n \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$. We will discover that much of the theory of Weyl-Heisenberg frames holds in this setting. We will start with the Weyl-Heisenberg frame identity which has a rich history. It was originally done for Weyl-Heisenberg frames by Daubechies [16] who used it to produce sufficient conditions for (g, a, b) to be a Weyl-Heisenberg frame. Walnut [42] (See also [29]) used Wiener amalgum spaces and the WH-frame identity to give other sufficient conditions to have Weyl-Heisenberg frames and introduced what is now called the **Walnut representation** of the frame operator. The WH-Frame Identity was extended to translation invariant systems in the fundamental paper of Janssen [31], Proposition 1.2.1. A deep analysis of the convergence properties of the sums in the identity appears in [8]. The minimal conditions for the identity to hold appear in [12]. The style of the proof for the WH-Frame Identity we give is an amalgum of the proofs from [31, 8, 12, 6]. We first need a Proposition whose proof is modeled after [8].

Proposition 4.1. *Let $\{x_m\}_{m \in \mathbb{Z}}$ and $\{y_n\}_{n \in \mathbb{Z}}$ be relatively separated sequences of real numbers. If $g \in L^2(\mathbb{R})$ is bounded and compactly supported then $\{E_{x_m} T_{y_n} g\}_{m,n \in \mathbb{Z}}$ is a Bessel sequence.*

Proof. Let g be supported on $[-a, a]$. Since $\{x_n\}_{n \in \mathbb{Z}}$ is relatively separated, $\{E_{x_n}\}_{n \in \mathbb{Z}}$ is a Bessel sequence in $L^2[a-b, a+b]$ for all $b \in \mathbb{R}$ with Bessel bound say C ([44, Theorem 4, page 135]). Let B be the upper bound for $|g|$ a.e. We define the synthesis operator $L : \ell_2 \oplus \ell_2 \rightarrow L^2(\mathbb{R})$ by:

$$L \left(\sum_{m,n \in \mathbb{Z}} a_{mn} e_{mn} \right) = \sum_{m,n \in \mathbb{Z}} a_{mn} E_{x_m} T_{y_n} g,$$

where $\{e_{mn}\}_{m,n \in \mathbb{Z}}$ is an orthonormal basis for $\ell_2 \oplus \ell_2$. We need to show that L

is a bounded operator. Fix n and compute

$$\begin{aligned}
 \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{x_m} T_{y_n} g \right\|^2 &= \int_{\mathbb{R}} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{x_m} T_{y_n} g(t) \right|^2 dt \\
 &= \int_{-a-y_n}^{a-y_n} \left| \sum_{m \in \mathbb{Z}} a_{mn} E_{x_m} \right|^2 |g(t-y_n)|^2 dt \\
 &\leq B \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{x_m} \right\|_{L^2[-a-y_n, a-y_n]}^2 \\
 &\leq BC \sum_{m \in \mathbb{Z}} |a_{mn}|^2.
 \end{aligned}$$

Since $\{y_n\}_{n \in \mathbb{Z}}$ is relatively separated, and g is compactly supported, we may write $\{y_n\}_{n \in \mathbb{Z}} = \cup_{j=1}^M \{y_n\}_{n \in I_j}$ where for each j and $k \neq \ell$ in I_j , $\text{supp } T_k g \cap \text{supp } T_\ell g = \emptyset$. Now

$$\begin{aligned}
 \left\| L \left(\sum_{m \in \mathbb{Z}} \sum_{n \in I_j} a_{mn} e_{mn} \right) \right\|^2 &= \left\| \sum_{m \in \mathbb{Z}} \sum_{n \in I_j} a_{mn} E_{x_m} T_{y_n} g \right\|^2 \\
 &= \sum_{n \in I_j} \left\| \sum_{m \in \mathbb{Z}} a_{mn} E_{x_m} T_{y_n} g \right\|^2 \\
 &\leq BC \sum_{n \in I_j} \sum_{m \in \mathbb{Z}} |a_{mn}|^2.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \left\| L \left(\sum_{m, n \in \mathbb{Z}} a_{mn} e_{mn} \right) \right\|^2 &\leq M \sum_{j=1}^M \left\| \sum_{m \in \mathbb{Z}} \sum_{n \in I_j} a_{mn} E_{x_m} T_{y_n} g \right\|^2 \\
 &\leq MBC \sum_{j=1}^M \sum_{m \in \mathbb{Z}} \sum_{n \in I_j} |a_{mn}|^2 \\
 &= MBC \sum_{m, n \in \mathbb{Z}} |a_{mn}|^2.
 \end{aligned}$$

Hence, L is a bounded operator. \square

Now we are ready for the Weyl-Heisenberg frame identity for semi-irregular Gabor frames.

Theorem 4.1 (Weyl-Heisenberg Frame Identity). *Let $g \in L^2(\mathbb{R})$, $0 < b$ and $\{y_n\}_{n \in \mathbb{Z}}$ a relatively separated sequence of real numbers satisfying $\sum |g(t-y_n)|^2 \leq$*

B a.e. For all bounded compactly supported functions f we have:

$$\begin{aligned} & \sum_{n,m \in \mathbb{Z}} | \langle f, E_{mb} T_{y_n} g \rangle |^2 = \\ & \frac{1}{b} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g(t - y_n - k/b)} dt = \\ & b^{-1} \int_{\mathbb{R}} |f(t)|^2 \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 dt + \\ & b^{-1} \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(t)} f(t - k/b) \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g(t - y_n - k/b)} dt < +\infty. \end{aligned}$$

Moreover, the series above are all unconditionally convergent.

Proof. For a fixed $n \in \mathbb{Z}$ and let

$$F_n(t) = \sum_{k \in \mathbb{Z}} f\left(t - \frac{k}{b}\right) \overline{g\left(t - y_n - \frac{k}{b}\right)}.$$

Since both f and g are bounded, $F_n \in L^1\left[0, \frac{1}{b}\right]$ and

$$\int_{\mathbb{R}} f(t) \overline{g(t - y_n)} e^{-2\pi i m b t} dt = \int_0^{\frac{1}{b}} F_n(t) e^{-2\pi i m b t} dt.$$

Hence,

$$\sum_{m \in \mathbb{Z}} | \langle f, E_{mb} T_{y_n} g \rangle |^2 = \sum_{m \in \mathbb{Z}} \left| \int_0^{\frac{1}{b}} f_n(t) e^{-2\pi i m b t} dt \right|^2 = b^{-1} \int_0^{\frac{1}{b}} |F_n(t)|^2 dt.$$

We will now do a calculation which involves interchanging infinite sums and integrals. We will act like all this makes sense and after the calculation explain why this is all okay.

$$\begin{aligned} \sum_n \sum_m | \langle f, E_{mb} T_{y_n} g \rangle |^2 &= b^{-1} \sum_n \int_0^{\frac{1}{b}} \left| \sum_k f\left(t - \frac{k}{b}\right) \overline{g\left(t - y_n - \frac{k}{b}\right)} \right|^2 dt \\ &= b^{-1} \sum_n \int_0^{\frac{1}{b}} \sum_{\ell} \overline{f\left(t - \frac{\ell}{b}\right) g\left(t - y_n - \frac{\ell}{b}\right)} \cdot \sum_k f\left(t - \frac{k}{b}\right) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt \\ &= b^{-1} \sum_n \sum_{\ell} \int_0^{\frac{1}{b}} \overline{f\left(t - \frac{\ell}{b}\right) g\left(t - y_n - \frac{\ell}{b}\right)} \cdot \sum_k f\left(t - \frac{k}{b}\right) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt \\ &= b^{-1} \sum_n \int_{\mathbb{R}} \overline{f(t)} g(t - y_n) \sum_k f\left(t - \frac{k}{b}\right) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt \\ &= b^{-1} \sum_k \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) \sum_n g(t - y_n) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt \\ &= b^{-1} \int_{\mathbb{R}} |f(t)|^2 \sum_n |g(t - y_n)|^2 dt \\ &+ b^{-1} \sum_{k \neq 0} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) \sum_n g(t - y_n) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt. \end{aligned}$$

Now we need to justify all the interchanges of sums and integrals above.

$$\begin{aligned}
 & \sum_n \int_0^{\frac{1}{b}} \sum_\ell \left| f\left(t - \frac{\ell}{b}\right) g\left(t - y_n - \frac{\ell}{b}\right) \right| \sum_k \left| f\left(t - \frac{k}{b}\right) g\left(t - y_n - \frac{k}{b}\right) \right| dt \\
 &= \sum_n \int_{\mathbb{R}} |f(t)g(t - y_n)| \cdot F_n(t) dt \\
 &= \sum_k \int_{\mathbb{R}} \left| f(t)f\left(t - \frac{k}{b}\right) \right| \cdot \sum_n \left| g(t - y_n)g\left(t - y_n - \frac{k}{b}\right) \right| dt
 \end{aligned}$$

Now, for each k ,

$$\begin{aligned}
 \sum_n \left| g(t - y_n)g\left(t - y_n - \frac{k}{b}\right) \right| &\leq \left(\sum_n |g(t - y_n)|^2\right)^{\frac{1}{2}} \left(\sum_n |g\left(t - y_n - \frac{k}{b}\right)|^2\right)^{\frac{1}{2}} \\
 &\leq B^{\frac{1}{2}} \cdot B^{\frac{1}{2}} = B.
 \end{aligned}$$

Since f is compactly supported, $f(t)f\left(t - \frac{k}{b}\right) = 0$ for all large k . That is, the sums over k are finite sums. It follows easily from here that all our series are unconditionally convergent which justifies interchanging sums and integrals.

Finally, since f is bounded and compactly supported,

$$\sum_{n,m} |\langle f, E_{mb}T_{y_n}g \rangle|^2 = \sum_{n,m} |\langle E_{mb}T_{y_n}f, g \rangle|^2 < \infty,$$

by Proposition 4. □

Theorem 4.1 leads to a simple condition guaranteeing the existence of semi-irregular Weyl-Heisenberg frames. To simplify the notation a little, if $g \in L^2(\mathbb{R})$ is bounded, $0 < b$, and $\{y_n\}_{n=1}^\infty$ is a relatively separated sequence in \mathbb{R} we let

$$G_0(t) = \sum_{n \in \mathbb{Z}} |g(t - y_n)|^2$$

and for $0 \neq k \in \mathbb{Z}$

$$G_k(t) = \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g\left(t - y_n - \frac{k}{b}\right)}.$$

We now have the following result from [6]. The Condition stated here is sometimes referred to as the **CC-condition**. That is:

$$\sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |G_k(t)| \leq B \quad a.e. \ t$$

Theorem 4.2. . Let $g \in L^2(\mathbb{R})$, $0 < b$ and $\{y_n\}_{n \in \mathbb{Z}}$ be a relatively separated sequence in \mathbb{R} . If

$$B = \sup_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |G_k(t)| < \infty$$

and

$$A = \inf_{t \in \mathbb{R}} \left[G_0(t) - \sum_{0 \neq k \in \mathbb{Z}} |G_k(t)| \right] > 0,$$

then $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{b}, \frac{B}{b}$.

Moreover, the series in the WH-frame identity converge unconditionally for all $f \in L^2(\mathbb{R})$.

Proof. Let f be a bounded, compactly supported function on \mathbb{R} . By the WH-frame identity and the hypothesis of the theorem we have

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} | \langle f, E_{mb}T_{y_n}g \rangle |^2 &= \\ \left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g\left(t - y_n - \frac{k}{b}\right)} dt \right| & \\ \geq \frac{1}{b} \int_{\mathbb{R}} |f(t)|^2 \left[\sum_{n \in \mathbb{Z}} |g(t - y_n)|^2 - \sum_{k \neq 0} \left| \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g\left(t - y_n - \frac{k}{b}\right)} \right| \right] dt & \\ \geq \frac{A}{b} \|f\|^2. & \end{aligned}$$

A similar argument gives the upper frame bound. Since those two estimates hold on a dense subset of $L^2(\mathbb{R})$, they hold on $L^2(\mathbb{R})$. \square

Next we will show that the Gabor Frame Identity (even the form for irregular frames) has a pointwise form. This result is due to Janssen [31]. The first part of this result is also contained in a form disguised by the notation in [13], Corollary 6.7.

Theorem 4.3. Let $g \in L^2(\mathbb{R})$, $b > 0$ and $\{y_n\}_{n \in \mathbb{Z}}$ be a relatively separated sequence of real numbers. The following are equivalent:

- (1) $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a Gabor frame with frame bounds A, B .
- (2) For all bounded, compactly supported $f \in L^2(\mathbb{R})$ we have:

$$\begin{aligned} A \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 &\leq \\ = \frac{1}{b} \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - k/b - y_n)} \right|^2 &\leq B \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 \quad \text{a.e.} \end{aligned}$$

Proof. The family $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a Gabor frame with frame bounds A, B if and only if for all bounded compactly supported functions f on \mathbb{R} and all bounded $1/b$ -periodic functions ϕ on \mathbb{R} we have

$$A\|\phi f\|^2 \leq \sum_{n,m \in \mathbb{Z}} |\langle \phi f, E_{mb}T_{y_n}g \rangle|^2 \leq B\|\phi f\|^2.$$

Now we will break down each of the terms above.

$$A\|\phi f\|^2 = A \int_{\mathbb{R}} |\phi(t)|^2 |f(t)|^2 dt = A \int_0^{1/b} |\phi(t)|^2 \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 dt.$$

Next,

$$B\|\phi f\|^2 = B \int_{\mathbb{R}} |\phi(t)|^2 |f(t)|^2 dt = B \int_0^{1/b} |\phi(t)|^2 \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 dt.$$

Finally, from the first equality in the proof of Theorem 4.1 we have

$$\begin{aligned} & \sum_{m,n \in \mathbb{Z}} |\langle \phi f, E_{mb}T_{y_n}g \rangle|^2 \\ &= \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_0^{1/b} |\phi(t)|^2 \left| \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - y_n - k/b)} \right|^2 dt \\ &= \frac{1}{b} \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}} |\phi(t)|^2 \overline{f(t - \ell/b)} f(t - \ell/b - j/b) \cdot \\ & \quad \sum_{n \in \mathbb{Z}} g(t - y_n - \ell/b) \overline{g(t - y_n - (\ell + j)/b)} dt \\ &= \frac{1}{b} \int_{\mathbb{R}} \overline{\phi(t)} f(t) \phi(t - k/b) f(t - k/b) \cdot \sum_{n \in \mathbb{Z}} g(t - y_n) \overline{g(t - y_n - k/b)} dt. \end{aligned}$$

Putting this altogether, we have that $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a Gabor frame with frame bounds A, B if and only if for all bounded compactly supported functions f on \mathbb{R} and all bounded $1/b$ -periodic functions ϕ on \mathbb{R} we have

$$\begin{aligned} & A \int_0^{1/b} |\phi(t)|^2 \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 dt \leq \\ & \frac{1}{b} \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}} |\phi(t)|^2 \overline{f(t - \ell/b)} f(t - \ell/b - j/b) \cdot \\ & \quad \sum_{n \in \mathbb{Z}} g(t - y_n - \ell/b) \overline{g(t - y_n - (\ell + j)/b)} dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b} \sum_{n \in \mathbb{Z}} \int_0^{1/b} |\phi(t)|^2 \left| \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - y_n - k/b)} \right|^2 dt \\
&\leq B \int_0^{1/b} |\phi(t)|^2 \sum_{j \in \mathbb{Z}} |f(t - j/b)|^2 dt.
\end{aligned}$$

Since ϕ is an arbitrary bounded $1/b$ -periodic function, the equivalence of (1) and (2) of the theorem follows. \square

Remark: It is worthwhile to observe that the middle term in (2) from Theorem 4.3 can be written

$$\begin{aligned}
&\frac{1}{b} \sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} f(t - k/b) \overline{g(t - k/b - y_n)} \right|^2 = \\
&\frac{1}{b} \sum_{j \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \overline{f(t - \ell/b)} f(t - \ell/b - j/b) \sum_{n \in \mathbb{Z}} g(t - y_n - \ell/b) \overline{g(t - y_n - (\ell + j)/b)}.
\end{aligned}$$

We can now give non-compactly supported functions yielding irregular Gabor frames. Choose $a_n > 0$ so that $\sum_{n \in \mathbb{Z}} a_n = 2$, $a_0 = 1$ and

$$\sum_{k \neq 0} \sum_{n \in \mathbb{Z}} a_n a_{n+k} \leq \frac{1}{2}.$$

Now choose any $y_n \in [n, n + \frac{1}{2}]$ and let $b = 1$. For every $t \in \mathbb{R}$, there is an n so that $t - y_n \in [0, 1]$, so $1 \leq \sum_n |g(t - y_n)|^2$. Also, for any $t \in \mathbb{R}$, if $t - y_n \in [\ell, \ell + 1]$ then $t - y_n - k \notin [\ell, \ell + 1]$, for every $0 \neq k \in \mathbb{Z}$, so

$$\sum_{0 \neq k \in \mathbb{Z}} |G_k(t)| \leq \sum_{0 \neq k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} a_n a_{n+k} \leq \frac{1}{2}.$$

By Theorem 4.2, $\{E_{mb} T_{y_n} g\}_{m, n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.

In general, the convergence properties of the series in the WH -frame identity can be very delicate [8]. In particular, the series may not converge for some f ; the series may converge symmetrically for all f but fail to converge for other f ; or the series may converge for all f but not converge unconditionally.

The WH -frame identity also yields a simple characterization of Parseval semi-irregular Weyl-Heisenberg frames. The corresponding result for regular Weyl-Heisenberg frames can be found e.g. in [9].

Theorem 4.4. *Let $g \in L^2(\mathbb{R})$, $0 < b$ and $\{y_n\}_{n=1}^{\infty}$ a relatively separated sequence in \mathbb{R} . The following are equivalent:*

(1) $\{E_{mb} T_{y_n} g\}_{m, n \in \mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{R})$.

(2) We have

(a) $G_0(t) = b$ a.e.

(b) $G_k(t) = 0$ a.e. for all $0 \neq k \in \mathbb{Z}$

Proof. (1) \Rightarrow (2): Fix an interval $I \subset \mathbb{R}$ with $|I| < \frac{1}{b}$ and let $f \in L^2(I)$. Then for all $0 \neq k \in \mathbb{Z}$, $\overline{f(t)}f(t - \frac{k}{b}) = 0$. Hence, by the *WH*-frame identity,

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{y_n}g \rangle|^2 = b^{-1} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt = b^{-1} \int_0^{1/b} |f(t)|^2 G_0(t) dt = \|f\|^2.$$

It follows from here that $G_0(t) = b$ a.e. on I , and since I was arbitrary, this equality holds on \mathbb{R} . Now, again by the *WH*-frame identity, for all bounded compactly supported $f \in L^2(\mathbb{R})$

$$\begin{aligned} \|f\|^2 &= b^{-1} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt + \sum_{0 \neq k \in \mathbb{Z}} b^{-1} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) G_k(t) dt \\ &= \|f\|^2 + \sum_{0 \neq k \in \mathbb{Z}} b^{-1} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) G_k(t) dt. \end{aligned}$$

Hence,

$$\sum_{0 \neq k \in \mathbb{Z}} b^{-1} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) G_k(t) dt = 0.$$

Fix $k_0 \neq 1$, let I be an interval in \mathbb{R} of length $\leq \frac{1}{b}$ and define $f \in L^2(\mathbb{R})$ by

$$f(t) = e^{i \arg G_{k_0}(t)}, \quad \text{for all } t \in I,$$

and $f(t - \frac{k_0}{b}) = 1$, for $t \in I$ and $f = 0$ otherwise. Then from the above,

$$\begin{aligned} 0 &= \sum_{0 \neq k \in \mathbb{Z}} b^{-1} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) G_k(t) dt \\ &= b^{-1} \sum_{k \geq 1} 2 \operatorname{Re} \int_{\mathbb{R}} \overline{f(t)} f\left(t - \frac{k}{b}\right) G_k(t) dt \\ &= b^{-1} 2 \int_I |G_{k_0}(t)| dt. \end{aligned}$$

Hence, $G_{k_0} = 0$ on I and since k_0, I were arbitrary, we have (b).

(2) \Rightarrow (1): By (2) and the *WH*-frame identity, for all bounded, compactly supported $f \in L^2(\mathbb{R})$,

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_{y_n}g \rangle|^2 = b^{-1} \int_{\mathbb{R}} |f(t)|^2 G_0(t) dt = \int_{\mathbb{R}} |f(t)|^2 dt = \|f\|^2.$$

So $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a Parseval frame for $L^2(\mathbb{R})$. \square

We also can capture the **Walnut representation** of the frame operator [42, 29] for half-irregular *WH*-frames.

Theorem 4.5. *Let $g \in L^2(\mathbb{R})$ $0 < b$, $\{y_n\}_{n \in \mathbb{Z}}$ a relatively separated sequence and let $(g, b, \{y_n\}_{n \in \mathbb{Z}})$ be a frame for $L^2(\mathbb{R})$ which satisfies the CC-condition. Then we have the following (Walnut) representation for the frame operator S for the frame $\{E_{mb}T_{y_n}g\}_{m,n \in \mathbb{Z}}$;*

$$Sf = b^{-1} \sum_{n \in \mathbb{Z}} T_{y_n} f \cdot G_n.$$

Moreover, the series converges unconditionally in $L^2(\mathbb{R})$.

Proof. Fix $f \in L^2(\mathbb{R})$. For an arbitrary $h \in L^2(\mathbb{R})$ we can follow the proof of the WH-frame identity to get

$$\left\langle b^{-1} \sum_n T_{y_n} f \cdot G_n, h \right\rangle = \sum_{m,n} \langle f, E_{mb}T_{y_n}g \rangle \langle E_{mb}T_{y_n}g, h \rangle = \langle Sf, h \rangle.$$

Since this is true for all h , the result follows. The unconditional convergence of the series follows from Theorem 4.4. \square

5 General Irregular Weyl-Heisenberg Frames

The general case is frames of the form $\{E_{x_n}T_{y_n}g\}_{n \in \mathbb{Z}}$ where $(x_n, y_n)_{n \in \mathbb{Z}}$ is a relatively separated sequence in \mathbb{R}^2 . Computations here are difficult since even nonharmonic Fourier series techniques do not apply. Since the results in this section are technical in nature, we will only outline what they are and where they can be found. The starting point for studying irregular Weyl-Heisenberg frames is a fundamental result of Feichtinger and Gröchenig [20]:

Theorem 5.1. *Let $0 \neq g \in L^2(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\langle E_x T_y g, g \rangle| dx dy < \infty. \quad (5.2)$$

Then there exists an open set $U \subset \mathbb{R}^2$ such that $\{E_{x_n}T_{y_n}g\}_{n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ for every separated set $\{(x_n, y_n)\}_{n \in \mathbb{Z}} \subset \mathbb{R}^2$ for which

$$\cup[(x_n, y_n) + U] = \mathbb{R}^2.$$

Theorem 5.4 is very important because it says we can find a box Q in \mathbb{R}^2 so that when we tile the plane with Q by say $\{Q_n\}_{n \in \mathbb{Z}}$, then whenever we choose $(x_n, y_n \in Q_n$, then $\{E_{x_n}T_{y_n}g\}_{n \in \mathbb{Z}}$ is a Weyl-Heisenberg frame for $L^2(\mathbb{R})$. For applications of this result, it is important to know the “size” of the box we can use for tiling the plane. Much work has gone into estimating the admissible boxes for Theorem 5.4. In particular, Gröchenig [28] (Theorems T and S, pages 24, 25) gives somewhat simple conditions for identifying Q . See also [4, 36] for estimates

on the size of Q . Each of these approaches has advantages in certain cases, e.g. where g is compactly supported or g is in a modulation space [26, 28].

As in the case of the Weyl-Heisenberg frames, the Wiener Amalgam spaces play an important role in the irregular cases. Given a function g , we say $g \in W[L^\infty, \ell^p]$, for $1 \leq p < \infty$ if for some $a > 0$

$$\begin{aligned} \|g\|_{w,a} &= \left(\sum_n \|g \cdot \chi_{[an, a(n+1)]}\|_\infty^p \right)^{\frac{1}{p}} \\ &= \left(\sum_n \|T_{na}g \cdot \chi_{[0,a]}\|_\infty^p \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

$\omega(L^\infty, \ell^1)$ is a Banach space. The subspace consisting of continuous functions were studied by Wiener [43]. Generalizations to L^p, ℓ^p can be found in [24]. Feichtinger [19] gave these and even more general classes the name **Wiener Type spaces**. Wiener type spaces were used by Feichtinger [25] to define **modulation spaces** which now play a major role in this area.

An alternative approach to finding the boxes in Theorem 5.4 (in the irregular lattice case) using the Wiener amalgam space $W(L^\infty, \ell^1)$ is presented in [6]. This approach also shows some restrictions on g which are necessary for the Feichtinger/Gröchenig Theorem to hold.

The following results of Casazza and Christensen shows the use of the Wiener space for constructing Weyl-Heisenberg frames over irregular lattices and has as a consequence that Theorem 5.4 can hold only if the function g is bounded below on an interval. The equivalence of (1) and (2) comes from [7] and the equivalence of (1) and (3) comes from [6].

Theorem 5.2. *Let $g \in W(L^\infty, \ell^1)$. Then the following are equivalent:*

- (1) *g is bounded below on an interval.*
- (2) *There exist $a, b_0, A > 0$ such that for all $b \in [0, b_0]$, $x_m \in [mb, (m_1)b]$, and $y_n \in [na, (n_1)a]$, $\{E_{x_m}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound A .*
- (3) *There exist $a, b_0, A > 0$ such that for all $b \in (0, b_0]$, $x_m \in [mb, (m+1)b]$, and all $y_n \in [na, (n+1)a]$, the family $\{E_{x_m}T_{y_n}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with lower frame bound A .*

The proof of the equivalence of (1) and (3) above is quite technical. This is caused by the fact that we are not assuming that our sequences $\{x_n\}$ and $\{y_n\}$ are separated and that g has compact support. Under these assumptions, Gröchenig [27] has more general and stronger results.

It is not known if $g \in W(L^\infty, \ell^1)$ is necessary for Theorem 5.2. But, in [6] it is shown:

Theorem 5.3. *Let $g \in L^2(\mathbb{R})$ and assume there exist $a, b > 0$ such that for all $y_n \in [an, a(n+1)]$, $\{E_{mb}T_{y_n}\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A, B > 0$. Then*

$$bA \leq \sum_{n \in \mathbb{Z}} \operatorname{ess\,inf}_{an \leq t \leq a(n+1)} |\chi_{[an, a(n+1)]} g(t)|^2 \leq \sum_{n \in \mathbb{Z}} \|\chi_{[an, a(n+1)]} g\|_\infty^2 \leq bB.$$

In particular, $g \in W(L^\infty, \ell^2)$.

A main ingredient in the proof of Theorem 5.2 is a new approach to doing Weyl-Heisenberg frame calculations due to Casazza and Christensen [6]. The idea is to transfer these calculations to $L^2[0, b]$ where we have many more results (especially perturbation theory [1, 14]) to work with. This idea has not been fully explored yet and we offer it here as a tool for further study.

Theorem 5.4. *Let $g \in L^2(\mathbb{R})$, $b > 0$ and $x_m, y_n \in \mathbb{R}$ with $\{x_n\}$ relatively separated. For each $k \in \mathbb{Z}$, let $R_k : \ell^2(\mathbb{Z}) \rightarrow L^2[0, \frac{1}{b}]$ be the synthesis operator for $\{\sqrt{b}\chi_{[0, \frac{1}{b}]} T_{\frac{k}{b}} E_{x_m}\}_{m \in \mathbb{Z}}$. Then for all $f \in L^2(\mathbb{R})$ we have*

$$\begin{aligned} & \sum_{n, m \in \mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g \rangle|^2 = \\ & \frac{1}{b} \sum_{n, k, \ell \in \mathbb{Z}} \left\langle R_k^* \left(\chi_{[0, \frac{1}{b}]} T_{\frac{k}{b}} f \cdot \overline{T_{y_n + \frac{k}{b}} g} \right), R_\ell^* \left(\chi_{[0, \frac{1}{b}]} T_{\frac{\ell}{b}} f \cdot \overline{T_{y_n + \frac{\ell}{b}} g} \right) \right\rangle_{\ell^2}. \end{aligned}$$

Proof. We compute:

$$\begin{aligned} & \sum_{m, n \in \mathbb{Z}} |\langle f, E_{x_m} T_{y_n} g \rangle|^2 = \sum_{m, n \in \mathbb{Z}} \left| \int_{\mathbb{R}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right|^2 \\ & = \sum_{m, n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \int_{\frac{k}{b}}^{\frac{k+1}{b}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right|^2 \\ & = \sum_{m, n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} \int_{\frac{k}{b}}^{\frac{k+1}{b}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right) \cdot \\ & \quad \left(\sum_{\ell \in \mathbb{Z}} \int_{\frac{\ell}{b}}^{\frac{\ell+1}{b}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right) \\ & = \sum_{n, m, k, \ell \in \mathbb{Z}} \left(\int_{\frac{k}{b}}^{\frac{k+1}{b}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right) \left(\int_{\frac{\ell}{b}}^{\frac{\ell+1}{b}} f(t) \overline{T_{y_n} g(t)} e^{-2\pi i x_m t} dt \right) \\ & = \frac{1}{b} \sum_{n, k, \ell \in \mathbb{Z}} \left[\sum_{m \in \mathbb{Z}} \left(\int_0^{\frac{1}{b}} (t \frac{k}{b} f(t)) (\overline{T_{y_n + \frac{k}{b}} g(t)} \sqrt{b} T_{\frac{k}{b}} E_{x_m}(t) dt) \right) \cdot \right. \\ & \quad \left. \left(\int_0^{\frac{1}{b}} (\overline{T_{\frac{\ell}{b}} f(t)}) (T_{y_n + \frac{\ell}{b}} g(t)) (\sqrt{b} T_{\frac{\ell}{b}} E_{x_m}(t) dt) \right) \right] \\ & = \frac{1}{b} \sum_{n, k, \ell \in \mathbb{Z}} \left[\left\langle \left(\left\langle \chi_{[0, \frac{1}{b}]} T_{\frac{k}{b}} f \cdot \overline{T_{y_n + \frac{k}{b}} g}, \sqrt{b} T_{\frac{k}{b}} E_{x_m} \right\rangle \right)_{m \in \mathbb{Z}}, \right. \right. \\ & \quad \left. \left. \left(\left\langle \chi_{[0, \frac{1}{b}]} T_{\frac{\ell}{b}} f \cdot \overline{T_{y_n + \frac{\ell}{b}} g}, \sqrt{b} T_{\frac{\ell}{b}} E_{x_m} \right\rangle_{\ell \in \mathbb{Z}} \right) \right\rangle_{\ell^2} \right] \\ & = \frac{1}{b} \sum_{n, k, \ell \in \mathbb{Z}} \left\langle R_k^* \left(\chi_{[0, \frac{1}{b}]} T_{\frac{k}{b}} f \cdot \overline{T_{y_n + \frac{k}{b}} g} \right), R_\ell^* \left(\chi_{[0, \frac{1}{b}]} T_{\frac{\ell}{b}} f \cdot \overline{T_{y_n + \frac{\ell}{b}} g} \right) \right\rangle_{\ell^2}. \end{aligned}$$

□

The importance of Theorem 5.4 is that it allows us to transfer Weyl-Heisenberg frame calculations from $L^2(\mathbb{R})$ to $L^2[0, \frac{1}{b}]$ where we have very strong perturbation results of Balan [1] and Christensen [14].

There is still more work to be done for fully implementing Theorem 5.4. In particular, perturbation theory should play a role here and this is just now being investigated (See Section 6).

6 Concluding Remarks

There has been much progress made recently on perturbations of Gabor frames. The results of [11] hold in the half-irregular case to show that when $(g, b, \{x_n\})$ is a frame for $L^2(\mathbb{R})$ and $\|g - h\|_{w,a}$ is small enough then $(h, b, \{x_n\})$ is also a frame. The important point here is that g does not need to be in $W(L^\infty, \ell^1)$. Until now there was little known about the joint perturbations of g and the parameters a, b . Recently, Lakey and Wang [34] have produced results in this direction. In a major paper, Feichtinger and Kablinger [21] have made a deep study of joint perturbations of WH -frames.

Brackett products have been very useful in this area [13, 39, 40]. There is an obvious generalization of Bracket products to the irregular case:

$$[f, g](t) = \sum_{n \in \mathbb{Z}} f(t - y_n) \overline{g(t - y_n)}.$$

It would be interesting to see if this could be used to advance the theory of irregular WH -frames.

Janssen and Strohmer [33] have given a very efficient method for finding the canonical tight window $S^{-\frac{1}{2}}g$ of a Weyl-Heisenberg frame. Perhaps these methods can be generalized to the irregular case. In the irregular case we have the problem that the dual frame and canonical tight frame need not be a (irregular) WH -frame. But perhaps these methods could identify the dual frame elements or the tight frame elements one at a time.

In an interesting development, Liu and Wang [35] have shown that whenever the generating function g of an orthonormal WH -basis is supported on an interval, then both the frequency and translation sets of the WH -basis must be lattices. They also give examples of compactly supported irregular orthonormal Gabor bases in higher dimensions.

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