

PATH CONNECTEDNESS OF WAVELET TRANSFORM

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ABSTRACT. This paper is a tutorial extension of the work by Daubechies and Sweldens on factoring wavelet transform into lifting steps using the Euclidean algorithm. For any pair of 1-dimensional, two channels, perfect reconstruction (PR) filter banks given by the filters \tilde{h}^0 , \tilde{g}^0 (analysis), h^0 , g^0 (synthesis), and \tilde{h}^1 , \tilde{g}^1 (analysis), h^1 , g^1 (synthesis), paths of PR filter banks \tilde{h}^t , \tilde{g}^t (analysis), h^t , g^t (synthesis) are constructed. The construction of such paths from the filter bank at $t = 0$ to the filter bank at $t = 1$ is simple and flexible, allowing for continuity and/or differentiability in the variable t . A MATLAB toolbox [1] for factorization into lifting steps, construction of such paths and experimentation is freely distributed by the authors. The results on path connectedness are generalized to M -channels, n -dimensional PR filterbanks given by lifting schemes.

1. INTRODUCTION

In [2] it is shown how any discrete wavelet transform or two band subband filtering with finite filters can be decomposed into a finite sequence of simple filtering steps called lifting steps. For the purpose of this paper perfect reconstruction is all that is needed, thus the terms wavelet transform and filter bank will be used with the same meaning. A two channel filter bank is given by two analysis filters \tilde{h} and \tilde{g} and two synthesis filters h and g .

The z -transform of a FIR filter h is given by the Laurent polynomial

$$(1.1) \quad h(z) = \sum_{k=k_b}^{k=k_e} h_k z^{-k}$$

If the *modulation matrix* is given by

$$M(z) = \begin{bmatrix} h(z) & h(-z) \\ g(z) & g(z) \end{bmatrix}$$

The perfect reconstruction condition is given by

$$\tilde{M}(z^{-1})^t M(z) = 2 \mathbf{I}$$

where $\tilde{M}(z)$ is defined as $M(z)$ in terms of \tilde{h} and \tilde{g} . In case of perfect reconstruction FIR $M(z)$ and $\tilde{M}(z)$ belong to $GL(2, \mathbf{R}[z, z^{-1}])$.

The *polyphase representation* of a filter is given by

$$h(z) = h_e(z^2) + z^{-1}h_0(z^2)$$

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where

$$h_e(z) = \sum_k h_{2k} z^{-k}, \quad h_o(z) = \sum_k h_{2k+1} z^{-k}$$

The *polyphase matrix* is defined as

$$(1.2) \quad P(z) = \begin{bmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{bmatrix}$$

Then,

$$P(z^2))^t = \frac{1}{2} M(z) \begin{bmatrix} 1 & z \\ 1 & -z \end{bmatrix}$$

If $\tilde{P}(z)$ is defined in similar terms, the perfect reconstruction condition becomes

$$(1.3) \quad P(z)\tilde{P}(z^{-1})^t = \mathbf{I}.$$

A *lifting step* is multiplication by a matrix

$$\begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}$$

and a *dual lifting step* is multiplication by a matrix

$$\begin{bmatrix} 1 & 0 \\ q(z) & 1 \end{bmatrix}$$

where

$$(1.4) \quad \begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -s(z) \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ q(z) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -q(z) & 1 \end{bmatrix}$$

In [2] it is proved that any one-dimensional wavelet transform with finite filters can be factored into a product of liftings and dual liftings.

In this paper, the lifting factorization will be used to construct paths of analysis filters \tilde{h}_t, \tilde{g}_t and synthesis filters h_t, g_t from the PR filters $\tilde{h}_0, \tilde{g}_0, h_0, g_0$ to the PR filters $\tilde{h}_1, \tilde{g}_1, h_1, g_1$ with the property that $\tilde{h}_t, \tilde{g}_t, h_t, g_t$ are PR filters for every $t \in [0, 1]$. The construction is simple and gives room for experimentation with different paths.

Because the path connectivity relies heavily on the lifting factorization, and the lifting factorization is made possible by the Euclidean algorithm on Laurent polynomials, a review of some details in [2] will be helpful.

The degree of the Laurent polynomial $h(z) = \sum_{k=k_b}^{k=k_e} h_k z^{-k}$ is given by

$$|h| = k_e - k_b$$

or just the length of the filter minus 1. A Laurent Polynomial is invertible if and only if it is a monomial. For two Laurent polynomials $a(z)$ and $b(z) \neq 0$ with $|a| \geq |b|$ an Euclidean algorithm giving $c(z) = \gcd(a(z), b(z))$ is constructed in [2], Theorem 5. Moreover, the algorithm gives the identity

$$\begin{bmatrix} a(z) \\ b(z) \end{bmatrix} = \prod_{i=1}^n \begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c(z) \\ 0 \end{bmatrix}$$

The only invertible Laurent polynomials are monomials. If the polyphase matrix (1.2) is invertible, its determinant is an invertible Laurent polynomial. If the determinant is not 1 then one can divide $g_e(z)$ and $g_o(z)$ by the determinant. Thus without loss of generality we assume the determinant is 1. In such case, there is a scalar K such that

$$(1.5) \quad \gcd(h_e(z), h_o(z)) = K.$$

A filter pair (h, g) is *complementary* in case the corresponding polyphase matrix $P(z)$ has determinant 1. Then, [2], Theorem 3, any other filter g^{new} such that (h, g^{new}) is *complementary* is given by

$$g^{new}(z) = g(z) + h(z)s(z^2)$$

where $s(z)$ is a Laurent polynomial. The new polyphase matrix is given by

$$P^{new}(z) = P(z) \begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}$$

Therefore the polyphase matrix of (h, g^{new}) differs from the polyphase matrix of (h, g) by a lifting.

The factorization of the polyphase representation of a PR filter bank into lifting steps is as follows [2], Theorem 7. Without loss of generality assume that determinant of

$$P(z) = \begin{bmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{bmatrix}$$

is 1. Then (1.5) holds. Thus the Euclidean algorithm gives

$$\begin{bmatrix} h_e(z) \\ h_o(z) \end{bmatrix} = \prod_{i=1}^n \begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix}$$

If $g_e(z)$, $g_o(z)$ and $P^0(z)$ are given by

$$P^0(z) = \begin{bmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{bmatrix} = \prod_{i=1}^n \begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix}$$

then determinant of $P^0(z)$ is 1. As detailed in [2], Theorem 7, it can be assumed that n is even. Using

$$\begin{bmatrix} q_i(z) & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & q_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_i(z) & 1 \end{bmatrix}$$

one has

$$P^0(z) = \prod_{i=1}^{i=n/2} \begin{bmatrix} 1 & q_{2i-1}(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{2i}(z) & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix}$$

Since $P(z)$ and $P^0(z)$ differ by a lifting,

$$P(z) = P^0(z) \begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}$$

By taking $m = \frac{n}{2} + 1$, $t_m(z) = 0$ and $s_m(z) = K^2 s(z)$, the factorization of $P(z)$ into lifting steps is given by

$$(1.6) \quad P(z) = \prod_{i=0}^m \begin{bmatrix} 1 & s_i(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z) & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix}$$

From the condition (1.3) it follows that

$$(1.7) \quad \tilde{P}(z) = \prod_{i=0}^m \begin{bmatrix} 1 & 0 \\ -s_i(z^{-1}) & 1 \end{bmatrix} \begin{bmatrix} 1 & -t_i(z^{-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \frac{1}{K} \end{bmatrix}$$

2. PATH CONNECTEDNESS

Before constructing paths between PR filters we establish a partial generalization [2] Theorem 7 for an n -dimensional signal decimated by a factor of M . We will show that an analysis of such signal given by a lifting scheme admits a synthesis, thus giving PR, and the synthesis is given by a lifting scheme.

A *lifting step* for such signal is given by an $M \times M$ matrix

$$(2.1) \quad S(\bar{z}) = \begin{bmatrix} 1 & s_{1,2}(\bar{z}) & \dots & s_{1,M}(\bar{z}) \\ 0 & 1 & \dots & s_{2,M}(\bar{z}) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & s_{M-1,M}(\bar{z}) \\ 0 & \dots & \dots & 1 \end{bmatrix}$$

where $s_{i,j}(\bar{z})$ is a Laurent polynomial on the n -dimensional complex variable $\bar{z} = (z_1, \dots, z_n)$. A *dual lifting step* is similarly defined in terms of lower triangular matrices. The fact that such matrices admit a simple left inverse allows to reconstruct any signal filtered by a sequence of such matrices.

Lemma 2.1. *Let $M \geq 2$ and let $S(\bar{z})$ be as in (2.1). Then it has a left inverse $T(\bar{z})$ which is upper triangular with ones in the diagonal entries. If all entries in $S(\bar{z})$ have finitely many coefficients, so does $T(\bar{z})$. The same applies for lower triangular case.*

Proof. If $M = 2$ the result is true by (1.4). Assume the result is true for $M - 1$ and let $H(\bar{z})$ be the $(M - 1) \times (M - 1)$ matrix given by the lemma for the top right $(M - 1) \times (M - 1)$ block of $S(\bar{z})$. Let

$$T(\bar{z}) = \left[\begin{array}{ccc|c} & & & h_{1,M}(\bar{z}) \\ & H(\bar{z}) & & h_{2,M}(\bar{z}) \\ & & & \vdots \\ & & & h_{M-1,M}(\bar{z}) \\ \hline 0 & \dots & \dots & 1 \end{array} \right]$$

where

$$\begin{aligned} h_{1,M}(\bar{z}) &= -(h_{1,1}s_{1,M}(\bar{z}) + \cdots + h_{1,M-1}s_{M-1,M}(\bar{z})) \\ h_{2,M}(\bar{z}) &= -(h_{2,1}s_{1,M}(\bar{z}) + \cdots + h_{2,M-1}s_{M-1,M}(\bar{z})) \\ &\dots \\ h_{M-1,M}(\bar{z}) &= -(h_{M-1,1}s_{1,M}(\bar{z}) + \cdots + h_{M-1,M-1}s_{M-1,M}(\bar{z})) \end{aligned}$$

Then

$$H(\bar{z})S(\bar{z}) = \mathbf{I}$$

and from the construction, all entries on $H(\bar{z})$ have finitely many coefficients. \square

One remark about the lemma is that no separability condition is required for the Laurent polynomial entries.

Now suppose that an n -dimensional signal is decimated into M channels and then, for two values $\rho = 0$ and $\rho = 1$, it is filtered by a sequence of liftings and dual liftings, and then dilated on each channel. Then, the two analysis are given by the $M \times M$ matrices (1.7),

$$(2.2) \quad \widetilde{P}^0(z^{-1})^0 = \Lambda^0 \prod_{i=0}^{m^0} \begin{bmatrix} 1 & 0 \\ -t_i^0(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -s_i^0(z) \\ 0 & 1 \end{bmatrix}, \quad \widetilde{P}^1(z^{-1})^t = \Lambda^1 \prod_{i=0}^{m^1} \begin{bmatrix} 1 & 0 \\ -t_i^1(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -s_i^1(z) \\ 0 & 1 \end{bmatrix}.$$

and the two synthesis are given by the $M \times M$ matrices (1.6),

$$(2.3) \quad P^0(z) = \prod_{i=0}^{m^0} \begin{bmatrix} 1 & s_i^0(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i^0(z) & 1 \end{bmatrix} (\Lambda^0)^{-1}, \quad P^1(z) = \prod_{i=0}^{m^1} \begin{bmatrix} 1 & s_i^1(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i^1(z) & 1 \end{bmatrix} (\Lambda^1)^{-1},$$

where Λ^ρ are diagonal matrices with positive entries in the diagonal, and all the the nonzero non-diagonal entries in the remaining matrices are Laurent polynomials. Without loss of generality one can assume that $m^0 = m^1$ since the identity matrix is upper and lower triangular.

There are numerous ways to connect two Laurent polynomials $a^0(z)$ and $a^1(z)$ via a path of Laurent polynomials on the variable ρ , and the path can be continuous on the variable ρ . One simple choice is

$$a^\rho(z) = \rho a^1(z) + (1 - \rho)a^0(z)$$

Then, a path of filtering schemes given by

$$\widetilde{P}^\rho(z^{-1})^t = (\rho\Lambda^1 + (1 - \rho)\Lambda^0) \prod_{i=0}^m \begin{bmatrix} 1 & 0 \\ -\rho t_i^1(z) - (1 - \rho)t_i^0(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho s_i^1(z) - (1 - \rho)s_i^0(z) \\ 0 & 1 \end{bmatrix},$$

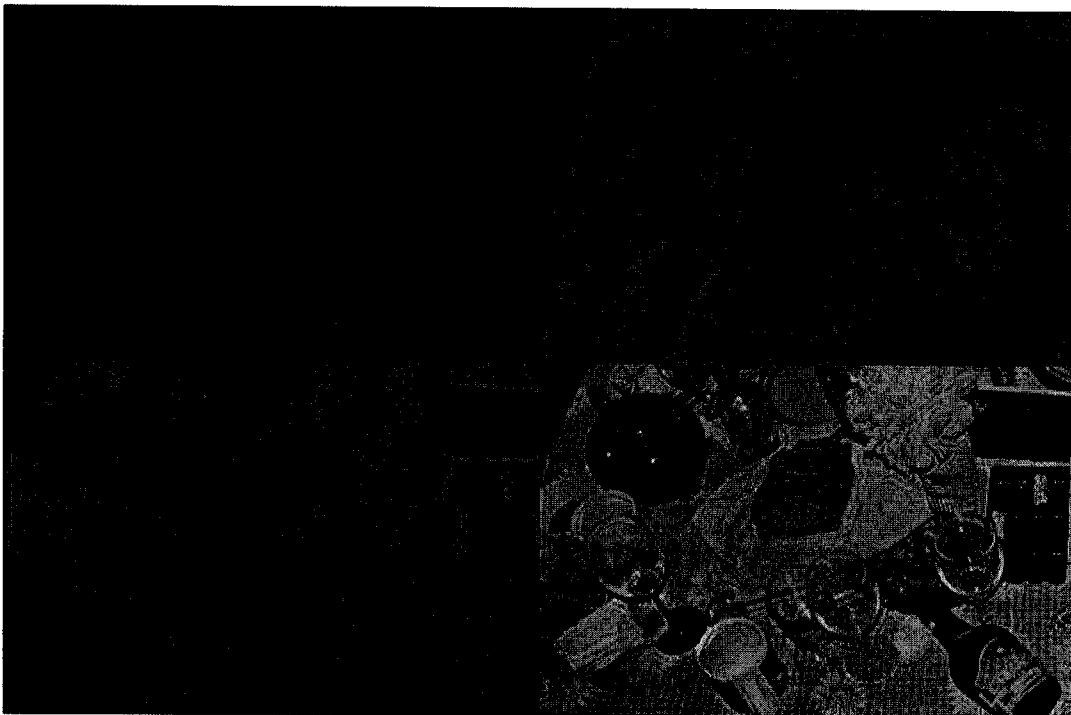
and the analysis are given by

$$(2.4) \quad P^\rho(z) = \prod_{i=0}^m \begin{bmatrix} 1 & \rho s_i^1(z) + (1 - \rho)s_i^0(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho t_i^1(z) + (1 - \rho)t_i^0(z) & 1 \end{bmatrix} (\rho\Lambda^1 + (1 - \rho)\Lambda^0)^{-1}$$

We illustrate this construction using the 512×768 image in (7). For two one dimensional different PR filters banks corresponding to $\rho = 0$ and $\rho = 1$, a path of one dimensional PR filter banks $\widetilde{h}^t, \widetilde{g}^t$ (analysis), h^t, g^t (synthesis) is constructed. For each $\rho \in [0, 1]$ a two dimensional PR separable filter bank is constructed. The transformed signal for the filter banks at $\rho = 0, \rho = 0.2, \rho = 0.4, \rho = 0.6, \rho = 0.8$ and $\rho = 1$ are shown in Figures 2 to 7.



FIGURE 1. Signal

FIGURE 2. Transform at $t = 0$

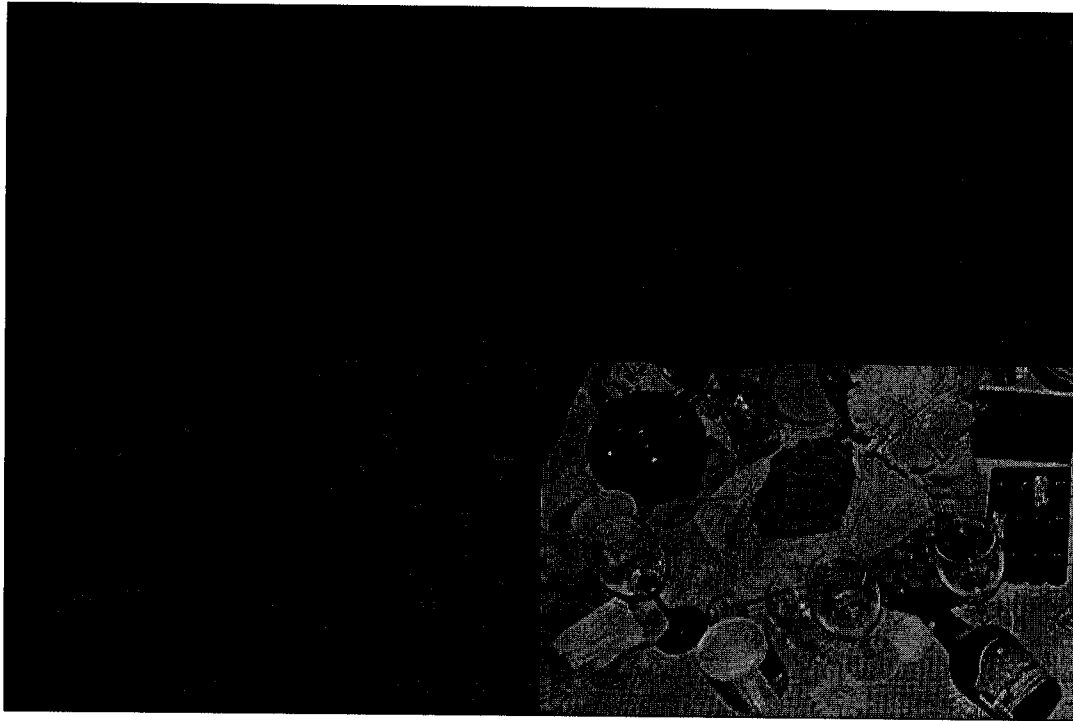


FIGURE 3. Transform at $t = 0.2$

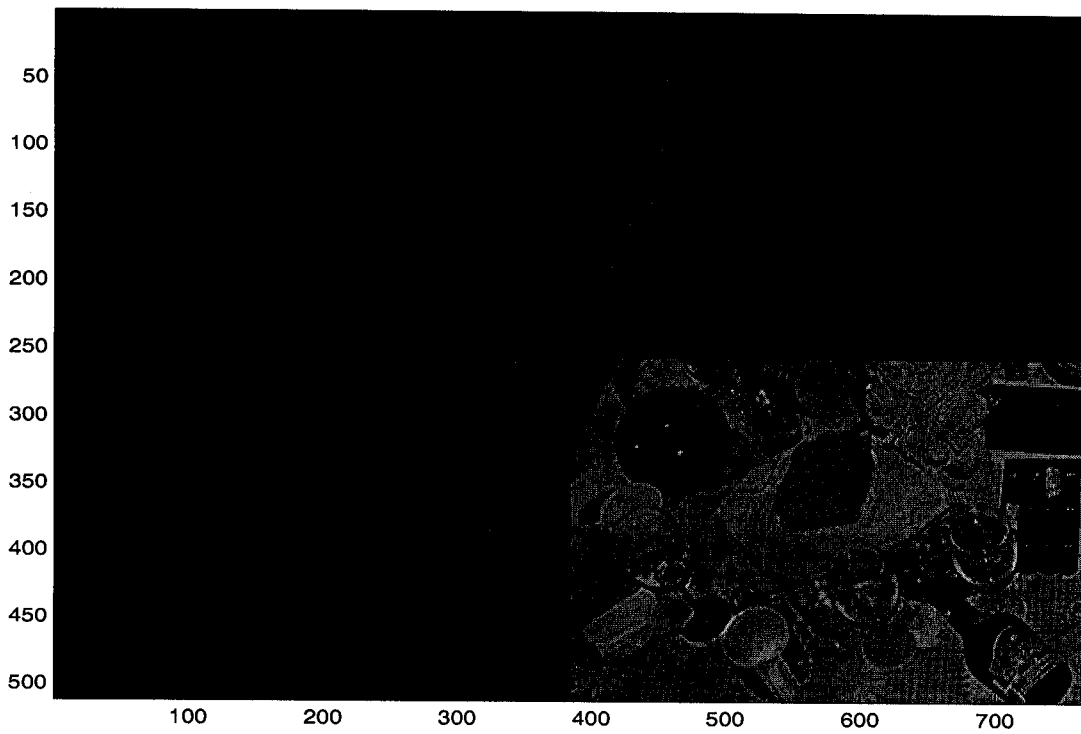
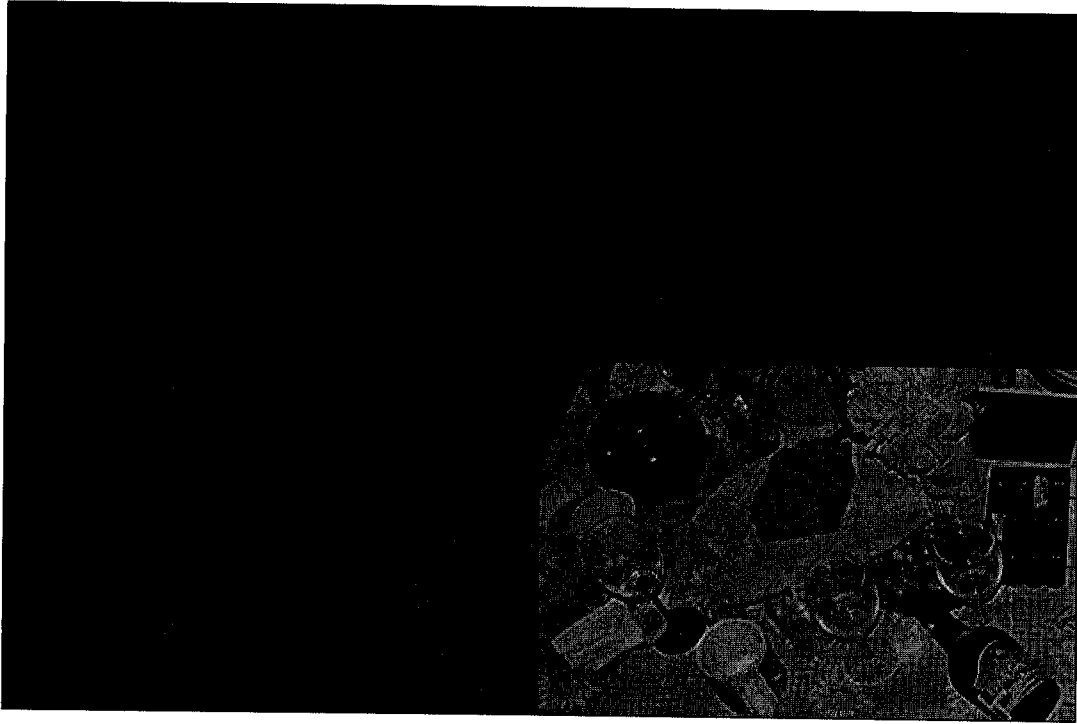
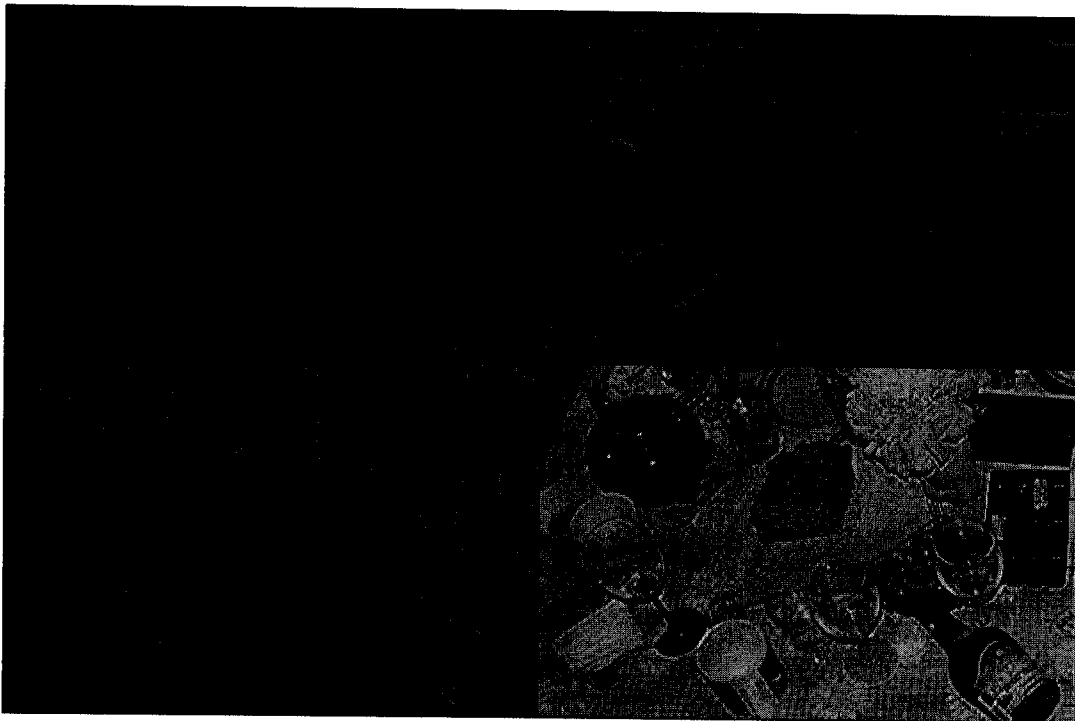
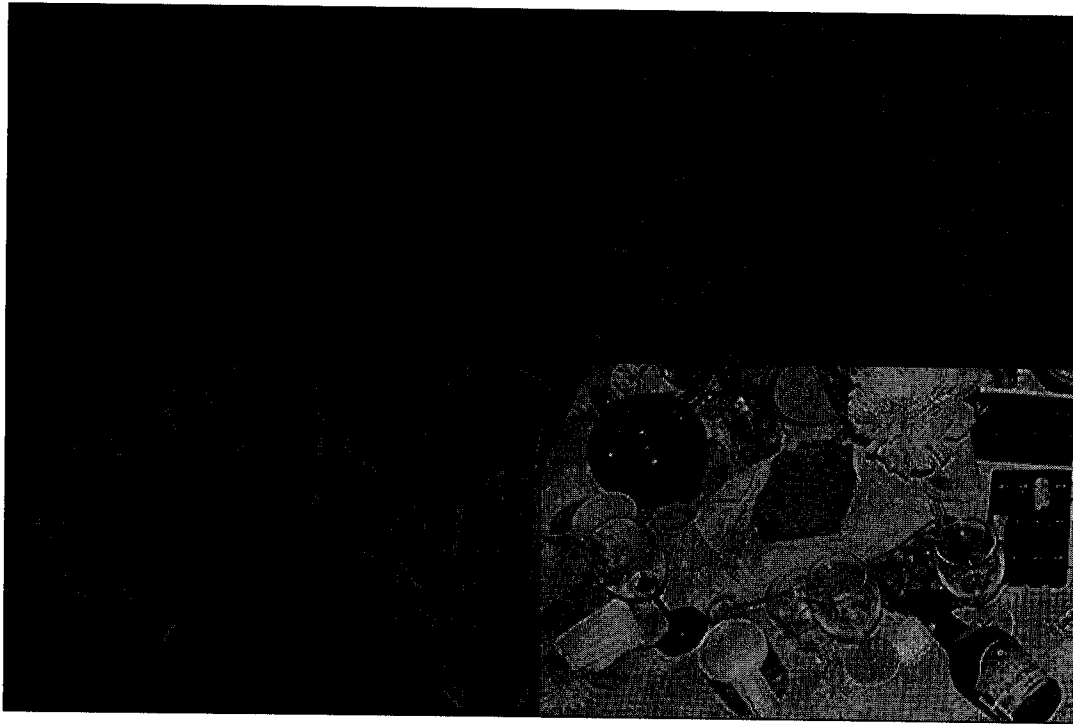


FIGURE 4. Transform at $t = 0.4$

FIGURE 5. Transform at $t = 0.6$ FIGURE 6. Transform at $t = 0.8$

FIGURE 7. Transform at $t = 1$

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