Perturbation of operators and applications to frame theory.

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Abstract

A celebrated classical result states that an operator U on a Banach space is invertible if it is close enough to the identity operator I in the sense that $||I - U|| < 1$. Here we show that U actually is invertible under a much weaker condition. As an application we prove new theorems concerning stability of frames (and frame-like decompositions) under perturbation in both Hilbert spaces and Banach spaces.

1 Perturbation of operators.

In this section, $\mathcal{X}$ and $\mathcal{Y}$ denote Banach spaces. The set of linear, bounded invertible operators from $\mathcal{X}$onto $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, or $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$.

We begin with a condition for an operator between Banach spaces to be invertible. Most of the work involves showing that the condition implies that the operator is surjective. This result is due to Hilding [H1]. Hilding was only interested in the case of an operator on a Hilbert space, but his proof actually works in the more general setting discussed here. For completeness

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we will give (a slight simplification of) Hildings proof. For an operator theoretic proof and generalizations we refer the reader to [CK,E].

Lemma 1.1: *Let $U : \mathcal{X} \to \mathcal{X}$ be a linear operator, and assume that there exist constants $\lambda_1, \lambda_2 \in [0;1[$ such that*

$$||Ux - x|| \leq \lambda_1||x|| + \lambda_2||Ux||, \forall x \in \mathcal{X}.$$

Then $U \in \mathcal{L}(\mathcal{X})$, and

$$\frac{1-\lambda_1}{1+\lambda_2}||x|| \leq ||Ux|| \leq \frac{1+\lambda_1}{1-\lambda_2}||x|| \, , \; \frac{1-\lambda_2}{1+\lambda_1}||x|| \leq ||U^{-1}x|| \leq \frac{1+\lambda_2}{1-\lambda_1}||x||, \; \forall x \in \mathcal{X}.$$

Proof: To simplify notation, let $\lambda = max\{\lambda_1, \lambda_2\}$, and $\epsilon = \frac{1-\lambda}{1+\lambda}$. We prove lemma 1.1 in two steps.
Step I For any $\alpha \leq 0$,

$$\|\alpha x - Ux\| \geq \epsilon\|x\|.$$

For the proof, we have

$$\|x - Ux\| \leq \lambda\|x\| + \lambda\|Ux\| \leq \lambda\|\alpha x - Ux\| + \lambda(1-\alpha)\|x\|.$$

Similarly,

$$\|x - Ux\| = \|(1-\alpha)x + (\alpha x - Ux)\| \geq (1-\alpha)\|x\| - \|\alpha x - Ux\|.$$

Combining our two inequalities above we have,

$$\|\alpha x - Ux\| \geq \frac{(1-\lambda)(1-\alpha)}{1+\lambda}\|x\| \geq \epsilon\|x\|.$$

Step II U is onto.
Let

$$E = \{\alpha \leq 0 : \|\alpha x^* - U^*x^*\| \geq \frac{\epsilon}{2}\|x^*\|, \forall x^* \in X^*\}.$$

Now, E is closed and E is non-empty since $\alpha = -(1 + \|U^*\|) \in E$. Also, if $\alpha \in E$, then $(\alpha I - U)^*$ is one-to-one and so $\alpha I - U$ is an onto isomorphism. Thus, by step I,

$$\|(\alpha I^* - u^*)^{-1}\| = \|(\alpha I - U)^{-1}\| \leq \frac{1}{\epsilon}.$$

Therefore, for all $x^* \in X^*$ we have

$$\|(\alpha I^* - U^*)x^*\| \geq \epsilon \|x^*\|,$$

and so

$$\|((\alpha + \frac{\epsilon}{2})I^* - U^*)x^*\| \geq \|(\alpha I^* - U^*)x^*\| - \frac{\epsilon}{2}\|x^*\| \geq \frac{\epsilon}{2}\|x^*\|.$$

The above shows that whenever $\alpha \in E$ we have $E \cap [\alpha, \alpha + \frac{\epsilon}{2}) \subset E$. It now follows that $0 \in E$, so U^* is one-to-one and hence U is onto.
Now we finish by proving the norm estimates. Given $x \in \mathcal{X}$,

$$||Ux|| \leq ||Ux - x|| + ||x|| \leq (1 + \lambda_1)||x|| + \lambda_2||Ux||$$

which leads to the upper estimate for $||Ux||$. Similary the lower estimate follows from

$$||Ux|| \geq ||x|| - ||Ux - x|| \geq (1 - \lambda_1)||x|| - \lambda_2||Ux||.$$

In particular $U \in \mathcal{L}(\mathcal{X})$. By replacing x by $U^{-1}x$ in the above inequalities, we get the rest of the lemma. **Q.E.D.**

Remark: Observe that the result is much stronger than the classical result that an operator U is invertible if $||I - U|| < 1$. We demonstrate this by an example:

Example 1.2: Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. We define an operator $U : \mathcal{H} \to \mathcal{H}$ by $Ue_i := e_i + \frac{1}{i}e_{i+1}, i = 1, 2, ...$ (extended by linearity). The action on an element $f = \sum_{i=1}^{\infty} < f, e_i > e_i$ is $Uf = \sum_{i=1}^{\infty} < f, e_i > (e_i + \frac{1}{i}e_{i+1})$. Now

$$||Uf - f|| = ||\sum_{i=1}^{\infty} < f, e_i > \frac{1}{i}e_{i+1}|| \leq ||f||, \quad \forall f \in \mathcal{H}.$$

Since $Ue_1 - e_1 = e_2$, it follows that $||I - U|| = 1$. So the classical result does not show that U is in fact invertible. But we can show this using Lemma 1.1:

First observe that $||U|| \leq ||U-I|| + ||I|| = 2$ and that

$$||Uf|| = || < f, e_1 > e_i + \sum_{i=2}^{\infty} [< f, e_i > + \frac{1}{i-1} < f, e_{i-1} >] e_i ||$$

$$\geq | < f, e_1 > |, \quad \forall f \in \mathcal{H}.$$

So

$$||Uf - f||^2 = \sum_{i=1}^{\infty} | \frac{1}{i} < f, e_i > |^2 \leq | < f, e_1 > |^2 + \frac{1}{4} ||f||^2$$

$$\leq ||Uf||^2 + \frac{1}{4} ||f||^2 \leq (||Uf|| + \frac{1}{2} ||f||)^2.$$

It follows that

$$||Uf - f|| \leq ||Uf|| + \frac{1}{2} ||f||$$

$$\leq \frac{7}{8} ||Uf|| + \frac{1}{8} ||U|| \cdot ||f|| + \frac{1}{2} ||f|| = \frac{7}{8} ||Uf|| + \frac{3}{4} ||f||.$$

So by Lemma 1.1, $U \in \mathcal{L}(\mathcal{H})$. As a consequence, $\{e_i + \frac{1}{i} e_{i+1}\}_{i=1}^{\infty}$ is a Riesz basis.

More generally we can consider an operator V which is close to some invertible operator U:

Theorem 1.3: *Let $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and let $V : \mathcal{X} \to \mathcal{Y}$ be linear. If there exist two constants $\lambda_1, \lambda_2 \in [0; 1[$ such that*

$$(1) \qquad ||Ux - Vx|| \leq \lambda_1 ||Ux|| + \lambda_2 ||Vx||, \ \forall x \in \mathcal{X}$$

then $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and

$$\frac{1-\lambda_1}{1+\lambda_2} ||Ux|| \leq ||Vx|| \leq \frac{1+\lambda_1}{1-\lambda_2} ||Ux|| \ , \ \forall x \in \mathcal{X}$$

and

$$\frac{1-\lambda_2}{1+\lambda_1} \frac{1}{||U||} ||y|| \leq ||V^{-1} y|| \leq \frac{1+\lambda_2}{1-\lambda_1} \cdot ||U^{-1}|| \cdot ||y||, \ \forall y \in \mathcal{Y}.$$

Proof: Define a linear mapping

$$L : \mathcal{Y} \to \mathcal{Y}, \quad Ly := VU^{-1}y.$$

Using (1) with $x = U^{-1}y$ we obtain that

$$||y - Ly|| \leq \lambda_1 ||y|| + \lambda_2 ||Ly||, \ \forall y \in \mathcal{Y}.$$

So $L \in \mathcal{L}(\mathcal{Y})$ by Lemma 1.1, and therefore $V = LU \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Furthermore, the norm estimates in Lemma 1.1 give that

$$\frac{1-\lambda_1}{1+\lambda_2}||Ux|| \leq ||Vx|| = ||LUx|| \leq \frac{1+\lambda_1}{1-\lambda_2}||Ux||, \ \forall x \in \mathcal{X}.$$

Using those inequalities with $x = V^{-1}y$, where $y \in \mathcal{Y}$, we obtain that

$$\frac{1-\lambda_2}{1+\lambda_1}||y|| \leq ||UV^{-1}y|| \leq \frac{1+\lambda_2}{1-\lambda_1}||y||$$

from which the result follows. **Q.E.D.**

In a special case we are even allowed to take $\lambda_2 = 1$:

Corollary 1.4: *If $U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{V} : \mathcal{X} \to \mathcal{Y}$ is linear and bounded and if there exists a constant $\lambda_1 \in [0; 1[$ such that*

$$||Ux - Vx|| \leq \lambda_1 ||Ux|| + ||Vx||, \ \forall x \in \mathcal{X},$$

then $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and

$$||V^{-1}|| \leq \frac{2}{1-\lambda_1}||U^{-1}||$$

Proof: Let $\epsilon > 0$. Repeating the proof of Theorem 1.3 we get

$$||y - VU^{-1}y|| \leq \lambda_1 ||y|| + ||VU^{-1}y|| \leq (\lambda_1 + \epsilon ||VU^{-1}||)||y|| + (1-\epsilon)||VU^{-1}y||, \ \forall y \in \mathcal{Y}.$$

If we choose ϵ small enough, then $\lambda_1 + \epsilon ||VU^{-1}|| < 1$, implying that VU^{-1} is invertible. Therefore V is invertible. Also, for those small ϵ,

$$||(VU^{-1})^{-1}|| \leq \frac{1 + 1 - \epsilon}{1 - (\lambda_1 + \epsilon ||VU^{-1}||)}$$

from which the norm estimate follows by letting $\epsilon \to 0$. **Q.E.D.**

Remark: Condition (1) implies that U and V share many properties. For example, if U has closed range, then V has closed range. Indeed, if $Vx_n \to y$ as $n \to \infty$, then $\{Vx_n\}$ is a Cauchy sequence, implying that $\{Ux_n\}$ is a Cauchy sequence by Theorem 1.3. So if U has closed range, Ux_n converges to an element in the range of U, say, Ux. Since

$$||Vx - Vx_n|| \leq \frac{1+\lambda_1}{1-\lambda_2}||Ux - Ux_n||$$

we obtain that $Vx_n \to y = Vx$, so V has closed range. In a similar way one can show that U has dense range iff V has dense range, that U is a quotient map iff V is a quotient map, and so on.

Our frame applications concerns an even more general condition than (1). Let $U : \mathcal{X} \to \mathcal{Y}$ be a bounded operator, $\mathcal{X}_0$ a dense subspace of $\mathcal{X}$, and $V : \mathcal{X} \to \mathcal{Y}$ a linear mapping. If

$$(2) \quad ||Ux - Vx|| \leq \lambda_1||Ux|| + \lambda_2||Vx|| + \mu||x||, \ \forall x \in \mathcal{X}_0,$$

where $\lambda_2 \in [0, 1[$, then V extends continously to an operator on $\mathcal{X}$ satisfying (2) on $\mathcal{X}$. This is a consequence of the triangle inequality: (2) yields $||Vx|| \leq ||Ux - Vx|| + ||Ux|| \leq (1 + \lambda_1)||Ux|| + \lambda_2||Vx|| + \mu||x||$ for all $x \in \mathcal{X}_0$, so

$$||Vx|| \leq \frac{1+\lambda_1}{1-\lambda_2}||Ux|| + \frac{\mu}{1-\lambda_2}||x||.$$

Now, V is a bounded linear operator on a dense subspace of $\mathcal{X}$ and hence has a unique extension to a bounded linear operator (of the same norm) on $\mathcal{X}$.

2 Applications to frame theory.

Let us begin with a short introduction to the part of frame theory which we need. For more information about general frame theory we refer to [HW, Y] , and for information about the special topics we discuss here we refer to [H2 , C1,C2,CC1,CC2].

Let $\mathcal{H}$ be a separable Hilbert space with inner product $< \cdot, \cdot >$ linear in the first entry. A family of elements $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *Bessel sequence* if

$$\exists B > 0 : \quad \sum_{i=1}^{\infty} | < f, f_i > |^2 \leq B \cdot ||f||^2, \ \forall f \in \mathcal{H}.$$

A Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is a *frame* if

$$\exists A > 0 : \quad A \cdot ||f||^2 \leq \sum_{i=1}^{\infty} | < f, f_i > |^2, \ \forall f \in \mathcal{H}.$$

If $\{f_i\}_{i=1}^{\infty}$ is a frame (or just a Bessel sequence) one can define a bounded operator

$$U : \mathcal{H} \to \ell^2(N), \ Uf = \{< f, f_i >\}_{1=1}^{\infty}.$$

The adjoint operator is

$$T : \ell^2(\mathbf{N}) \to \mathcal{H}, \ T\{c_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} c_i f_i.$$

One can show that the so called *frame operator* $S := TU$ is invertible if $\{f_i\}_{i=1}^{\infty}$ is a frame. This leads to the important *frame decomposition*:

$$f = SS^{-1}f = \sum_{i=1}^{\infty} < f, S^{-1}f_i > f_i, \ \forall f \in \mathcal{H}.$$

A *Riesz basis* is a family of the form $\{f_i\}_{i=1}^{\infty} = \{Te_i\}_{i=1}^{\infty}$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$. It is not difficult to prove that

$$\{f_i\}_{i=1}^{\infty} \text{ is a Riesz basis}$$

$$\Updownarrow$$

$$\{f_i\}_{i=1}^{\infty} \text{ is a frame and } \sum_{i=1}^{\infty} c_i f_i = 0, \{c_i\}_{i=1}^{\infty} \in \ell^2(N) \Rightarrow c_i = 0, \ \forall i.$$

In applications of frames it is important to know how far a frame is from being a Riesz basis. One approach is due to Holub [H2], who defines a *near-Riesz basis* as a frame consisting of a Riesz basis plus finitely many elements.

A second can be found in [C1], where a *Riesz frame* is defined as a frame, where every subfamily is a frame for its closed span, with bounds A, B which are common to all those frames. Riesz frames share many properties with Riesz bases, and they always contain a Riesz basis as a subfamily, cf. [C1, CC1].

In this section we demonstrate how Lemma 1.1 leads to a significant improvement of a result previously published by the second named author. This former result states the following, cf. [C2]: let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. If there exist $\lambda_1, \mu \geq 0$ such that $\lambda_1 + \frac{\mu}{\sqrt{A}} < 1$ and

$$(2) \quad ||\sum_{i=1}^{n} c_i(f_i - g_i)|| \leq \lambda_1 ||\sum_{i=1}^{n} c_i f_i|| + \mu [\sum_{i=1}^{n} |c_i|^2]^{1/2}$$

for all finite sequences $c_1, ..c_n$ $(n \in N)$, then $\{g_i\}_{i=1}^{\infty}$ is a frame.

As mentioned in [C2], this result is the best possible, in the sense that one can construct examples where the conclusion fails already if (2) is only satisfied for $\lambda_1 = 1, \mu = 0$ (or $\lambda_1 = 0, \mu = \sqrt{A}$). From this point of view it is very surprising, that we can prove Theorem 2.1 below, stating that we can actually add a whole term on the right hand side of (2) for free:

Theorem 2.1: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B.Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and assume that there exist constants $\lambda_1, \lambda_2, \mu \geq 0$ such that $max(\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1$ and*

$$(3) \quad ||\sum_{i=1}^{n} c_i(f_i - g_i)|| \leq \lambda_1 ||\sum_{i=1}^{n} c_i f_i|| + \lambda_2 ||\sum_{i=1}^{n} c_i g_i|| + \mu [\sum_{i=1}^{n} |c_i|^2]^{1/2}$$

for all $c_1, ..c_n$ $(n \in N)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds

$$A\left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2}\right)^2, \quad B\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2}\right)^2.$$

Proof: $\{f_i\}_{i=1}^{\infty}$ is a frame, so we can define a bounded linear operator

$$T : \ell^2(N) \to \mathcal{H}, \quad T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i.$$

Furthermore $||T|| \le \sqrt{B}$. As in the last remark of section 1, the condition (3) implies that

$$||\sum_{i=1}^{n} c_i g_i|| \le \frac{1+\lambda_1}{1-\lambda_2}||\sum_{i=1}^{n} c_i f_i|| + \frac{\mu}{1-\lambda_2}[\sum_{i=1}^{n} |c_i|^2]^{1/2}$$

for all finite sequences, and we can define a bounded operator

$$U : \ell^2(N) \to \mathcal{H}, \quad U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i.$$

Also, (3) holds for all sequences in $\ell^2(N)$, and

$$||U\{c_i\}_{i=1}^{\infty}|| \le \frac{1+\lambda_1}{1-\lambda_2}||T\{c_i\}_{i=1}^{\infty}|| + \frac{\mu}{1-\lambda_2}||\{c_i\}_{i=1}^{\infty}||$$

$$\le \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}||\{c_i\}_{i=1}^{\infty}||, \quad \forall\{c_i\}_{i=1}^{\infty} \in \ell^2(N).$$

This estimate shows that $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence with the upper bound

$$(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2})^2 = B(1+\frac{\lambda_1+\lambda_2+\frac{\mu}{\sqrt{B}}}{1-\lambda_2})^2$$

Now we prove that $\{g_i\}_{i=1}^{\infty}$ has a lower frame bound. Since $\{f_i\}_{i=1}^{\infty}$ is a frame, the frame operator $S = TT^*$ is invertible, and we can define

$$T^{\dagger} : \mathcal{H} \to \ell^2(N), \quad T^{\dagger}f := T^*(TT^*)^{-1}f = \{< f, (TT^*)^{-1}f_i >\}_{i=1}^{\infty}.$$

$\{(TT^*)^{-1}f_i\}_{i=1}^{\infty}$ is the dual frame of $\{f_i\}_{i=1}^{\infty}$, so $||T^{\dagger}f|| \le \frac{1}{\sqrt{A}}||f||$, $\forall f \in \mathcal{H}$. Using (3) on the sequence $\{c_i\}_{i=1}^{\infty} = T^{\dagger}f$ we obtain that

$$||f - UT^{\dagger}f|| \le \lambda_1||f|| + \lambda_2||UT^{\dagger}f|| + \mu||T^{\dagger}f||$$

$$\le (\lambda_1 + \frac{\mu}{\sqrt{A}})||f|| + \lambda_2||UT^{\dagger}f||, \ \forall f \in \mathcal{H}.$$

By Lemma 1.1, the operator $UT^{\dagger}$ is invertible, and

$$||UT^{\dagger}|| \le \frac{1+\lambda_1+\frac{\mu}{\sqrt{A}}}{1-\lambda_2}, \quad ||(UT^{\dagger})^{-1}|| \le \frac{1+\lambda_2}{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}.$$

Every $f \in \mathcal{H}$ can be written as

$$f = UT^{\dagger}(UT^{\dagger})^{-1}f = \sum_{i=1}^{\infty} < (UT^{\dagger})^{-1}f, (TT^*)^{-1}f_i > g_i$$

implying that

$$||f||^4 =< f, f >^2 = |\sum_{i=1}^{\infty} < (UT^{\dagger})^{-1}f, (TT^*)^{-1}f_i >< g_i, f > |^2$$

$$\leq \sum_{i=1}^{\infty} | < (UT^{\dagger})^{-1}f, (TT^*)^{-1}f_i > |^2 \cdot \sum_{i=1}^{\infty} | < g_i, f > |^2$$

$$\leq \frac{1}{A}||(UT^{\dagger})^{-1}f||^2 \cdot \sum_{i=1}^{\infty} | < g_i, f > |^2$$

$$\leq \frac{1}{A}[\frac{1+\lambda_2}{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}]^2 \cdot ||f||^2 \cdot \sum_{i=1}^{\infty} | < g_i, f > |^2, \ \forall f \in \mathcal{H}.$$

So

$$\sum_{i=1}^{\infty} | < g_i, f > |^2 \geq A(\frac{1-(\lambda_1+\frac{\mu}{\sqrt{A}})}{1+\lambda_2})^2 ||f||^2$$

$$= A(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1+\lambda_2})||f||^2, \ \forall f \in \mathcal{H}.$$

Q.E.D.

Remark: For readers familar with operator theory we mention that the operator $T^{\dagger}$ from the proof of Theorem 2.1 is in fact the *pseudo-inverse* of T. By stressing this point we are able to show a stronger version of the theorem in section 3. The reason for stating Theorem 2.1 in the way we have done it is that this form is more applicable to other areas of research - especially to wavelets. Already the "old version" (with $\lambda_2 = 0$) has been shown to be very useful in connection with wavelet frames and Weyl-Heisenberg frames, cf. [FZ].

In the case $\mu = 0$ there is a long tradition for studying the condition (3). Pollard ([P], see also [B, p.488]) showed that completeness of $\{f_i\}_{i=1}^{\infty}$ implies completeness of $\{g_i\}_{i=1}^{\infty}$, but only for the case $max(\lambda_1, \lambda_2) < \frac{1}{\sqrt{2}}$. Hilding ([H1]) relaxed the condition to $max(\lambda_1, \lambda_2) < 1$. From our point of view the problem with the case $\mu = 0$ is that the condition

$$||\sum_{i=1}^{n} c_i(f_i - g_i|| \leq \lambda_1 ||\sum_{i=1}^{n} c_i f_i|| + \lambda_2 ||\sum_{i=1}^{n} c_i g_i||, \ \forall \{c_i\}_{i=1}^{n}$$

(still with $max(\lambda_1, \lambda_2) < 1$) implies that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same linear dependence in the sense that

$$\sum_{i=1}^{\infty} c_i f_i = 0 \Leftrightarrow \sum_{i=1}^{\infty} c_i g_i = 0.$$

This is a very inconvenient restriction in the context of frames.

Corollary 2.2: *If $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis and the condition in Theorem 2.1 is satisfied, then $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis.*

Proof: Let A denote a lower frame bound for $\{f_i\}_{i=1}^{\infty}$. Then $||\sum_{i=1}^{\infty} c_i f_i|| \geq \sqrt{A}||\{c_i\}_{i=1}^{\infty}||, \ \forall \{c_i\}_{i=1}^{\infty} \in \ell^2(N)$. Given $\{c_i\}_{i=1}^{\infty} \in \ell^2(N)$,

$$||\sum_{i=1}^{\infty} c_i g_i|| \geq ||\sum_{i=1}^{\infty} c_i f_i|| - ||\sum_{i=1}^{\infty} c_i (f_i - g_i)||$$

$$\geq ||\sum_{i=1}^{\infty} c_i f_i|| - \lambda_1 ||\sum_{i=1}^{\infty} c_i f_i|| - \lambda_2 ||\sum_{i=1}^{\infty} c_i g_i|| - \mu ||\{c_i\}_{i=1}^{\infty}||$$

$$\geq (1 - \lambda_1)||\sum_{i=1}^{\infty} c_i f_i|| - \lambda_2 ||\sum_{i=1}^{\infty} c_i g_i|| - \mu ||\{c_i\}_{i=1}^{\infty}||$$

$$\geq ((1 - \lambda_1)\sqrt{A} - \mu)||\{c_i\}_{i=1}^{\infty}|| - \lambda_2 ||\sum_{i=1}^{\infty} c_i g_i||.$$

So

$$||\sum_{i=1}^{\infty} c_i g_i|| \geq \frac{1 - (\lambda_1 + \frac{\mu}{\sqrt{A}})}{1 + \lambda_2} \sqrt{A} ||\{c_i\}_{i=1}^{\infty}||.$$

So $\sum_{i=1}^{\infty} c_i g_i = 0 \Rightarrow c_i = 0, \forall i$. **Q.E.D.**

Our result in example 1.2 (that $\{e_i + \frac{1}{i} e_{1+1}\}_{i=1}^{\infty}$ is a Riesz basis) could also have been obtained using Corollary 2.2. We leave the details to the reader.

We now turn the attension to the question about excess of frames. First we extend Theorem 12 from [CC2] to the setting where the λ_2-term is present:

Theorem 2.3: *The assumptions in Theorem 2.1 implies that*

$$\{f_i\}_{i=1}^{\infty} \text{ is a near-Riesz basis} \Leftrightarrow \{g_i\}_{i=1}^{\infty} \text{ is a near-Riesz basis}$$

in which case $\{f_i\}_{i=1}^{\infty}$ *and* $\{g_i\}_{i=1}^{\infty}$ *have the same excess.*

Proof: Choose $\epsilon > 0$ such that $\lambda_1 + \frac{\mu}{\sqrt{A-\epsilon}} < 1$. By [CC2, Prop. 11], there exists a number m such that $\{f_i\}_{i=m}^{\infty}$ is a Riesz basis for $\overline{span}\{f_i\}_{i=m}^{\infty}$, with lower bound $A-\epsilon$. Define an operator

$$T : span\{f_i\}_{i=m}^{\infty} \to span\{g_i\}_{i=m}^{\infty}, \quad Tf_i := g_i, \quad i = m, m+1, ...$$

extended by linearity. Every $f \in span\{f_i\}_{i=m}^{\infty}$ has a unique representation as $f = \sum_{i=m}^{\infty} c_i f_i$ where only finitely many of the coefficients $\{c_i\}_{i=m}^{\infty}$ are nonzero, and

$$||f - Tf|| = ||\sum_{i=m}^{\infty} c_i(f_i - g_i)||$$

$$\leq \lambda_1 \cdot ||\sum_{i=m}^{\infty} c_i f_i|| + \lambda_2 \cdot ||\sum_{i=m}^{\infty} c_i g_i|| + \mu \cdot ||\{c_i\}_{i=m}^{\infty}||$$

$$\leq (\lambda_1 + \frac{\mu}{\sqrt{A-\epsilon}})||\sum_{i=m}^{\infty} c_i f_i|| + \lambda_2 \cdot ||\sum_{i=m}^{\infty} c_i g_i||$$

$$= (\lambda_1 + \frac{\mu}{\sqrt{A-\epsilon}})||f|| + \lambda_2 \cdot ||Tf||.$$

So T extends to a bounded operator from $\overline{span}\{f_i\}_{i=m}^{\infty}$ into $\overline{span}\{g_i\}_{i=m}^{\infty}$, which we again call T. The above extimates still holds, and as in the proof of Lemma 1.1, it follows that

$$\frac{1 - (\lambda_1 + \frac{\mu}{\sqrt{A-\epsilon}})}{1 + \lambda_2}||f|| \leq ||Tf|| \leq \frac{1 + \lambda_1 + \frac{\mu}{\sqrt{A-\epsilon}}}{1 - \lambda_2}||f||, \ \forall f \in \overline{span}\{f_i\}_{i=m}^{\infty}.$$

T has closed range, so $T(\overline{span}\{f_i\}_{i=m}^{\infty}) = \overline{span}\{g_i\}_{i=m}^{\infty}$. By [CK], $codim(\overline{span}\{f_i\}_{i=m}^{\infty}) = codim(\overline{span}\{g_i\}_{i=m}^{\infty})$, that is, we need to add the same numbers of element to $\{f_i\}_{i=m}^{\infty}$, resp. $\{g_i\}_{i=m}^{\infty}$ to get Riesz bases for $\mathcal{H}$, from which it follows that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess. **Q.E.D.**

Remark: It is important that Theorem 2.3 is restricted to the case of finite excess. In fact, as shown in [CC2] there exist examples where the perturbation condition is satisfied and $\{f_i\}_{i=1}^{\infty}$ has infinite excess, but where $\{g_i\}_{i=1}^{\infty}$ does not contains a Riesz basis. As we show now, this can only happen if the μ-term is present:

Proposition 2.4: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame containing a Riesz basis and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. If there exist constants $\lambda_1, \lambda_2 \in [0;1[$ such that*

$$||\sum_{i=1}^{n} c_i(f_i - g_i)|| \leq \lambda_1 ||\sum_{i=1}^{n} c_i f_i|| + \lambda_2 ||\sum_{i=1}^{n} c_i g_i||$$

for all $c_1, ..c_n (n \in N)$, then $\{g_i\}_{i=1}^{\infty}$ is a frame containing a Riesz basis. Furthermore $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess.

Proof: Choose $I \subseteq N$ such that $\{f_i\}_{i \in I}$ is a Riesz basis. Then $\{g_i\}_{i \in I}$ is a Riesz basis by Corollary 2.2, from which the result follows. **Q.E.D.**

For Riesz frames the problem mentioned can not occur:

Proposition 2.5: *Let $\{f_i\}_{i=1}^{\infty}$ be a Riesz frame with lower bound A, and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ satisfy the hypotheses of Theorem 2.1. Then $\{g_i\}_{i=1}^{\infty}$ contains a Riesz basis and $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}$ have the same excess.*

Proof: By [C2] the frame $\{f_i\}_{i=1}^{\infty}$ contains a Riesz basis $\{f_i\}_{i \in I}$ with lower bound A. By Corollary 2.2 , the corresponding family $\{g_i\}_{i \in I}$ is a Riesz basis. **Q.E.D.**

Remark: For families $\{g_i\}_{i=1}^{\infty}$ which are known to be Bessel sequences, the results presented here can be slightly extended. The reason is, that in this case the operator

$$U : \ell^2(N) \to \mathcal{H} \ , \ \ U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i$$

is bounded, so we are allowed to work with Corollary 1.4 instead of Lemma 1.1. So in this case we can take $\lambda_2 = 1$.

Although the results in Section 1 are stated in a Banach space, we have restricted us to the Hilbert space setting in the present section, to make the theory direct applicable. But in fact, as shown by Gröchenig the notion of a frame can be extended to Banach space. Corresponding to a Banach space $\mathcal{X}$ we denote the dual space by $\mathcal{X}'$.

Definition: Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ an associated Banach space of scalar-valued sequences. Let $\{y_i\}_{i=1}^{\infty} \subseteq \mathcal{X}'$ and $T : \mathcal{X}_d \to \mathcal{X}$ be a bounded linear operator. If

(a) $\{< x, y_i >\}_{i=1}^{\infty} \in \mathcal{X}_d, \ \forall x \in \mathcal{X}$

(b) $\exists A, B > 0 : A||x|| \leq ||\{< x, y_i >\}_{i=1}^{\infty}|| \leq B||x||, \forall x \in \mathcal{X}$

(c) $T\{< x, y_i >\}_{i=1}^{\infty} = x, \ \forall x \in \mathcal{X}$,

then $(\{y_i\}_{i=1}^{\infty}, T)$ is a *Banach frame* for $\mathcal{X}$ with respect to $\mathcal{X}_d.A, B$ are called *frame bounds*. Note, that in the case of a frame in a Hilbert space, they are the squareroots of the usual frame bounds. The analogue of Theorem 2.1 now reads

Theorem 2.6: *Suppose that $(\{y_i\}_{i=1}^{\infty}, T)$ is a Banach frame for X with respect to $\mathcal{X}_d$. Denote the corresponding bounds by A, B. Consider an operator $S : \mathcal{X}_d \to \mathcal{X}$ and suppose that there exist $\lambda_1, \lambda_2, \mu \geq 0$ such that $max(\lambda_2, \lambda_1 + \mu B) < 1$ and*

$$||T\{c_i\}_{i=1}^{\infty} - S\{c_i\}_{i=1}^{\infty}|| \leq \lambda_1||T\{c_i\}_{i=1}^{\infty}|| + \lambda_2||S\{c_i\}_{i=1}^{\infty}|| + \mu||\{c_i\}_{i=1}^{\infty}||, \ \forall\{c_i\}_{i=1}^{\infty} \in \mathcal{X}_d.$$

Then there exists a sequence $\{z_i\}_{i=1}^{\infty} \in \mathcal{X}'$ such that $(\{z_i\}_{i=1}^{\infty}, S)$ is a Banach frame for $\mathcal{X}$ w.r.t. $\mathcal{X}_d$, with bounds $A\frac{1-\lambda_2}{1+\lambda_1+\mu B}, B\frac{1+\lambda_2}{1-(\lambda_1+\mu B)}$.

Proof: We check the conditions in the definition. Let $x \in \mathcal{X}$. We use the assumption on the sequence $\{c_i\}_{i=1}^{\infty} = \{< x, y_i >\}_{i=1}^{\infty}$:

$$||x - S\{< x, y_i >\}_{i=1}^{\infty}|| \leq \lambda_1||x|| + \lambda_2||S\{< x, y_i >\}_{i=1}^{\infty}|| + \mu B||x||.$$

By our Lemma 1.1, the mapping $Lx := S\{< x, y_i >\}_{i=1}^{\infty}$ is an isomorphism of $\mathcal{X}$ onto $\mathcal{X}$, and

$$\frac{1 - \lambda_2}{1 + \lambda_1 + \mu B}||x|| \leq ||L^{-1}x|| \leq \frac{1 + \lambda_2}{1 - (\lambda_1 + \mu B)}||x||.$$

So

$$x = LL^{-1}x = S\{< L^{-1}x, y_i >\}_{i=1}^{\infty}, \ \forall x \in \mathcal{X}.$$

The mapping $x \longmapsto < L^{-1}x, y_i >$ is an element of $\mathcal{X}'$, which we call z_i. Clearly $\{< x, z_i >\}_{i=1}^{\infty} = \{< L^{-1}x, y_i >\}_{i=1}^{\infty} \in \mathcal{X}_d$, and $x = S\{< x, z_i >\}_{i=1}^{\infty}, \ \forall x \in \mathcal{X}$.

Finally,

$$||\{< x, z_i >\}_{i=1}^{\infty}|| = ||\{< L^{-1}x, y_i >\}_{i=1}^{\infty}|| \geq A||L^{-1}x|| \geq A\frac{1-\lambda_2}{1+\lambda_1+\mu B}||x||$$

and

$$||\{< x, z_i >\}_{i=1}^{\infty}|| \leq B||L^{-1}x|| \leq B\frac{1+\lambda_2}{1-(\lambda_1+\mu B)}||x||, \ \forall x \in \mathcal{X}.$$

Q.E.D.

For more results about perturbation of Banach frames we refer to [CH].

3 Extension of frame theory.

The main feature of a frame $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is that every element in $\mathcal{H}$ can be represented as a linear combination of the frame elements. But there exist natural examples of families $\{f_i\}_{i=1}^{\infty}$ which have this property without beeing a frame. A theory for such families has been developed in [C3]. The aim of the present section is to show how the results from section 1 can be used in this setting.

Let us briefly recall the situation from [C3]. Given a family $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ we consider the -maybe unbounded- operator

$$T : \mathcal{D}(T) \subseteq \ell^2(N) \to \mathcal{H}, \ \ T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i,$$

where the domain $\mathcal{D}(T) = \{\{c_i\}_{i=1}^{\infty} \in \ell^2(N) \mid \sum_{i=1}^{\infty} c_i f_i \text{ converges}\}$. We assume that T is closed and surjective. Then there exists a unique operator $T^\dagger : \mathcal{H} \to \mathcal{D}(T) \subseteq \ell^2(N)$ such that

$$N_{T^\dagger} = 0 \ (= {R_T}^\perp), \ \ \overline{R_{T^\dagger}} = {N_T}^\perp \text{ and } TT^\dagger f = f, \ \forall f \in \mathcal{H}.$$

$T^\dagger$ is called the *pseudo-inverse* of T. As shown in [C3, Th. 4.1] , $T^\dagger$ has the form $T^\dagger f = \{< f, h_i >\}_{i=1}^\infty$, where $\{h_i\}_{i=1}^\infty$ is a Bessel sequence in $\mathcal{H}$. In particular, T is bounded, and since $N_{T^\dagger} = 0$, $T^\dagger$ is injective. The *generalized frame decomposition* is s immediate consequence of th definition of $T^\dagger$ and the form it has:

$$f = TT^\dagger f = \sum_{i=1}^\infty < f, h_i > f_i, \quad \forall f \in \mathcal{H}.$$

It is not difficult so show that the theory described here can be used for any Schauder basis satisfying the extra condition that the expansion coefficients are square summable. Now let $\{g_i\}_{i=1}^\infty \subseteq \mathcal{H}$. We assume that $\sum_{i=1}^\infty c_i g_i$ is convergent for all $\{c_i\}_{i=1}^\infty \in \mathcal{D}(T)$ and define the operator

$$U : \mathcal{D}(U) := \mathcal{D}(T) \to \mathcal{H}, \quad U\{c_i\}_{i=1}^\infty := \sum_{i=1}^\infty c_i g_i.$$

In the present setting, the question is to find conditions such that U is closed and surjective. As we shall see now, the analog of Theorem 2.1 is true:

Theorem 3.1: *Assume that T is closed and surjective. Let $\{g_i\}_{i=1}^\infty \subseteq \mathcal{H}$ and define the corresponding operator U as above. Suppose that there exist $\lambda_1, \lambda_2, \mu \geq 0$ such that $max(\lambda_1 + \mu \cdot ||T^\dagger||, \lambda_2) < 1$ and*

$$||T\{c_i\}_{i=1}^\infty - U\{c_i\}_{i=1}^\infty|| \leq \lambda_1 ||T\{c_i\}_{i=1}^\infty|| + \lambda_2 ||U\{c_i\}_{i=1}^\infty|| + \mu ||\{c_i\}_{i=1}^\infty||, \ \forall \{c_i\}_{i=1}^\infty \in \mathcal{D}(T).$$

Then U is closed and surjective, and

$$\sum_{i=1}^\infty |< g_i, f >|^2 \geq \left(\frac{1 - (\lambda_1 + \mu ||T^\dagger||)}{1 + \lambda_2}\right)^2 \frac{1}{||T^\dagger||^2} ||f||^2, \ \forall f \in \mathcal{H}.$$

Proof: By [K, p.191], U is closed. Observe that we can use the assumption on $\{c_i\}_{i=1}^\infty := T^\dagger f$ since $R_{T^\dagger} \subseteq \mathcal{D}(U)$. We obtain that

$$||f - UT^\dagger f|| \leq \lambda_1 ||f|| + \lambda_2 ||UT^\dagger f|| + \mu \cdot ||T^\dagger|| \cdot ||f||, \ \forall f \in \mathcal{H}.$$

So $UT^\dagger$ is invertible and therefore surjective. Also, by Lemma 1.1

$$||(UT^\dagger)^{-1}|| \leq \frac{1 + \lambda_2}{1 - (\lambda_1 + \mu \cdot ||T^\dagger||)}, \quad ||UT^\dagger|| \leq \frac{1 + \lambda_1 + \mu \cdot ||T^\dagger||}{1 - \lambda_2}.$$

This leads to

$$||f||^4 = | < UT^\dagger (UT^\dagger)^{-1} f, f > |^2$$

$$\leq ||T^\dagger (UT^\dagger)^{-1} f||^2 \sum_{i=1}^{\infty} | < g_i, f > |^2$$

$$\leq ||T^\dagger||^2 \cdot ||f||^2 (\frac{1+\lambda_2}{1-(\lambda_1 + \mu \cdot ||T^\dagger||)})^2 \sum_{i=1}^{\infty} | < g_i, f > |^2, \quad \forall f \in \mathcal{H}.$$

from which the estimate for the lower bound follows. **Q.E.D.**

Since $UT^\dagger (UT^\dagger)^{-1} = I$, it is natural to ask whether the operator $T^\dagger (UT^\dagger)^{-1}$ is the pseudo inverse of U? It is immediate, that

$$N_{T^\dagger (UT^\dagger)^{-1}} = \{ f \in \mathcal{H} \mid (UT^\dagger)^{-1} f \in N_{T^\dagger} = {R_T}^\perp = \{0\} \} = \{0\} = {R_U}^\perp.$$

and that

$$\overline{R_{T^\dagger (UT^\dagger)^{-1}}} = \overline{R_{T^\dagger}} = {N_T}^\perp.$$

So $U^\dagger = T^\dagger (UT^\dagger)^{-1}$ if and only if $\overline{R_{U^\dagger}} = {N_T}^\perp$, i.e., iff $N_T = N_U$.

In applications of the generalized frame theory, the most difficult point is to show the closedness of the operator T. Here we present a few results related to this question. For later convenience (and since the proof does not use the Hilbert space structure) we state the main result in the Banach space setting:

Theorem 3.2: *Let $\mathcal{X}$ be a Banach space, and let $\mathcal{X}_d$ be a Banach sequence space of scalars. We assume that convergence in $\mathcal{X}_d$ implies coordinatewise convergence, and that $\mathcal{X}_d$ contains the canonical unit vectors. Then, given a family of elements $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{X}$, (1) and (2) below are equivalent:*

(1) There exists a number m such that $\{f_i\}_{i=m}^{\infty}$ is a basis for its closed span.

(2) For every choice of numbers $\alpha_i \in R$, the operator

$$T_\alpha : \mathcal{D}(T_\alpha) := \{ \{c_i\}_{i=1}^{\infty} \in \mathcal{X}_d \mid \sum_{i=1}^{\infty} c_i \alpha_i f_i \ \ \textit{converges} \} \to \mathcal{X}, \ \ T_\alpha \{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i \alpha_i f_i$$

is closed.

Proof: First assume that (1) is satisfied. From

$\{c_i^n\}_{i=1}^\infty \in \mathcal{D}(T_\alpha), \;\; ||\{c_i^n\}_{i=1}^\infty - \{c_i\}_{i=1}^\infty|| \to 0, \;\; ||T\{c_i^n\}_{i=1}^\infty - g|| \to 0$ for $n \to \infty$

we want to conclude that $\{c_i\}_{i=1}^\infty \in \mathcal{D}(T_\alpha)$ and that $T\{c_i\}_{i=1}^\infty = g$. Choose m such that $\{f_i\}_{i=m}^\infty$ is a basis for its closed span. Then

$$\sum_{i=m}^{\infty} \alpha_i c_i^n f_i \to g - \sum_{i=1}^{m-1} \alpha_i c_i f_i \text{ for } n \to \infty.$$

For suitable coefficients $d_i, \; i = m, m+1, ..$ we have $g - \sum_{i=1}^{m-1} \alpha_i c_i f_i = \sum_{i=m}^\infty d_i f_i$, and $\alpha_i c_i^n \to d_i$. It follows that $d_i = \alpha_i c_i, i = m, m+1, ..,$ so $\{0, 0, ...c_m, c_{m+1}, ...\} \in \mathcal{D}(T_\alpha)$. It follows that $\{c_i\}_{i=1}^\infty \in \mathcal{D}(T_\alpha)$ and that $T_\alpha\{c_i\}_{i=1}^\infty = g$.

Now we prove (2) $\Rightarrow$ (1). Suppose the result fails,i.e., $\{f_i\}_{i=m}^\infty$ is not a basic sequence for any choice of m. By induction there exist numbers

$$n_0 = 0 < m_1 < n_1 < m_2 < n_2 ...$$

and a sequence $\{c_i\}_{i=1}^\infty$ such that

$$(5) \qquad \forall j: \quad ||\sum_{i=n_j+1}^{m_{j+1}} c_i f_i|| = 1$$

and

$$(6) \qquad \forall j: \quad ||\sum_{i=n_j+1}^{n_{j+1}} c_i f_i|| \leq \frac{1}{2^{j+2}}.$$

We only give the induction step, since the first case follows from the same argument. So assume that we have choosen $\{n_k\}_{k=1}^j$ and $\{m_k\}_{k=1}^j$ satisfying (5) and (6). By [LT, prop.1.a.3] there exist natural numbers m_{j+1} and n_{j+1} and scalars $\{d_i\}_{i=n_j+1}^{n_{j+1}}$ such that $n_j < m_{j+1} < n_{j+1}$ and

$$||\sum_{i=n_j+1}^{m_{j+1}} d_i f_i|| \geq 2^{j+2} ||\sum_{i=n_j+1}^{n_{j+1}} d_i f_i||.$$

If we divide this inequality by $||\sum_{i=n_j+1}^{m_{j+1}} d_i f_i||$ and relable the resulting scalars as c_i, we get the result.
Now, let $\{e_i\}_{i=1}^{\infty}$ denote the unit vector basis of $\mathcal{X}_d$ and choose

$$\alpha_j = 2^j||\sum_{i=n_j+1}^{n_{j+1}} c_i e_i||$$

and consider the family

$$\{\alpha_0 f_1, ..., \alpha_0 f_{n_1}, \alpha_1 f_{n_1+1}, .., \alpha_1 f_{n_2}, ...\} = \{f_i'\}_{i=1}^{\infty}$$

and let

$$T' : \mathcal{D}(T') \to \mathcal{X}, \quad T'\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i'.$$

We now show that T' is not closed. For $k \in N$, define

$$x_k := \{\frac{c_1}{\alpha_0}, ...\frac{c_{n_1}}{\alpha_0}, \frac{c_{n_1+1}}{\alpha_1}, ..., \frac{c_{n_2}}{\alpha_1}, .., \frac{c_{n_k+1}}{\alpha_k}, \frac{c_{n_{k+1}}}{\alpha_k}, 0, 0...\} \in \mathcal{X}_d.$$

$\{x_k\}$ is a Cauchy sequence: if $k < l$ then

$$||x_l - x_k|| \leq \sum_{i=k+1}^{l} ||\{0, 0, ..., \frac{c_{n_i+1}}{\alpha_i}, ..., \frac{c_{n_{i+1}}}{\alpha_i}, 0, 0, , , \}||$$

$$\leq \sum_{i=k+1}^{l} \frac{1}{2^i} \leq \frac{1}{2^k} \to 0 \text{ as } k \to \infty.$$

So by assumption,

$$x_k \to x := \{\frac{c_1}{\alpha_0}, ..., \frac{c_{n_1}}{\alpha_0}, \frac{c_{n_1+1}}{\alpha_1}, ..., \frac{c_{n_2}}{\alpha_1}, .., \frac{c_{n_k+1}}{\alpha_k}, \frac{c_{n_{k+1}}}{\alpha_k}, ...\} \text{ for } k \to \infty.$$

The sequence $\{T'x_k\}_{k=1}^{\infty}$ is convergent. To prove this we only need to observe, that if $k \leq l$, then

$$||\sum_{i=k}^{l} \sum_{j=n_i+1}^{n_{i+1}} \frac{c_j}{\alpha_i} \alpha_i f_j|| \leq \sum_{i=k}^{\infty} ||\sum_{j=n_i+1}^{n_{i+1}} c_j f_j||$$

$$\leq \sum_{i=k}^{\infty} \frac{1}{2^{i+2}} = \frac{1}{2^{k+1}} \to 0 \text{ as } k \to \infty.$$

Now we finish the proof by showing that x is not in $\mathcal{D}(T')$ For each k,

$$||\sum_{i=n_k+1}^{m_k} \frac{c_i}{\alpha_k}\alpha_k f_i|| = ||\sum_{i=n_k+1}^{m_k} c_i f_i|| = 1.$$

Therefore the series

$$\text{“}T'\{x\}\text{“} = \frac{c_1}{\alpha_0}\alpha_0 f_1 + \frac{c_2}{\alpha_0}\alpha_0 f_2 + ...$$

is not a Cauchy sequence in $\mathcal{X}$, hence x is not in $\mathcal{D}(T')$. **Q.E.D.**

In a Hilbert space, Theorem 3.2 can be interpreted as a statement about near-Riesz bases:

Corollary 3.3: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for the Hilbert space $\mathcal{H}$. Then (1) and (2) below are equivalent:*

(1) $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis .

(2) For every choice of numbers $\alpha_i \in R$, the operator

$$T_\alpha : \mathcal{D}(T_\alpha) := \{\{c_i\}_{i=1}^{\infty} \in \ell^2(N) \mid \sum_{i=1}^{\infty} c_i\alpha_i f_i \ \ \textit{converges}\} \rightarrow \mathcal{H},\ \ T_\alpha\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i\alpha_i f_i$$

is closed.

Proof: We only need to prove (2) $\Rightarrow$ (1). Choose m such that $\{f_i\}_{i=m}^{\infty}$ is a basis for its closed span. By [C4], $\{f_i\}_{i=m}^{\infty}$ is at the same time a frame for its closed span, and therefore a Riesz basis for its closed span. **Q.E.D.**

Corollary 3.4: *Let $\{f_i\}_{i=1}^{\infty}$ be a near-Riesz basis. Then, for every choice of numbers $\alpha_i$such that $\alpha_i \geq \epsilon$ for some $\epsilon > 0$, the operator*

$$T_\alpha : \mathcal{D}(T_\alpha) := \{\{c_i\}_{i=1}^{\infty} \in \ell^2(N) \mid \sum_{i=1}^{\infty} c_i\alpha_i f_i \ \ \textit{converges}\} \rightarrow \mathcal{H},\ \ T_\alpha\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i\alpha_i f_i$$

is closed and surjective.

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