Perturbations and Irregular Sampling Theorems for Frames

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***Abstract*— This paper gives a perturbation theorem for frames in a Hilbert space which is a generalization of a result by Casazza and Christensen. Then this result is applied to the Perturbation of regular sampling in shift-invariant spaces. Irregular sampling theorems for frames in wavelet subspaces are established for which it is easy to derive explicit formulas and algorithms to calculate the ranges of the perturbations for frames. Furthermore, improved estimates for the Perturbations are derived for the Perturbations of regular sampling in shift-invariant spaces.**

***Index Terms*— Riesz basis , frame , sampling , irregular sampling, perturbation, wavelets.**

I. Introduction

IN digital signal and image processing, digital communication, etc., a continuous signal is usually represented and processed by using its discrete samples. For a band-limited signal, the classical Shannon sampling theorem provides an exact representation by its uniform samples with a sampling rate higher than its Nyquist rate. The Shannon sampling theorem is one of the most powerful results in signal analysis. This classical theorem has broad application in signal processing and communication theory and has been generalized to many other forms. Recently it has also been extended to wavelet subspaces in [1], [2], [3]. Irregular sampling is also useful in practice. A reconstruction from more general sets of points is necessary if the measurements cannot be made at uniform points $\{n\}$. In [2], [4] and [5], irregular sampling theorems in translation invariant subspaces were proved. The authors of [6], [7] studied an algorithm to treat the Perturbation of regular sampling in wavelet spaces assuming φ is a stable generator, i.e., $\{\varphi(\cdot - k)\}_k$ is a Riesz basis. We will show that the assumptions in these papers on the values $\{\delta_n\}$ of deviation from the integral points can be weakened.

In [8], [9] and [10] perturbations about frames have been studied. Our objective in this paper is to find a generalization of several perturbation results for frames. We find that the result of Casazza and Christensen becomes a corollary of this result. By applying this result to the Perturbations of regular sampling in shift-invariant spaces, irregular sampling theorems for frames in general wavelet subspaces are established. Based on these results, improved estimates for the Perturbations are derived. The derived estimates are easy to calculate.

The following are some notations used in this paper. The Fourier transform of $f \in L^2(R)$ is defined by $\hat{f}(\omega) = \int_R f(t)e^{-it\omega}dt$. Let

$$\langle f, g\rangle = \int_R f(t)g(t)dt$$

$$\| f \| = \sqrt{\langle f, f\rangle}$$

$$\| f \|_* = (\int_0^{2\pi} |f(t)|^2 dt)^{1/2}.$$

II. Perturbation Theorems for Frames

Let $\mathbb{H}$ be a Hilbert space and $\{f_i\}_{i\in\mathbb{Z}}$ be a sequence in $\mathbb{H}$. We say that $\{f_i\}_{i\in\mathbb{Z}}$ is a *frame* for $\mathbb{H}$ with *lower frame bound* A and *upper frame bound* B if for all $f \in \mathbb{H}$ we have:

$$A\|f\|^2 \le \sum_{i\in\mathbb{Z}} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

If only the right-hand inequality is satisfied for all $f \in \mathbb{H}$, then $\{f_i\}_{i\in\mathbb{Z}}$ is called a *Bessel sequence* with bound B. We define the *synthesis operator* $T : \ell^2(\mathbb{Z}) \to \mathbb{H}$ by:

$$T\{c_i\} = \sum_i c_i f_i.$$

The conjugate operator $T^* : \mathbb{H} \to \ell^2(\mathbb{Z})$ of T is called the *analysis operator* and satisfies:

$$T^* f = \{\langle f, f_i\rangle\}_{i\in\mathbb{Z}}.$$

If $\{f_i\}$ is a frame for $\mathbb{H}$, the operator $S = TT^*$ is a bounded, positive, self-adjoint invertible operator on $\mathbb{H}$ called the *frame operator* and satisfies

$$Sf = \sum_i \langle f, f_i\rangle f_i.$$

In this case, $\{S^{-1}f_i\}$ is called the *canonical dual frame* for $\{f_i\}$. The canonical dual frame gives the reconstruction formula

$$f = \sum_i \langle f, S^{-1}f_i\rangle f_i = \sum_i \langle f, f_i\rangle S^{-1}f_i.$$

If $\{g_i\}$ is another sequence in $\mathbb{H}$, define an operator L by

$$Lf = \sum_i \langle f, f_i\rangle g_i.$$

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*Lemma*1: If L is a bounded operator on $\mathbb{H}$ then

$$L^*f = \sum_i \langle f, g_i \rangle f_i, \quad \text{for all } f \in \mathbb{H}.$$

Proof: For all $f, g \in \mathbb{H}$ we have

$$\langle g, L^*f \rangle = \langle Lg, f \rangle = \langle \sum_i \langle g, f_i \rangle g_i, f \rangle$$

$$= \sum_i \langle g, f_i \rangle \langle g_i, f \rangle = \langle g, \sum_i \langle f, g_i \rangle f_i \rangle. \qquad \square$$

Theorem 1: Let $\{f_i\}_{i\in\mathbb{Z}}$ be a frame for a Hilbert space $\mathbb{H}$ with frame bounds A, B and let $\{g_i\}_{i\in\mathbb{Z}}$ be a sequence in $\mathbb{H}$. If the operator L is invertible on $\mathbb{H}$, then $\{g_i\}_{i\in\mathbb{Z}}$ has lower frame bound $\frac{1}{B\|L^{-1}\|^2}$. Hence, if $\{g_i\}$ is a Bessel sequence then $\{g_i\}$ is a frame.

Proof: For any $f \in \mathbb{H}$ we have

$$\frac{1}{\|L^{-1}\|}\|f\| \le \|L^*f\|$$

$$= \quad \|\sum_i \langle f, g_i \rangle f_i\| \le \sqrt{B}\left(\sum_i |\langle f, g_i \rangle|^2\right)^{1/2}.$$

Hence, $\{g_i\}_i$ has the the prescribed lower frame bound. $\square$

We note that the assumption that $\{g_i\}$ is Bessel sequence is necessary for us to conclude that $\{g_i\}$ is a frame.

*Example*1: Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal basis for $\mathbb{H}$ and define:

$$f_{2i} = e_i, \qquad f_{2i-1} = 0,$$

and

$$g_{2i} = e_i, \qquad g_{2i-1} = ie_i.$$

Then $\{f_i\}$ is a frame for $\mathbb{H}$ while $\{g_i\}$ does not have a finite upper frame bound and hence is not a frame for $\mathbb{H}$. But it is immediate that

$$f = Lf = \sum_{i\in\mathbb{Z}} \langle f, f_i \rangle g_i, \quad \text{for all } f \in \mathbb{H},$$

and so L is an invertible operator on $\mathbb{H}$.

The main perturbation theorem in [10] now follows from Theorem 1.

Theorem 2: Let $\{f_i\}_{i\in\mathbb{Z}}$ be a frame for a Hilbert space $\mathbb{H}$ with frame bounds A and B. Let $\{g_i\}_{i\in\mathbb{Z}}$ be a sequence in $\mathbb{H}$ and assume there exist constants $\lambda_1, \lambda_2, \mu \ge 0$ such that $\max(\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2) < 1$ and

$$\|\sum_i c_i(f_i - g_i)\|$$

$$\le \quad \lambda_1\|\sum_i c_i f_i\| + \lambda_2\|\sum_i c_i g_i\| + \mu\left[\sum_i |c_i|^2\right]^{1/2}$$

for all $\{c_i\}_i$. Then $\{g_i\}_i$ is a frame for $\mathbb{H}$ with frame bounds

$$A\left(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2}\right)^2, \quad B\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 - \lambda_2}\right)^2.$$

Proof: Let T (respectively, U) be the synthesis operator for $\{f_i\}_i$ (respectively, $\{g_i\}$). We compute:

$$\|U\{c_i\}\| \le \|\sum_i c_i f_i\| + \|\sum_i c_i(f_i - g_i)\| \le$$

$$\|\sum_i c_i f_i\| + \lambda_1\|\sum_i c_i f_i\| + \lambda_2\|\sum_i c_i g_i\| + \mu\|\{c_i\}\|.$$

Hence,

$$\|U\{c_i\}\| \le \frac{1+\lambda_1}{1-\lambda_2}\|T\{c_i\}\| + \frac{\mu}{1-\lambda_2}\|\{c_i\}\| \le$$

$$\frac{(1+\lambda_1)\sqrt{B} + \mu}{1-\lambda_2}\|\{c_i\}\|.$$

It follows that $\{g_i\}$ is a Bessel sequence with Bessel bound

$$\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2 = B\left(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2}\right)^2.$$

Let S be the frame operator for $\{f_i\}_i$. Let $T^+f = T^*S^{-1}f$. So $TT^+f = f$ and

$$UT^+f = \sum_i \langle f, S^{-1}f_i \rangle g_i.$$

For $f \in \mathbb{H}$ let $\{c_i\} = T^+f$. We compute

$$\|f - UT^+f\| \le \lambda_1\|f\| + \lambda_2\|UT^+f\| + \frac{\mu}{\sqrt{A}}\|f\|$$

$$= \left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)\|f\| + \lambda_2\|UT^+f\|.$$

By Lemma 1 of [10] it follows that UT^+ is invertible and

$$\|(UT^+)^{-1}\| \le \frac{1+\lambda_2}{1 - \left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)}.$$

Since $\{S^{-1}f_i\}$ is a frame for $\mathbb{H}$, an application of Theorem 1 shows that $\{g_i\}$ is a frame for $\mathbb{H}$. $\square$

Corollary 1 is a generalization of a classical result of Paley and Wiener concerning perturbations of Riesz bases.

III. Irregular Sampling Theorems

We now introduce some notations used in the following. We write

$$E_\varphi = \{\omega \in R | \sum_{n=-\infty}^{+\infty} |\hat{\varphi}(\omega + 2n\pi)|^2 > 0\}.$$

$$G_\varphi(\omega) = (\sum_{n=-\infty}^{+\infty} |\hat{\varphi}(\omega + 2n\pi)|^2)^{1/2}.$$

$$q(t,s) = \sum_k \varphi(t-k)\tilde{\varphi}(s-k).$$

where

$$\hat{\tilde{\varphi}}(\omega) = \begin{cases} \frac{\hat{\varphi}(\omega)}{\sum_n |\hat{\varphi}(\omega+2n\pi)|^2}, & \omega \in E_\varphi \\ 0, & \omega \notin E_\varphi \end{cases}$$

$Z_\varphi(x,\omega) = \sum_n \varphi(x+n)e^{-in\omega}$ is the Zak transform of φ. $\hat{\varphi}^*(\omega) = Z_\varphi(0,\omega)$.

$$\chi_E = \begin{cases} 1, & t \in E \\ 0, & \text{otherwise} \end{cases}$$

In this section, let T (respectively, U) be the synthesis operator for $\{q(k,\cdot)\}_k$ (respectively, $\{q(k+\delta_k,\cdot)\}_k$). Let

$$Sf = TT^*f = \sum_k \langle f, q(k,\cdot) \rangle q(k,\cdot).$$

We consider the shift-invariant subspace $V(\varphi)$ generated by $\varphi(t)$

$$V(\varphi)=\{\sum_{k\in R}c_k\varphi(t-k)|\{c_k\}_{k\in Z}\in l^2\}\subset L^2(R).$$

First, we give a necessary and sufficient condition on the samples $\{f(n+\delta_n)\}_n$.

Theorem 3: Let $V(\varphi)$ be a closed subspace of $L^2(R)$ and $\{\varphi(\cdot-n)|n\in\mathbb{Z}\}$ be a frame for $V(\varphi)$ with bounds A and B. Suppose that φ is continuous, $\sup_{x\in R}\sum_{n\in\mathbb{Z}}|\varphi(x+n)|^2<+\infty$, and that there exist two constants $c_1,c_2>0$ such that

$$c_1\chi_{E_\omega}(\omega)\le|\hat{\varphi}^*(\omega)|\le c_2\chi_{E_\omega}(\omega),\quad \text{a.e.}\quad \omega\in R$$

Suppose that there exist a sequence $\{\delta_k\}_{k\in\mathbb{Z}}$ such that operator

$$Lf=\sum_n\langle f,S^{-1}q(n,\cdot)\rangle q(n+\delta_n,\cdot).$$

is invertible, and $\{q(n+\delta_n,\cdot)\}_n$ is Bessel sequence if and only if there are constants $C,D>0$ such that

$$C\sum_n|f(n)|^2\le\sum_n|f(n+\delta_n)|^2\le D\sum_n|f(n)|^2 \qquad (1)$$

holds for any $f\in V(\varphi)$.

Proof: Let

$$q(t,s)=\sum_k\varphi(t-k)\tilde{\varphi}(s-k).$$

We know that $\{q(n,\cdot)\}=\{q(0,\cdot-n)\}$ is a frame for $V(\varphi)$ with bounds $\frac{c_1^2}{B}$ and $\frac{c_2^2}{A}$(reference [4]). On the other hand, for any $t\in R$

$$\sum_k\varphi(t-k)\tilde{\varphi}(s-k)\quad\text{converges to}\quad q(t,\cdot)$$

in $L^2(R)$. Since $\{\tilde{\varphi}(\cdot-n)\}$ is the canonical dual frame for $\{\varphi(\cdot-n)\}$, we have that $\langle f,q(t,\cdot)\rangle=f(t)$.

Necessity. The operator

$$Lf=\sum_n\langle f,S^{-1}q(n,\cdot)\rangle q(n+\delta_n,\cdot).$$

is invertible, $\{q(n+\delta_n,\cdot)\}_n$ is Bessel sequence. $\{S^{-1}q(n,\cdot)\}_n$ is the canonical dual frame for $\{q(n,\cdot)\}_n$, By Theorem 1, we have that $\{q(n+\delta_n,\cdot)\}_n$ is a frame that there exist two constants $M_1,M_2>0$ such that

$$M_1\parallel f\parallel^2\le\sum_n\langle f,q(n+\delta_n,\cdot)\rangle^2\le M_2\parallel f\parallel^2 .$$

holds for any $f\in V(\varphi)$. Then

$$M_1\parallel f\parallel^2\le\sum_n|f(n+\delta_n)|^2\le M_2\parallel f\parallel^2 .$$

holds for any $f\in V(\varphi)$. Since $\{q(n,\cdot)\}=\{q(0,\cdot-n)\}$ is a frame for $V(\varphi)$, we have that

$$\frac{c_1^2}{B}\parallel f\parallel^2\le\sum_n|f(n)|^2\le\frac{c_2^2}{A}\parallel f\parallel^2 .$$

holds for any $f\in V(\varphi)$. Then

$$M_1\frac{A}{c_2^2}\sum_n|f(n)|^2\le\sum_n|f(n+\delta_n)|^2\le M_2\frac{B}{c_1^2}\sum_n|f(n)|^2$$

holds for any $f\in V(\varphi)$.

Sufficiency. Since

$$C\sum_n|f(n)|^2\le\sum_n|f(n+\delta_n)|^2\le D\sum_n|f(n)|^2$$

holds for any $f\in V(\varphi)$ and $\{q(n,\cdot)\}=\{q(0,\cdot-n)\}$ is a frame for $V(\varphi)$, we have that

$$\frac{c_1^2}{B}\parallel f\parallel^2\le\sum_n|f(n)|^2\le\frac{c_2^2}{A}\parallel f\parallel^2 .$$

holds for any $f\in V(\varphi)$. Then

$$C\frac{c_1^2}{B}\parallel f\parallel^2\le\sum_n\langle f,q(n+\delta_n,\cdot)\rangle^2$$
$$=\quad\sum_n|f(n+\delta_n)|^2\le D\frac{c_2^2}{A}\parallel f\parallel^2 .$$

holds for any $f\in V(\varphi)$. So $\{q(n+\delta_n,\cdot)\}_n$ is Bessel sequence. By Lemma 1,

$$L^*f=\sum_n f(n+\delta_n)S^{-1}q(n,\cdot).$$

$\{S^{-1}q(n,\cdot)\}_n$ is the canonical dual frame for $\{q(n,\cdot)\}_n$, with bounds $\frac{A}{c_2^2}$ and $\frac{B}{c_1^2}$. Then

$$\parallel L^*f\parallel^2\ge\frac{A}{c_2^2}\sum_n|f(n+\delta_n)|^2\ge C\frac{A}{c_2^2}\frac{c_1^2}{B}\parallel f\parallel^2 .$$

So

$$\parallel Lf\parallel^2\ge C\frac{A}{c_2^2}\frac{c_1^2}{B}\parallel f\parallel^2 .$$

Hence, the operator Lf is invertible. □

We now look for algorithms to reconstruct a continuous signal $f\in V(\varphi)$ by using its discrete samples $\{f(n+\delta_n)\}_n$. Obviously the samples cannot be arbitrary; that is, some constraints should be imposed on the sampling points $\{n+\delta_n\}_n$ or the samples $\{f(n+\delta_n)\}_n$. The weaker the constraints are the better the reconstruction method is evaluated. For irregular sampling on more general wavelet subspaces, we have the following theorem.

Theorem 4: Let $V(\varphi)$ be a closed subspace of $L^2(R)$ and $\{\varphi(\cdot-n)|n\in\mathbb{Z}\}$ be a frame for $V(\varphi)$ with bounds A and B. Suppose that φ is continuous, $\sup_{x\in R}\sum_{n\in\mathbb{Z}}|\varphi(x+n)|^2<+\infty$, and that there exist two constants $c_1,c_2>0$ such that

$$c_1\chi_{E_\omega}(\omega)\le|\hat{\varphi}^*(\omega)|\le c_2\chi_{E_\omega}(\omega),\quad \text{a.e.}\quad \omega\in R$$

Suppose that there exist a sequence $\{\delta_k\}_{k\in\mathbb{Z}}$ and two constants $C,D>0$ such that

$$C\sum_n|f(n)|^2\le\sum_n|f(n+\delta_n)|^2\le D\sum_n|f(n)|^2$$

holds for any $f\in V(\varphi)$. then there exists a frame $\{S_n\}_{n\in\mathbb{Z}}$ for $V(\varphi)$ such that for any $f\in V(\varphi)$

$$f(t)=\sum_n f(n+\delta_n)S_n(t) \qquad (2)$$

where the convergence is both in $L^2(R)$ and uniform on R.

Proof: $\{S^{-1}q(n,\cdot)\}_n$ is the canonical dual frame for $\{q(n,\cdot)\}_n$. By Theorem 3 and Theorem 1, Hence $\{q(n+$

$\delta_n, \cdot)\}_n$ is a frame for V_φ. Due to Theorem 1 of [2], there is a dual frame $\{S_n(t)\}$ of $\{q(n+\delta_n, \cdot)\}_n$ such that

$$f(t) = \sum_n \langle f, q(n+\delta_n)\rangle S_n(t) = \sum_n f(n+\delta_n)S_n(t)$$

for any $f \in V_\varphi$. □

Our objective is to find explicit formulas and algorithms to calculate the ranges of the perturbations for frames. So we now discuss a special case of the irregular sampling Theorem.

Theorem 5: Let $V(\varphi)$ be a closed subspace of $L^2(R)$ and $\{\varphi(\cdot - n)|n \in \mathbb{Z}\}$ be a frame for $V(\varphi)$ with bounds A and B. Suppose that φ is continuous, $\sup_{x\in R}\sum_{n\in\mathbb{Z}} |\varphi(x+n)|^2 < +\infty$, and that there exist two constants $c_1, c_2 > 0$ such that

$$c_1\chi_{E_\omega}(\omega) \le |\hat{\varphi}^*(\omega)| \le c_2\chi_{E_\omega}(\omega), \quad \text{a.e.} \quad \omega \in R$$

Suppose that there exist a sequence $\{\delta_k\}_{k\in\mathbb{Z}}$ and there is a positive number $\theta < 1$ such that

$$\sum_n |f(n+\delta_n) - f(n)|^2 \le \theta^2 \sum_n |f(n)|^2 \qquad (3)$$

holds for any $f \in V(\varphi)$. then there exists a frame $\{S_n\}_{n\in\mathbb{Z}}$ for $V(\varphi)$ such that for any $f \in V(\varphi)$

$$f(t) = \sum_n f(n+\delta_n)S_n(t)$$

where the convergence is both in $L^2(R)$ and uniform on R.

Proof: If

$$\sum_n |f(n+\delta_n) - f(n)|^2 \le \theta^2 \sum_n |f(n)|^2$$

holds for any $f \in V(\varphi)$, then

$$(1-\theta)^2\sum_n |f(n)|^2 \le \sum_n |f(n+\delta_n)|^2 \le (1+\theta)^2\sum_n |f(n)|^2.$$

By Theorem 4, Theorem 5 is shown. □

IV. Irregular Sampling Theorems and Algorithms for Wavelet Subspaces

In order to establish the algorithm for perturbations of regular sampling in shift-invariant spaces, we need to introduce the function class $L^\lambda_\sigma[a,b]$ $(\lambda > 0, \sigma \in [0,1), 0 \in [a,b] \subset [-1,1])$ defined and used in [6] and [7].

Definition 1: $L^\lambda_\sigma[a,b](\lambda > 0, \sigma \in [0,1)$ and $0 \in [a,b] \subset [-1,1])$ consists of all the measurable functions f, for which the norm

$$\begin{aligned} & \| f \|_{L^\lambda_\sigma[a,b]} \\ = \ & \sup_{\{r_k\}_k\subset[a,b]} \frac{\sum_k |f(k+\sigma+r_k) - f(k+\sigma)|}{\sup_k |r_k|^\lambda} < \infty. \end{aligned}$$

Theorem 6: Let $V(\varphi)$ be a closed subspace of $L^2(R)$ and $\{\varphi(\cdot - n)|n \in \mathbb{Z}\}$ be a frame for $V(\varphi)$ with bounds A and B. Suppose that φ is continuous, $\sup_{x\in R}\sum_{n\in\mathbb{Z}} |\varphi(x+n)|^2 < +\infty$, $\varphi \in L^\lambda_0[a,b]$ and that there exist two constants $C_1, C_2 > 0$ such that

$$C_1\chi_{E_\omega}(\omega) \le |\hat{\varphi}^*(\omega)| \le C_2\chi_{E_\omega}(\omega), \quad \text{a.e.} \quad \omega \in R$$

Then for any $\{r_n\}_n \subset [-r_\varphi, r_\varphi]\bigcap[a,b]$, there is a frame $\{S_n\}_n$ for $V(\varphi)$ such that $f = \sum_n f(n+r_n)S_n$ holds in $L^2(R)$ for every $f \in V(\varphi)$ if

$$r_\varphi < \left(\frac{C_1}{\| \varphi \|_{L^\lambda_0[a,b]}}\right)^{1/\lambda}. \qquad (4)$$

Proof: For any $f \in V(\varphi)$,there is a scalar sequence $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ holds in $L^2(R)$. Let $t_k = k + r_k$

$$\begin{aligned} \triangle &= \sum_k |f(t_k) - f(k)|^2 \\ &= \sum_k |\sum_l c_l(\varphi(t_k - l) - \varphi(k-l))|^2 \\ &= \sum_n \sum_{k,l} c_k c_l(\varphi(t_n - k) - \varphi(n-k)) \\ &\quad \times(\varphi(t_n - l) - \varphi(n-l)) \\ &= \sum_{k,l} c_k c_l \sum_n(\varphi(t_n - k) - \varphi(n-k)) \\ &\quad \times(\varphi(t_n - l) - \varphi(n-l)). \end{aligned}$$

Take

$$a_{k,l} = \sum_n(\varphi(t_n - k) - \varphi(n-k))(\varphi(t_n - l) - \varphi(n-l)).$$

Then $a_{k,l} = a_{l,k}$ holds for any $k, l \in \mathbb{Z}$. Following the argument in [6], we have

$$\triangle = \sum_{k,l} a_{kl}c_kc_l \le (\sum_k c_k^2)\text{sup}_k \sum_l |a_{kl}|$$

and

$$\text{sup}_k \sum_l |a_{kl}| \le (\| \varphi \|_{L^\lambda_0[a,b]} \ \text{sup}_\beta |r_\beta|^\lambda)^2.$$

Hence

$$\triangle \le (\sum_k c_k^2)(\| \varphi \|_{L^\lambda_0[a,b]} \ \text{sup}_\beta |r_\beta|^\lambda)^2.$$

Since

$$\begin{aligned} \sum_k |f(k)|^2 &= \frac{1}{2\pi} \| \sum_k \sum_l c_l\varphi(k-l)e^{-ik\omega} \|_*^2 \\ &= \frac{1}{2\pi} \| \hat{\varphi}^* \sum_k c_k e^{-ik\omega} \|_*^2 \\ &\ge C_1^2 \sum_k c_k^2. \end{aligned}$$

By Theorem 5, we only need to show

$$(\| \varphi \|_{L^\lambda_0[a,b]} \ \text{sup}_\beta |r_\beta|^\lambda)^2 \le \theta^2 C_1^2.$$

But this is the assumption in the theorem. So Theorem 6 is shown. □

If $\varphi \notin L^\lambda_0[a,b]$,we use another algorithm to calculate the ranges of the perturbations for a frame.

Theorem 7: Let $V(\varphi)$ be a closed subspace of $L^2(R)$ and $\{\varphi(\cdot - n)|n \in \mathbb{Z}\}$ be a frame for $V(\varphi)$ with bounds A and B. Suppose that φ is continuous, $\sup_{x\in R}\sum_{n\in\mathbb{Z}} |\varphi(x+n)|^2 < +\infty$, and that there exist two constants $C_1, C_2 > 0$ such that

$$C_1\chi_{E_\omega}(\omega) \le |\hat{\varphi}^*(\omega)| \le C_2\chi_{E_\omega}(\omega), \quad \text{a.e.} \quad \omega \in R$$

Moreover, suppose that $\{\varphi'(\cdot - n)|n \in \mathbb{Z}\}$ is a Bassel sequence with bound M. Suppose that $\delta > 0$ and $\delta\lceil 2\delta\rceil < C_1^2/M$. Then, for any sequence $\{\delta_n\}$ with $\sup_n|\delta_n| \leq \delta$, there exists a frame $\{S_n\}$ for $V(\varphi)$ such that for any $f \in V(\varphi)$

$$f(t) = \sum_n f(n + \delta_n)S_n(t)$$

where the convergence is both in $L^2(R)$ and uniform on R and $\lceil x\rceil$ is the smallest integer that is larger than or equal to x.

Proof: For any $f \in V(\varphi)$, there is a scalar sequence $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ holds in $L^2(R)$. Following the argument in [4], we have

$$\sum_n |f(n + \delta_n) - f(n)|^2 \leq \delta M\lceil 2\delta\rceil \sum_n |c_n|^2.$$

Recalling the proof of Theorem 6, we know

$$C_1^2 \sum_k |c_k|^2 \leq \sum_k |f(k)|^2.$$

Therefore, by Theorem 5, we only need to show that for some $0 < \theta < 1$ we have

$$\delta M\lceil 2\delta\rceil \sum_n |c_n|^2 \leq \theta^2 C_1^2 \sum_n |c_n|^2$$

That is, $\delta M\lceil 2\delta\rceil \leq \theta^2 C_1^2$. Again, this inequality is our assumption in the theorem which completes the proof. □

Compared to the estimate $\delta\lceil 2\delta\rceil < AC_1^2/BM$ derived in [4], since lower frame bound A and upper frame bound B satisfies $A \leq B$, the present result does improve the result of [4].

V. Applications of the Algorithms

Example 2 (see [12]): Consider the B-spline of order 1 scaling function

$$N_1(t) = t\chi_{[0,1)} + (2 - t)\chi_{[1,2)}.$$

In this case, $\hat{N}_1^* = e^{i\omega}$, $G_{N_1}(\omega) = (1/3 + 2/3\cos^2(\omega/2))^{1/2}$, $C_1 = C_2 = 1$. $A = \frac{1}{3}$ and $B = 1$. Since $\| N_1 \|_{L_0^1[-1,1]} = 3$, by Theorem 6, we derive an estimate $r_{N_1} < 1/3$.

*Example*3: Consider the Gaussian kernel $K = 1/\sqrt{2\pi}e^{-t^2/2}$. In this case, $\hat{K} = e^{-\omega^2/2}$ and

$$G_K^2 = \sum_k e^{-(\omega+2k\pi)^2} = e^{-\omega^2} + 2\sum_{k=1}^{\infty} e^{-(\omega+2k\pi)^2}.$$

For any $\omega \in [0, 2\pi]$, we derive

$$G_K^2 \geq e^{-4\pi^2} + 2\int_2^{\infty} e^{-x^2}dx = e^{-4\pi^2} + \sqrt{2\pi}/e^2$$

and

$$G_K^2 \leq 1 + 2\int_0^{\infty} e^{-x^2}dx = 1 + \sqrt{2\pi}.$$

By the Poisson summation formula, we know that

$$\hat{K}^* = 1/\sqrt{2\pi}\sum_k e^{-(\omega+2k\pi)^2/2}.$$

Then we have $e^{-2\pi}/\sqrt{2\pi} + 2/e \leq \hat{K}^* \leq 1/\sqrt{2\pi} + 2$ and

$$\| K \|_{L_0^1[-1,1]} \leq 1/\sqrt{2\pi}(\sqrt{2}e^{-1/4} + 4e^{-1/2}).$$

By Theorem 6, we derive an estimate

$$r_K < (e^{1/2-2\pi} + 2e^{-1/2}\sqrt{2\pi})/(\sqrt{2}e^{1/4} + 4) \approx 0.35.$$

*Example*4: Let $\hat{\varphi}(\omega) = \chi_{[-2a\pi,2a\pi)}(\omega)$, $0 < a < \frac{1}{2}$. Then

$$G_\varphi(\omega) = \chi_{[-2a\pi,2a\pi)}(\omega) \text{ on } [-\pi, \pi).$$

Since $\hat{\varphi}^*(\omega) = \sum_k \hat{\varphi}(\omega + 2k\pi)$ in $L^2(0, 2\pi)$ (see reference [14]), we derive

$$\hat{\varphi}^*(\omega)\chi_{[-\pi,\pi)} = \chi_{[-2a\pi,2a\pi)}(\omega).$$

Then $C_1 = C_2 = 1$. It is easy to verify that $\{\varphi(\cdot - n)|n \in Z\}$ is a frame for $V_0(\varphi)$ (see reference [15]). By reference [14], $\{\varphi(\cdot - n)|n \in Z\}$ is not a Riesz basis for the subspace $V_0(\varphi)$. Since $\hat{f}'(\omega) = i\omega\hat{f}(\omega)$, $M = \sup_\omega \sum_k |\hat{\varphi'}(\omega + 2k\pi)|^2 = (2a\pi)^2$, $0 < a < \frac{1}{2}$. By Theorem 7, we derive an estimate

$$\sup_n|\delta_n| < \frac{1}{(2a\pi)^2}.$$

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