A Physical Interpretation for Finite Tight Frames

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Abstract

Though finite tight frames arise in many applications, they have often proved difficult to understand and construct. We investigate the nonlinear problem of finding a tight frame for which the lengths of the frame elements have been prescribed in advance. Borrowing several ideas from Classical Mechanics, we show that this problem has a natural, intuitive interpretation. In particular, we show that such frames may be characterized as the minimizers of a potential energy function, and justify their interpretation as "maximally orthogonal" sequences. By exploiting this idea, we are able to show that such frames always exist, provided the requisite lengths satisfy a "fundamental inequality." In so doing, we characterize those sequences of nonnegative numbers which arise as the lengths of a tight frame's elements.

1 Introduction

For over a century, orthonormal bases have proved invaluable to mathematicians, scientists, and engineers, by allowing us to decompose vectors with ease. In particular, if $\{e_n\}_{n=1}^{N}$ is an orthonormal basis for a Hilbert space $\mathbb{H}_N$, we

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have the *Parseval-Plancherel identity*,

$$f = \sum_{n=1}^{N} \langle f, e_n \rangle e_n, \tag{1}$$

for all $f \in \mathbb{H}_N$. However, in many applications, orthonormality is never explicitly needed. Rather, what we truly care about is the decomposition formula (1), and orthonormality is only a means to an end.

The theory of frames is an attempt to understand decompositions similar to (1), but without having to prescribe orthogonality *a priori*. To be precise, we say that a sequence of M vectors $\{f_m\}_{m=1}^M$ within an N-dimensional Hilbert space $\mathbb{H}_N$ is a *tight frame* for $\mathbb{H}_N$ provided there exists an $A > 0$ such that,

$$f = \frac{1}{A} \sum_{m=1}^{M} \langle f, f_m \rangle f_m, \tag{2}$$

for all $f \in \mathbb{H}_N$. As we shall see in the following section, tight frames are a nontrivial generalization of orthonormal bases, even in finite dimensions.

In recent years, great progress has been made in the understanding and implementation of tight frames, some of which is described in detail below. Nevertheless, many fundamental and important questions about tight frames remain unanswered.

For instance, note that though the definition of an orthonormal basis requires all vectors to be of unit length, the definition of a tight frame makes no such *a priori* assumption. However, in the next section, we shall explicitly demonstrate how (2) contains an implicit restriction upon the lengths of a tight frame's elements. One therefore asks the question:

> Given positive integers M and N, for what sequences $\{a_m\}_{m=1}^M \subset [0, \infty)$ do there exist tight frames $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$, such that $\|f_m\| = a_m$ for all m?

We answer this question completely. In particular, we shall demonstrate that such a tight frame exists if and only if the sequence $\{a_m\}_{m=1}^M$ satisfies the *fundamental inequality*,

$$\max_{m=1,\ldots,M} a_m^2 \leq \frac{1}{N} \sum_{m=1}^{M} a_m^2.$$

Furthermore, when this inequality is violated, we determine the vector sequences of such norms which are as "close" to being tight frames as possible, in a natural, intuitive sense.

Of course, this problem does not exist in a vacuum. Frames have been a subject of interest for many years, both in theory and in applications. In recent years, several inquiries have been made into some of the deeper issues of finite tight frames.

The theory of frames was first introduced by Duffin and Schaeffer [13] in the 1950's, furthering the study of nonharmonic Fourier series and the time-frequency decompositions of Gabor [17]. Decades later, the subject was reinvigorated following a publication of Daubechies, Grossman and Meyer [12]. Frames have subsequently evolved into a state-of-the-art signal processing tool.

There are many excellent sources for those interested in the mathematics of frames, beginning with the original work of Duffin and Schaeffer [13], which introduces frames for arbitrary Hilbert spaces. In addition, particular classes of frames related to time-frequency and time-scale decompositions have also been extensively studied. In particular, Gabor (Weyl-Heisenberg) frames, are described by Heil and Walnut [24], while Daubechies [10,11] offers excellent introductions to both wavelet and Gabor frames.

The theory of frames provides both great liberties in the design of vector space decompositions, as well as quantitative measures on the computability and robustness of the corresponding reconstructions. As such, frames are increasingly popular in both pure and applied contexts, appearing in many places, and in many guises.

Frames are resilient against the corruptions of additive noise and quantization, while providing numerically stable reconstructions [8,11,20]. Perfect reconstruction oversampled filter banks have been extensively studied [5,9,27,30], and are equivalent to translation-invariant frames in $\ell^2(\mathbb{Z})$. In general, appropriate frame decompositions may reveal "hidden" signal characteristics, and have been employed as detection devices [2,4,33]. Though the Naimark [1] Theorem [1] has been applied to frame theory for several years [19,23], researchers have only recently begun to exploit this result to parse several results concerning quantum measurement in terms of frames [16,29]. Frames have also been used to design unitary space-time constellations for multiple-antenna wireless systems [25].

Specific types of finite tight frames have been studied to solve problems in information theory [26,32,36]. In addition, many techniques of constructing finite tight frames have been discovered, several of which involve group theory [34]. Researchers have also been interested in tight frames whose elements are restricted to spheres and ellipsoids [3,14], as well as the manifold structures of spaces of all such frames [15].

Many of our results followed from our study of John Benedetto and Matt

[1] A famous Russian mathematician, M.A. Naimark is often referred to by the alternative spelling *Neumark*.

Fickus' work on characterizing unit norm tight frames [3]. After making our discoveries, we subsequently learned that some of the results that we present here have been obtained independently, albeit in a very different setting, and with a completely different rationale.

To be precise, in December of 2002, Jelena Kovačević presented our results at the Department of Electrical Engineering at the University of Illinois at Urbana-Champaign. In discussions with Pramod Viswanath, they came to the understanding that our results are related to his work with Anantharam on optimal sequences and sum capacity of synchronous CDMA systems in the context of wireless communications [35]. Near the end of this paper, we will provide greater details on the relationship between the two approaches, and our belief that this work in a very practical setting is a validation of our results.

We begin in Section 2 with the basic theory of finite tight frames. In Section 3, we motivate our main results by introducing a physical interpretation of frame theory, extending the frame-equivalent notions of force and potential energy first introduced by Benedetto and Fickus. Section 4 contains several results concerning the minimization of this generalized frame potential, which serve to highlight the connection between optimal energy and tightness. These ideas culminate in Section 5 with the establishment of a fundamental inequality on the lengths of a tight frame's elements, and the physical interpretation thereof.

2 Finite Tight Frames

We begin our investigation with a brief introduction to the main ideas of the general theory of frames. By restricting ourselves to finite sequences of vectors within finite-dimensional spaces, much of this basic theory reduces to well-known concepts from numerical linear algebra. Working within this simple setting allows us to focus all of our energy upon the nonlinear, nontrivial problem of finding tight frames of predetermined lengths. We then conclude this section with the derivation of a "fundamental inequality" upon the lengths of a tight frame's elements. The remaining sections are dedicated to proving that this inequality indeed characterizes these lengths, and understanding what happens when the inequality is violated.

2.1 The rudiments of the theory of frames

Let $\{f_m\}_{m\in\mathcal{I}}$ be a sequence of vectors within a Hilbert space $\mathbb{H}$, where $\mathcal{I}$ is any countable indexing set. Intuitively, any vector $f \in \mathbb{H}$ may be "decomposed" in terms of $\{f_m\}_{m\in\mathcal{I}}$ by applying the corresponding *analysis operator*,

$$F : H \to \ell(\mathcal{I}), \quad (Ff)(m) = \langle f, f_m\rangle,$$

to f, where $\ell(\mathcal{I})$ is the space of all complex sequences indexed by $\mathcal{I}$. In order to recover arbitrary vectors $f \in \mathbb{H}$ from their decompositions $\{\langle f, f_m\rangle\}_{m\in\mathcal{I}}$, F, at the very least, needs to be one-to-one.

For this recovery process to be numerically stable, we require the analysis to be continuous. To be precise, $\{f_m\}_{m\in\mathcal{I}}$ is a *Bessel sequence* if there exists $B < \infty$ such that,

$$\sum_{m\in\mathcal{I}} |\langle f, f_m\rangle|^2 \le B\|f\|^2,$$

for all $f \in \mathbb{H}$. Under this assumption, F is a bounded operator into $\ell^2(\mathcal{I})$, and therefore possesses a corresponding adjoint *synthesis* operator,

$$F^* : \ell^2(\mathcal{I}) \to \mathbb{H}, \quad F^*g = \sum_{m\in\mathcal{I}} g(m) f_m.$$

Having this operator, we may reconstruct f from Ff by first synthesizing the coefficients into F^*Ff, and then inverting the *frame operator*,

$$F^*F : \mathbb{H} \to \mathbb{H}, \quad F^*F = \sum_{m\in\mathcal{I}} \langle f, f_m\rangle f_m.$$

This method of reconstruction is more stable numerically than any attempt to recover f from Ff directly. In particular, if numerical error perturbs Ff into a sequence g which is no longer within the range of F, then no direct solution of $Ff = g$ will exist. However, the process described above will nevertheless "reconstruct" f in the least-squares sense. That is, f is a solution to $F^*Ff = F^*g$ if and only if f is the minimizer of $\|Ff - g\|$.

In order to guarantee a unique, numerically stable solution to this least-squares problem, we place an assumption upon $\{f_m\}_{m\in\mathcal{I}}$ which is equivalent to the continuous invertibility of the frame operator. Specifically, a sequence $\{f_m\}_{m\in\mathcal{I}}$ is a *frame* for H if there exists constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{m\in\mathcal{I}} |\langle f, f_m\rangle|^2 \le B\|f\|^2, \tag{3}$$

for all $f \in \mathbb{H}$. Any such constants A and B are *lower* and *upper frame bounds* for $\{f_m\}$, respectively [2] . In such a situation, the corresponding frame operator satisfies $AI \le F^*F \le BI$, and is therefore a topological isomorphism from $\mathbb{H}$ onto itself. Thus, any $f \in \mathbb{H}$ may then be uniquely reconstructed from

[2] Often, the optimal such constants A and B are referred to as *the* lower and *the* upper frame bounds, respectively.

$\{\langle f, f_m \rangle\}_{m\in\mathcal{I}}$ in the least-squares sense,

$$f = (F^*F)^{-1}F^*Ff = \sum_{m\in\mathcal{I}} \langle f, f_m \rangle (F^*F)^{-1} f_m. \tag{4}$$

The sequence of vectors $\{(F^*F)^{-1}f_m\}_{m\in\mathcal{I}}$ used in this reconstruction is the *canonical dual frame* of $\{\langle f, f_m \rangle\}_{m\in\mathcal{I}}$, while the operator $(F^*F)^{-1}F^*$ is the *psuedo-inverse* of F.

To summarize, the assumption that $\{f_m\}_{m\in\mathcal{I}}$ is a frame guarantees decompositions of the form (4). However, this assumption does not guarantee that the $(F^*F)^{-1}$ may be evaluated in a reasonable amount of time. That is, despite the great theoretical value of (4), this formula may be of limited use in practical applications, as the cost of inverting the frame operator may be prohibitive.

As a remedy, we focus upon a particular class of frames whose frame operators are trivial. A *tight frame* is a frame whose lower and upper frame bounds are equal. To be precise, given $A > 0$, a sequence $\{f_m\}_{m\in\mathcal{I}}$ is an A-*tight frame* for $\mathbb{H}$ if

$$A\|f\|^2 = \sum_{m\in\mathcal{I}} |\langle f, f_m \rangle|^2,$$

for all $f \in \mathbb{H}$. By polarization, this condition is equivalent to the frame operator satisfying $F^*F = AI$. Thus, by (4), $\{f_m\}_{m\in\mathcal{I}}$ is an A-tight frame for $\mathbb{H}$ if and only if the decomposition-reconstruction formula,

$$f = \frac{1}{A}\sum_{m\in I} \langle f, f_m \rangle f_m,$$

holds for all $f \in \mathbb{H}$.

Having introduced the main concept of the theory of frames in general, we now narrow our focus to the issue at hand: finite tight frames.

2.2 The theory of finite frames

For our remaining work, we will consider only finite sequences of vectors within finite-dimensional Hilbert spaces. That is, we consider sequences of M vectors $\{f_m\}_{m=1}^M$ within N-dimensional real or complex Hilbert spaces $\mathbb{H}_N$, where M and N are positive integers. With the indexing set fixed as $\mathcal{I} = \{1, \dots, M\}$, we regard $\ell^2(\mathcal{I})$ as $\mathbb{K}^M$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Any finite sequence $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ is necessarily Bessel, guaranteeing the existence of the corresponding analysis, synthesis, and frame operators,

$$F : \mathbb{H}_N \to \mathbb{K}^M, \quad F^* : \mathbb{K}^M \to \mathbb{H}_N, \quad F^*F : \mathbb{H}_N \to \mathbb{H}_N,$$

which may be represented as $M\times N$, $N\times M$, and $N\times N$ matrices, respectively. To be precise, given an orthonormal basis $\{e_n\}_{n=1}^N$ for $\mathbb{H}_N$, the *coordinate vector* of $f\in\mathbb{H}_N$ with respect to $\{e_n\}_{n=1}^N$ is the column vector,

$$[f]\in\mathbb{K}^N,\quad [f](n)=\langle f,e_n\rangle.$$

With respect to this basis $\{e_n\}_{n=1}^N$ for $\mathbb{H}_N$, and the standard basis for $\mathbb{K}^M$, the matrix representations of the analysis and synthesis operators are,

$$[F]=\begin{bmatrix}[f_1]^*\\ \vdots\\ [f_M]^*\end{bmatrix},\quad [F]^*=\begin{bmatrix}[f_1]&\cdots&[f_M]\end{bmatrix}.$$

For example, the four vertices of a tetrahedron in $\mathbb{R}^3$,

$$\{[f_1],[f_2],[f_3],[f_4]\}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-1\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right\},$$

have corresponding analysis and synthesis operators,

$$[F]=\begin{bmatrix}1&1&1\\-1&1&-1\\-1&-1&1\\1&-1&-1\end{bmatrix},\quad [F]^*=\begin{bmatrix}1&-1&-1&-1\\1&1&-1&-1\\1&-1&1&-1\end{bmatrix}.$$

By computing the representation of the corresponding frame operator,

$$[F^*F]=[F]^*[F]=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix},$$

we find that $F^*F=4I$, that is, that these four vectors of length $\sqrt{3}$ form a 4-tight frame for $\mathbb{R}^3$.

Note that if rescaled by a factor of $1/2$, these tetrahedral vertices become a 1-tight frame of 4 vectors of $\mathbb{R}^3$. That is, this is an example of a vector sequence for which the Parseval-Plancherel identity (1) holds for all $f\in\mathbb{R}^3$, despite the fact that $\{f_m\}_{m=1}^M$ is not an orthonormal basis for $\mathbb{R}^3$.

In general, we note that $\{f_m\}_{m\in\mathcal{I}}$ is an orthonormal basis for $\mathbb{H}$ if and only if the analysis operator F is unitary, that is, if and only if F^* is the inverse of F. Meanwhile, $\{f_m\}_{m\in\mathcal{I}}$ is a 1-tight frame for $\mathbb{H}$ if and only if F^* is a *left*-inverse of F. Thus, in the finite-dimensional setting, tight frames are nontrivial generalizations of orthonormal bases, allowing the possibility of $M > N$.

In fact, it is precisely the possibility of having $M > N$ which has attracted much of the applied community to frame theory in the first place. In this setting, the action of the frame operator redundantly encodes N dimensions into M, making the transformed signal resilient against the destructive effects of additive noise.

In this setting, the problem of recovering f from Ff becomes the classically studied problem of solving overdetermined linear systems of the form $[F][f] = [g]$, where $[F]$ is an $M \times N$ matrix with $M > N$. Modern frame theory borrows liberally from the classical approach, by first establishing the "normal equations,"

$$[F]^*[F][f] = [F]^*[g],$$

and then attempting to invert the $N \times N$ matrix $[F]^*[F]$, which has the advantages of being square, positive semidefinite, and smaller than $[F]$.

For that matter, in the finite setting, the optimal lower and upper frame bounds A and B for a sequence $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$, are precisely the least and greatest eigenvalues of $[F]^*[F]$, respectively. In other words, $\sqrt{A}$ and $\sqrt{B}$ are lower and upper bounds for the *singular values* of $[F]$, and their ratio is the condition number of $[F]$.

Finite tight frames are therefore equivalent to matrices whose singular values are identical, that is, matrices whose condition number is 1. Other trivial characterizations of tight frames may be obtained by noting that the entries of $[F]^*[F]$, are the inner products of the columns of $[F]$. Thus, a finite sequence is A-tight if and only if its coordinate vectors in $\mathbb{K}^M$ are orthogonal and all of norm $\sqrt{A}$. Even for the general Hilbert space case, a classical result of Naimark [1] demonstrates that any A-tight frame is $\sqrt{A}$ times the projection of an orthonormal basis of a larger space.

Thus, the problem of characterizing all finite tight frames is linear and trivial. At the same time, many important issues concerning tight frames remain unresolved, both in theory and applications. These nonlinear, nontrivial problems arise when looking for a particular tight frame which satisfies some additional, application-specific conditions.

From a purely mathematical point of view, the four vectors in $\mathbb{R}^3$ obtained by taking the standard orthonormal basis along with the zero vector

$$\{f_1, f_2, f_3, f_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

is as equally valid as a tight frame for $\mathbb{R}^3$ as the four vertices of the tetrahedron discussed above. However, from an intuitive perspective, this sequence lacks the beautiful symmetry of the tetrahedral arrangement. Furthermore, this frame is inefficient from the applied point of view, as the last frame element is effectively useless.

Real-life applications of this theory require sequences that are more than just tight frames. In particular, these tight frames should be designed to produce a specific, predetermined distribution of information. This may be accomplished in part by prescribing the lengths of the tight frame's elements in advance.

For example, a completely uniform distribution of information should involve a tight frame whose elements are all of equal length. Unfortunately, there have been many inconsistencies in the frame literature with regard to the appropriate terminology for such objects. In May of 2002, at the DIMACS Workshop on Source Coding and Harmonic Analysis at Rutgers University, the self-elected "Frame Nomenclature Standardization Committee" consisting of Matt Fickus, John Benedetto, Radu Balan, Carlos Cabrelli, Chris Heil, Pete Casazza and Jelena Kovačević, agreed to standardize their terminology as follows:

(1) *Equal-norm frame (ENF)*: A frame whose elements have the same norm, that is $\|f_m\| = \|f_{m'}\|$ for all $m, m' \in \mathcal{I}$.
(2) *Unit-norm frame (UNF)*: A frame where all elements are of unit norm, that is $\|f_m\| = 1$ for all $m \in \mathcal{I}$.
(3) *A-tight frame (A-TF)*: A tight frame with frame bound A; the definition used in the discussion above.
(4) *Parseval tight frame (PTF)*: A tight frame with frame bound 1, which could also be denoted as a 1-tight frame.

At the time, this diplomatic solution was unanimously agreed to. Whether this standard will be followed in practice remains to be seen.

Regardless of terminology, obtaining meaningful characterizations of some of these classes has proven very difficult. For example, where M and N are arbitrary positive integers, there is no known parametrization of all unit-norm tight frames of M elements for N-dimensional spaces, despite the well-known parametrization of all orthonormal bases of size N. However, Benedetto and Fickus [3] were able to find a qualitative characterization of such frames, and much of their work will be described in greater detail below.

In this paper, we investigate the natural generalizations of unit-norm tight frames, namely tight frames whose lengths are predetermined, but of arbitrary values.

2.4 The fundamental inequality

Given a Hilbert space $\mathbb{H}_N$ and a sequence of nonnegative scalars $\{a_m\}_{m=1}^M$, we seek to characterize those sequences $\{f_m\}_{m=1}^M \subset \mathbb{H}_M$ which are both tight frames for $\mathbb{H}_N$ and satisfy $\|f_m\| = a_m$ for all m. At the same time, we wish to characterize those sequences $\{a_m\}_{m=1}^M$ for which such a frame exists.

As a first step in obtaining these characterizations, we use the frame bounds of an arbitrary sequence[3] $\{f_m\}_{m=1}^M$ to obtain trivial bounds upon $\{\|f_m\|\}_{m=1}^M$.

Proposition 1 *If $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ has frame bounds $A \leq B$, then,*

$$\max_{m=1,\dots,M} \|f_m\|^2 \leq B, \qquad AN \leq \sum_{m=1}^M \|f_m\|^2 \leq BN.$$

PROOF. To prove the first inequality, note that for any $m = 1, \dots, M$,

$$\|f_m\|^4 = |\langle f_m, f_m \rangle|^2 \leq \sum_{m'=1}^M |\langle f_m, f_{m'} \rangle|^2 \leq B\|f_m\|^2.$$

Thus, $\|f_m\|^2 \leq B$ for all $m = 1, \dots, M$. To prove the remaining inequalities, let $\{e_n\}_{n=1}^N$ be an orthonormal basis for $\mathbb{H}_N$. By the Parseval identity,

$$\sum_{m=1}^M \|f_m\|^2 = \sum_{m=1}^M \sum_{n=1}^N |\langle f_m, e_n \rangle|^2 = \sum_{n=1}^N \sum_{m=1}^M |\langle e_n, f_m \rangle|^2.$$

Meanwhile, the definition of the frame bounds guarantees

$$NA = \sum_{n=1}^N A\|e_n\|^2 \leq \sum_{n=1}^N \sum_{m=1}^M |\langle e_n, f_m \rangle|^2 \leq \sum_{n=1}^N B\|e_n\|^2 = NB,$$

[3] We do not require the sequences to be frames. In this context, *frame bounds* refer to any constants A and B satisfying (3), allowing the possibility of A, $B = 0$.

and the result is demonstrated. $\square$

By letting $A = B$ in the previous result, we find that a finite tight frame's frame constant is uniquely determined by the lengths of the frame elements. At the same time, we find an implicit relationship that the lengths of a finite tight frame's elements must satisfy.

Corollary 2 *If $\{f_m\}_{m=1}^M$ is an A-tight frame for $\mathbb{H}_N$, then,*

$$\sup_{m=1,\ldots,M} \|f_m\|^2 \leq A = \frac{1}{N}\sum_{m=1}^{M} \|f_m\|^2.$$

Thus, given positive integers M and N, the existence of a tight frame of lengths $\{a_m\}_{m=1}^M$ for an N-dimensional space necessitates these lengths satisfy,

$$\max_{m=1,\ldots,M} a_m^2 \leq \frac{1}{N}\sum_{m=1}^{M} a_m^2. \tag{5}$$

Though the process is long and difficult, we shall demonstrate that this inequality is sufficient as well. That is, for any sequence $\{a_m\}_{m=1}^M$ that satisfies (5), we shall show that there exists a tight frame of these lengths for any space of dimension N. As such, we refer to (5) as the *fundamental inequality on the lengths of a tight frame's elements*, or more simply, as the *fundamental inequality*, when the context is understood.

Having concluded our preliminary discussion, we now concentrate upon the proof of the sufficiency of (5), and the underlying intuition thereof.

3 A Physical Interpretation of Frame Theory

We continue our investigation into finite tight frames of arbitrary prescribed lengths by first focusing upon recent developments in a special case of the general situation. A *unit-norm tight frame* (UNTF) is a tight frame whose frame elements are all of unit length. Even in spaces of finite dimension, little is known about such frames, despite the fact that they are perhaps the most natural nontrivial generalizations of orthonormal bases.

Corollary 2 guarantees the frame constant of a UNTF of M elements for an N-dimensional space is necessarily the *redundancy* of the frame, M/N. However, only recently have such frames been found to exist for arbitrary $M \geq N$. One proof of existence involves the direct construction of such frames as projections of Fourier bases. This proof is efficient, yet unintuitive, and does not generalize in any way that helps us solve the more general problem.

Another proof is given by Benedetto and Fickus [3], and is but one consequence of their search for a deeper intuitive understanding of UNTFs[4]. Inspired by the "classical" examples of UNTFs, such as the vertices of the regular polygons in $\mathbb{R}^2$ and the vertices of the regular polyhedra in $\mathbb{R}^3$, their intention was to pursue general connections between frame theory and geometric regularity. Eventually failing in this pursuit, they were then led to explore connections between UNTFs and classical means of equally distributing sets of points on spheres. It was in this context that a particular idea from classical physics caught their eye.

By adapting this classical idea, Benedetto and Fickus were able to provide a qualitative characterization of UNTFs as sets of unit vectors which are "optimally orthogonal." After a brief review of the physical concepts that motivated their work, we extend their concepts of the *frame force* and *frame potential* to the general setting, laying the foundation upon which we shall later build our main results.

3.1 A classical motivation

Imagine a finite number of electrons placed upon a metallic spherical shell, repelling each other according to Coulomb's law. If the excess energy of the system begins to dissipate due to friction, each particle will further distance itself from the others, eventually leaving the points in a stable equilibrium. The final arrangement of points is considered "well-distributed[5]."

Mathematically, the electrons are represented as vertices $\{f_m\}_{m=1}^M \subset \mathbb{R}^3$, with $\|f_m\| = 1$ for all $m = 1, \ldots, M$. Each point f_n pushes against distinct points f_m according to the central, inverse-square Coulomb force,

$$\mathrm{CF}(f_m, f_n) = \frac{f_m - f_n}{\|f_m - f_n\|^3}.$$

The dissipation of energy due to friction corresponds to a loss of total potential energy, which is computed by summing the distinct pairwise potentials,

$$\mathrm{CP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{n \neq m} \frac{1}{\|f_m - f_n\|}.$$

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[4] Unit-norm tight frames are referred to as *normalized tight frames* in [3].

[5] Though not necessarily optimal, as discussed below.

Viewing the total potential as a function of the M vertices restricted to the unit sphere, the loss of energy corresponds to a descent in the direction of the potential's gradient. Upon encountering a local minimizer of the potential function, a stable equilibrium is achieved, while the global minimizer is the optimal means of distributing points equally in this context.

Benedetto and Fickus investigated whether this classical notion of equidistribution could be used to characterize UNTFs. They noted that no direct correlation between such sets may be made, as most of the known optimal arrangements of electrons do not yield tight frames, and even simple examples of UNTFs, such as orthonormal bases, are far from being well-distributed.

Instead, they observed that optimal arrangements of electrons are sets of points which are "as far away from each other as possible," while UNTFs are sets of vectors which are "as orthogonal to each other as possible." Though, the classical formalism was the correct approach philosophically, the application of the theory to the Coulomb force lead to the wrong notion of "distance." Instead, the classical theory needed to be applied to a new, nonnatural force.

3.2 The frame force and frame potential

For the sake of simplicity, the discussion below is given in terms of real Euclidean spaces. However, our subsequent definitions and results will be valid for arbitrary finite-dimensional real and complex Hilbert spaces. For a positive integer N, the *frame force* of $f_{m'} \in \mathbb{R}^N$ upon $f_m \in \mathbb{R}^N$ is,

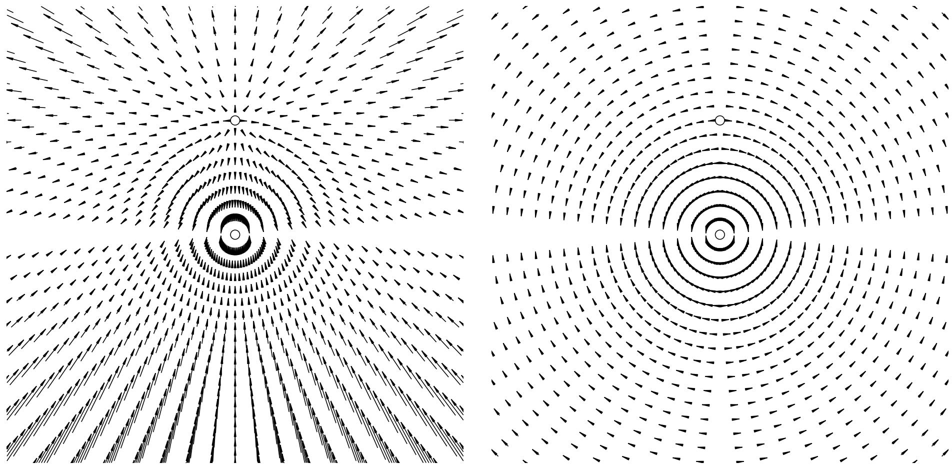
$$\mathrm{FF}(f_m, f_{m'}) \in \mathbb{R}^N, \quad \mathrm{FF}(f_m, f_{m'}) = 2\langle f_m, f_{m'}\rangle(f_m - f_{m'}). \tag{6}$$

This is essentially the same definition that appears in [3], but without the restriction of f_m and $f_{m'}$ to the unit sphere.

Recall that the Coulomb force is *central*, that is, the force between two points is directed along their axis. Clearly, the frame force is central as well. However, while the scalar portion of the Coulomb force depends only upon the distance between the points, the scalar portion of frame force is a function of their inscribed angle. As a result, the frame force acts quite differently than forces found in nature.

For a better idea of how the frame force behaves, consider the vector field created on $\mathbb{R}^2$ by fixing $f_{m'}$ as $(0, 1)$. A radial sampling of a rescaled version of this field is pictured on the left in Figure 1. As demonstrated in the figure, the frame force is defined with respect to a translation-variant coordinate system. That is, unlike "natural" forces, there must be a fixed origin that serves as a reference for measuring the angle between points. Furthermore, the frame

Fig. 1. The force field generated by $(0, 1)$ on $\mathbb{R}^2$, and its tangential component.



force between identical points is well-defined, being zero. Most importantly, the frame force "encourages orthogonality," in that the force is repulsive when the angle between the points is acute, attractive when the angle is obtuse, and zero when the points are orthogonal.

Paralleling the intuition from the Coulomb case, an "optimally orthogonal" sequence may be found by minimizing the associated total potential energy function. We now compute this potential using classical techniques.

A slight complication arises from the fact that the frame force field generated by a point is not conservative, that is, the work required to travel between two points depends upon the path taken. Fortunately, the force becomes conservative provided the movement of the points is restricted to concentric spheres centered at the origin.

This novel property of the frame force is exemplified by the second graph of Figure 1, in which only the portion of the frame force which lies parallel to the possible direction of movement is shown. Physically, this means that the work required to travel along a sphere is independent of path, but the work required to travel to other spheres is path-dependent.

Mathematically, this phenomenon is represented by the relative ease with which the anti-gradient of the frame force is computed after making the restrictions $\|f_m\| = a_m$ and $\|f_{m'}\| = a_{m'}$. In particular, a correspondence is established between the distance of two points and their inner product,

$$\|f_m - f_{m'}\|^2 = \|f_m\|^2 - 2\langle f_m, f_{m'}\rangle + \|f_{m'}\|^2 = a_m^2 - 2\langle f_m, f_{m'}\rangle + a_{m'}^2, \quad (7)$$

allowing us to write the scalar part of the force in terms of $\|f_m - f_{m'}\|$,

$$\mathrm{FF}(f_m, f_{m'}) = (a_m^2 + a_{m'}^2 - \|f_m - f_{m'}\|^2)(f_m - f_{m'}).$$

Taking the antigradient of this conservative field then reduces to antidifferentiating the "scalar force,"

$$p(x) = -\int (a_m^2 + a_{m'}^2 - x^2)x\,dx = \frac{1}{4}x^2[x^2 - 2(a_m^2 + a_{m'}^2)].$$

The pairwise potential $\mathrm{P}(f_m, f_{m'})$ between f_m and $f_{m'}$ is obtained by evaluating at $x = \|f_m - f_{m'}\|$, and using (7) to simplify,

$$\mathrm{P}(f_m, f_{m'}) = p(\|f_m - f_{m'}\|) = \langle f_m, f_{m'}\rangle^2 - \frac{1}{4}(a_m^2 + a_{m'}^2)^2. \qquad (8)$$

The total potential energy TP contained within a system of points $\{f_m\}_{m=1}^M$ restricted to spheres of radii $\{a_m\}_{m=1}^M$ is found by summing the distinct pairwise potentials,

$$\mathrm{TP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^{M} \sum_{m' \neq m} \langle f_m, f_{m'}\rangle^2 - \frac{1}{4}(a_m^2 + a_{m'}^2)^2. \qquad (9)$$

As additive constants within a potential energy function are physically meaningless, there is no harm done in omitting the terms $(1/4)(a_m^2 + a_{m'}^2)^2$ from (9). Similarly, we choose to include the diagonal terms $\langle f_m, f_m\rangle = a_m^4$.

Taking the natural extension of this idea to arbitrary real or complex finite-dimensional Hilbert spaces, we define the *frame potential* of $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$,

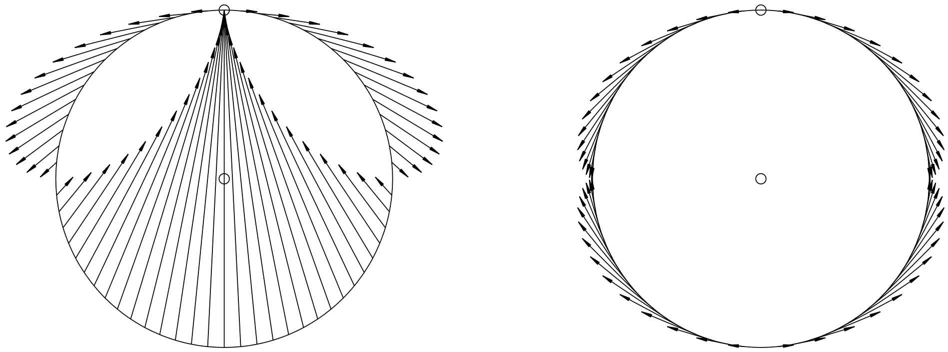
$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^{M} \sum_{m'=1}^{M} |\langle f_m, f_{m'}\rangle|^2. \qquad (10)$$

Physically, this potential function determines the amount of energy required to change states under the effects of the frame force. To be precise, given any two sequences $\{f_m\}_{m=1}^M$ and $\{g_m\}_{m=1}^M$ with $\|f_m\| = a_m = \|g_m\|$ for all m, the quantity $\mathrm{FP}(\{g_m\}) - \mathrm{FP}(\{f_m\})$ is the work required to deform $\{f_m\}$ into $\{g_m\}$, while remaining on spheres of radii $\{a_m\}$ at all times.

As the frame potential is the potential energy function of a force which encourages orthogonality, we are justified in referring to the minimizers of this function as *maximally orthogonal* sequences. Indeed, even without discussing the underlying physical intuition, the definition of the frame potential of a sequence is clearly a measurement ot its "total orthogonality."

Though our immediate priority is the continued development of this intuitive picture, we shall eventually demonstrate that the minimizers of the frame potential have great significance in terms of finite frame theory.

Fig. 2. A weighted frame force field created by a point, and its effective component.



3.3 How one charge, many radii becomes many charges, one radius

As described in detail above, the frame potential (10) is the total potential energy of a system of points under the frame force (6), where the movement of each point is restricted to a sphere of a certain radius.

To further develop this intuition, we must first admit the inherent difficulty in trying the visualize the effects of the frame force between points on many spheres of different radii. Fortunately, this intuitive picture may be simplified by "projecting" the dynamics onto the unit sphere. To be precise, we shall demonstrate how the action of the frame force upon $\{f_m\}_{m=1}^M$ restricted to spheres of radii $\{a_m\}_{m=1}^M$ may be related to the action of a modified frame force upon the *projected* vectors $\{f_m/\|f_m\|\}_{m=1}^M$.

Being inspired by the appearance of multiplicative *charge* terms in the classical Coulomb force between particles of distinct charges, we consider the *weighted frame force* between $g_m, g_{m'} \in S^{(N-1)} \equiv \{f \in \mathbb{R}^N : \|f\| = 1\}$ of corresponding *charges* $q_m, q_{m'} \geq 0$,

$$\mathrm{WFF}(q_m, q_{m'}, g_m, g_{m'}) = 2q_m, q_{m'}\langle g_m, g_{m'}\rangle(g_m - g_{m'}).$$

Figure 2 depicts the weighted force field created by a single point on the unit circle, contrasting with the nonweighted complexity of Figure 1. The length of the force vectors is now a bilinear function of the charges.

To determine the precise way in which the weighted projected dynamics are made equivalent to the nonweighted, nonprojected case, we first compute the total potential energy contained within a system of points $\{g_m\}_{m=1}^M \subset S^{(N-1)}$ of corresponding charges $\{q_m\}_{m=1}^M$. The computation is the same as that used above for the nonweighted frame force, except for the presence of multiplicative constants of the form $q_m q_{m'}$, and the fact that $a_m = 1$ for all m.

In particular, the pairwise frame potential (8) becomes,

$$\mathrm{WP}(q_m, q_{m'}, g_m, g_n) = q_m q_{m'}(\langle g_m, g_{m'}\rangle^2 - 1),$$

leading to the *weighted frame potential*,

$$\mathrm{WFP}(\{q_m\}_{m=1}^M, \{g_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{m'=1}^M q_m q_{m'} |\langle g_m, g_{m'}\rangle|^2.$$

To see how the weighted potential relates to the nonweighted version (10), note that for any $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ of respective lengths $\{a_m\}_{m=1}^M$,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{m'=1}^M |\langle f_m, f_{m'}\rangle|^2 = \sum_{m=1}^M \sum_{m'=1}^M a_m^2 a_{m'}^2 |\langle g_m, g_{m'}\rangle|^2,$$

where $\{g_m\}_{m=1}^M$ is the *projected sequence* of $\{f_m\}_{m=1}^M$,

$$\{g_m\}_{m=1}^M \subset S(1) \equiv \{f \in \mathbb{H}_N : \|f\| = 1\}, \quad g_m = f_m/\|f_m\|.$$

Thus, the frame potential of the sequence of points $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ lying on concentric spheres of radii $\{a_m\}_{m=1}^M$ is equivalent to the weighted frame potential of the projected sequence $\{g_m\}_{m=1}^M$, provided the corresponding charges $\{q_m\}_{m=1}^M$ are chosen as the *squares* of the radii. That is,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \mathrm{WFP}(\{a_m^2\}_{m=1}^M, \{g_m\}_{m=1}^M).$$

As a consequence, we forego the idea of the frame potential as the energy of points of equal charge on many spheres, in favor of the more intuitive concept as the energy of points of many charges on a single sphere.

To tantalize the reader, we note that the fundamental inequality (5), whose sufficiency we endeavor to prove, may now be interpreted as a measurement on the uniformity of the charges $\{a_m^2\}_{m=1}^M$.

Before continuing our exploration of the physical theory, we note the correspondence above is only possible due to the frame potential's unique ability to absorb multiplicative scalars, and is yet more evidence of the mathematical simplicity of the frame force, despite its bizarre natural behavior. For the sake of contrast, we observe that the Coulomb potential contained within a set of points of equal charge on concentric spheres is not at all equivalent to the potential contained within points of varying charge on a single sphere.

3.4 The effective component of the frame force

Physically, a change in energy requires work, and work requires movement. For example, though an object at rest on the ground experiences the force of gravity, no work is done, as the ground restricts its downward movement.

As with our computation of the frame potential above, we now restrict the movement of $\{f_m\}_{m=1}^M$ to spheres of radii $\{a_m\}_{m=1}^M$. The only physically significant portion of the frame force of $f_{m'}$ upon f_m,

$$\mathrm{FF}(f_m, f_{m'}) = 2\langle f_m, f_{m'}\rangle(f_m - f_{m'}),$$

is the component which lies parallel to the surface of the sphere of radius a_m at f_m. We may explicitly compute this *effective component* of the frame force by subtracting the normal component from the whole. In particular, we remove the projection of the force onto the axis spanned by f_m,

$$\mathrm{FF}(f_m, f_{m'}) - \mathrm{Proj}_{f_m}\mathrm{FF}(f_m, f_{m'}) = \mathrm{FF}(f_m, f_{m'}) - \frac{\langle \mathrm{FF}(f_m, f_{m'}), f_m\rangle}{a_m^2} f_m,$$

and simplify to obtain the *effective frame force* of $f_{m'} \in \mathbb{R}^N$ upon $f_m \in \mathbb{R}^N$,

$$\mathrm{EFF}(f_m, f_{m'}) = 2\langle f_m, f_{m'}\rangle \left(\frac{\langle f_{m'}, f_m\rangle}{a_m^2} f_m - f_{m'} \right).$$

A scaled version of effective frame force field generated in $\mathbb{R}^2$ by $(0, 1)$ is pictured on the left in Figure 1. Adding such fields corresponds to finding the net effective force created by several points, with the "perfectly balanced" arrangements leading to perfect cancellation. We now characterize such arrangements, and in so doing provide the first glimpses of a deep connection between our physical theory and the theory of finite tight frames.

Proposition 3 *A nonzero sequence $\{f_m\}_{m=1}^M \subseteq \mathbb{R}^N$ satisfies,*

$$\sum_{m=1}^{M} \mathrm{EFF}(f, f_m) = 0,$$

for all $f \in \mathbb{R}^N$, if and only if $\{f_m\}_{m=1}^M$ is a tight frame for $\mathbb{R}^N$.

PROOF. We first note that,

$$0 = \sum_{m=1}^{M} \mathrm{EFF}(f, f_m) = \sum_{m=1}^{M} 2\langle f, f_m\rangle \left(\frac{\langle f_m, f\rangle}{\|f\|^2} f - f_m \right),$$

for all $f \in \mathbb{R}^N$ if and only if,

$$\|f\|^2 \sum_{m=1}^{M} \langle f, f_m\rangle f_m = \sum_{m=1}^{M} |\langle f, f_m\rangle|^2 f, \tag{11}$$

for all $f \in \mathbb{R}^N$. This condition holds provided $\{f_m\}_{m=1}^M$ is an A-tight frame, as the sum on the left of (11) is Af, while the sum on the right is $A\|f\|^2$.

For the converse, let F be the analysis operator of $\{f_m\}_{m=1}^M$. Under this notation, (11) becomes,

$$\|f\|^2 F^*Ff = \|Ff\|^2 f,$$

for all $f \in \mathbb{R}^N$. Thus, every vector is an eigenvector of the frame operator F^*F. This is only possible if $F^*F = AI$ for some $A \in \mathbb{R}$. A is nonnegative since F^*F is positive semidefinite, and is nonzero since $\{f_m\}_{m=1}^M$ is nonzero by assumption. Thus, $\{f_m\}_{m=1}^M$ is an A-tight frame for $\mathbb{R}^N$. □

Effective forces may be also be used to reaffirm our intuition from the previous subsection. In particular, the *effective component* of the weighted frame force is the component of the force vector which is tangential to the surface of the unit sphere. The explicit computation of this vector is the same as that given above, except for the presence of a multiplicative constant $q_m q_{m'}$, and that $a_m = 1$ for all m. To be precise, the *effective weighted frame force* of $g_{m'} \in S^{(N-1)}$ of charge $q_{m'}$ against $g_m \in S^{(N-1)}$ of charge q_m is,

$$\mathrm{WEFF}(q_m, q_{m'}, g_m, g_{m'}) = 2q_m q_{m'} \langle g_m, g_{m'}\rangle \left(\langle g_{m'}, g_m\rangle g_m - g_{m'}\right).$$

By letting $g_m = f_m/a_m$ and $q_m = a_m^2$ for all m, we discover that the effective weighted force is equal to the effective force dilated by a_m:

$$\begin{aligned}\mathrm{WEFF}(a_m^2, a_{m'}^2, g_m, g_{m'}) &= 2\langle a_m g_m, a_{m'} g_{m'}\rangle \left(\langle a_{m'} g_{m'}, a_m g_m\rangle g_m - a_m a_{m'} g_{m'}\right), \\ &= 2\langle f_m, f_{m'}\rangle \left(\frac{\langle f_{m'}, f_m\rangle}{a_m^2} a_m f_m - a_m f_{m'}\right), \\ &= a_m \mathrm{EFF}(f_m, f_{m'}). \qquad (12)\end{aligned}$$

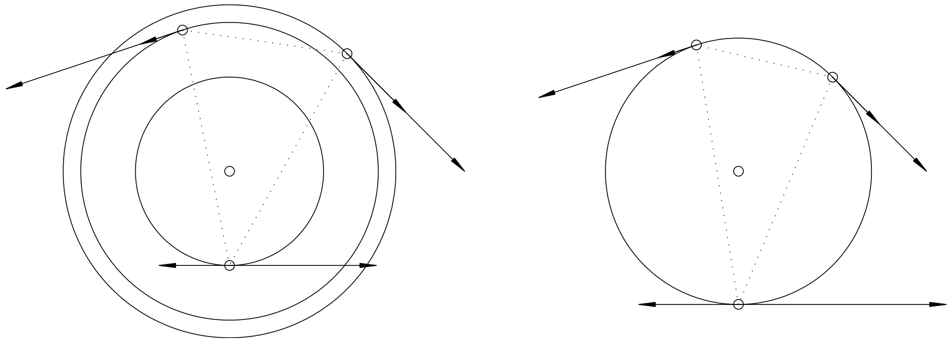
In accordance with Newton's Laws of Motion, the corresponding "acceleration" vectors are obtained by dividing the force vectors by the charge[6]. As f_m lies on a sphere of radius a_m, and is of charge 1, the acceleration of f_m due to $f_{m'}$ is simply $\mathrm{EFF}(f_m, f_{m'})$. Meanwhile, as g_m lies on a sphere of radius 1, and is of charge a_m^2, the acceleration of g_m due to $g_{m'}$ is $\mathrm{WEFF}(a_m^2, a_{m'}^2, f_m, f_{m'})/a_m^2$. Therefore, by (12), this acceleration is equal to $\mathrm{EFF}(f_m, f_{m'})/a_m$.

However, as $\mathrm{EFF}(f_m, f_{m'})$ is a vector tangential to the surface of the sphere of radius a_m, the quantity $\mathrm{EFF}(f_m, f_{m'})/a_m$ also represents the "projection" of this acceleration onto the unit sphere. That is, the projection of the frame force acceleration between points on arbitrary spheres is equal to the weighted frame force acceleration between the projected points on the unit sphere.

For an illustration of this phenomenon, consider the two images in Figure 3. The image on the left depicts three points on concentric spheres of distinct

[6] Here, the charge plays the role of the mass in Newton's Second Law. In fact, from this perspective, the frame force is better interpreted as a generalization of gravitation, instead of electromagnetism.

Fig. 3. Equivalent notions of acceleration.



radii. The two vectors based at each point are the acceleration vectors created by the push of the frame force from the other two. The image on the right depicts these points projected to the unit sphere, and the acceleration vectors created by the push of the weighted frame force. Note that for each point on the left, the proportion of the vector length to distance from the origin is preserved in the image on the right.

As a consequence, the frame force dynamics of a physical system on many spheres can always be understood in terms of the equivalent weighted frame force dynamics on the unit sphere. That is, the weighted frame force is truly the "projection" of the frame force.

Armed with this intuition, we now turn to a rigorous analysis of the "maximally orthogonal" minimizers of the frame potential. Though the results of the next section are near immediate consequences of the definition of the frame potential, the subsequent work requires a thorough understanding of the physical theory.

4 The Frame Potential

In the previous section, we introduced the frame potential of $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{m'=1}^M |\langle f_m, f_{m'}\rangle|^2,$$

as the total potential energy contained within the sequence under the frame force (6), provided the vertices were restricted to spheres of radii $\{a_m\}_{m=1}^M$. Intuitively, the minimizers of the frame potential over a given subset of $\mathbb{H}_N^M \equiv \mathbb{H}_N \times \cdots \times \mathbb{H}_N$ are the elements of that set which are as close as possible to being an orthogonal sequence.

We now begin the process of demonstrating that this physical generalization of orthogonality leads to many of the main ideas of frame theory. This process will continue into the next section, where eventually the physical interpretation is used to motivate, prove and explain new results in frame theory. In particular, the frame potential is used to show the existence of tight frames whose lengths are any sequence satisfying the fundamental inequality (5).

For now, we focus upon how alternative representations of the frame potential may be used to great effect in certain minimization problems. These representations are given in terms of the *trace* of an operator $T : \mathbb{H}_N \to \mathbb{H}_N$,

$$\operatorname{Tr}(T) = \sum_{n=1}^{N} \langle Te_n, e_n \rangle,$$

where $\{e_n\}_{n=1}^N$ is any orthonormal basis for $\mathbb{H}_N$. In particular, we rephrase a classical, trivial result to show that the frame potential is the trace of the square of the frame operator.

Lemma 4 *For any sequence $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ with corresponding analysis F,*

$$\operatorname{FP}(\{f_m\}_{m=1}^M) = \operatorname{Tr}((F^*F)^2).$$

In particular, if $\{\lambda_n\}_{n=1}^N$ are the eigenvalues of F^*F, then,

$$\operatorname{FP}(\{f_m\}_{m=1}^M) = \sum_{n=1}^{N} \lambda_n^2.$$

PROOF. For the first identity, observe,

$$\begin{aligned}
\operatorname{FP}(\{f_m\}_{m=1}^M) &= \sum_{m=1}^{M} \sum_{m'=1}^{M} \langle f_{m'}, f_m \rangle \langle f_m, f_{m'} \rangle, \\
&= \sum_{m=1}^{M} \sum_{m'=1}^{M} \langle \sum_{n=1}^{N} \langle f_{m'}, e_n \rangle e_n, f_m \rangle \langle f_m, f_{m'} \rangle, \\
&= \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{m'=1}^{M} \langle \langle e_n, f_m \rangle f_m, \langle e_n, f_{m'} \rangle f_{m'} \rangle, \\
&= \sum_{n=1}^{N} \langle F^*Fe_n, F^*Fe_n \rangle = \operatorname{Tr}((F^*F)^2).
\end{aligned}$$

For the second result, compute the trace with respect to the orthonormal basis of eigenvectors for the self-adjoint positive semidefinite frame operator F^*F. □

Thus, the frame potential is the square of the Hilbert-Schmidt[7] norm of the frame operator. Having these alternative representations, we show how a cou-

[7] Also known as the Frobenius norm.

ple of minimization problems naturally lead to the concept of the canonical dual frame, and to that of Parseval frames.

4.1 Minimizing the dual energy

Given any frame $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$, a corresponding *dual frame* is any frame $\{h_m\}_{m=1}^M$ for $\mathbb{H}_N$ such that,

$$f = \sum_{m=1}^{M} \langle f, f_m \rangle h_m,$$

for all $f \in \mathbb{H}_N$. Equivalently, $\{h_m\}_{m=1}^M$ is a dual frame for $\{f_m\}_{m=1}^M$ if $H^*F = I$, where H and F are the sequences' analysis operators, respectively.

Meanwhile, the *canonical dual frame* of $\{f_m\}_{m=1}^M$ is the sequence $\{\widetilde{f}_m\}_{m=1}^M$ that arises by solving the normal equations in a corresponding least-squares problem $Ff = g$. Specifically, $\widetilde{f}_m \equiv (F^*F)^{-1} f_m$ for all m, and has the analysis operator $\widetilde{F} = F(F^*F)^{-1}$. As we now demonstrate, the canonical dual is the unique dual frame which is as orthogonal as possible.

Proposition 5 *Given a frame $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$, the unique minimizer of the frame potential restricted to the dual frames of F,*

$$\mathrm{FP} : \{\{h_m\}_{m=1}^M \subset \mathbb{H}_N : H^*F = I\} \to \mathbb{R},$$

is the canonical dual frame $\{(F^*F)^{-1} f_m\}_{m=1}^M$.

PROOF. Let F be the analysis operator for $\{f_m\}_{m=1}^M$, while $\widetilde{F}$ and H are the analysis operators of the canonical dual and an arbitrary dual, respectively. Note the frame operator of the canonical dual is the inverse of the frame operator for $\{f_m\}_{m=1}^M$,

$$\widetilde{F}^*\widetilde{F} = (F^*F)^{-1} F^*F (F^*F)^{-1} = (F^*F)^{-1}.$$

Therefore, the frame operator of the arbitrary dual may be decomposed as,

$$\begin{aligned} H^*H &= (\widetilde{F} + (H - \widetilde{F}))^* (\widetilde{F} + (H - \widetilde{F})), \\ &= \widetilde{F}^*\widetilde{F} + (H - \widetilde{F})^* (H - \widetilde{F}), \end{aligned}$$

as the middle terms vanish,

$$(H - \widetilde{F})^* \widetilde{F} = H^*F(F^*F)^{-1} - \widetilde{F}^*\widetilde{F} = (F^*F)^{-1} - (F^*F)^{-1} = 0.$$

By properties of the trace, the frame potential of the arbitrary dual $\{h_m\}_{m=1}^M$ satisfies,

$$\begin{aligned}\mathrm{Tr}((H^*H)^2) &= \mathrm{Tr}(\,(\widetilde{F}^*\widetilde{F} + (H-\widetilde{F})^*(H-\widetilde{F}))^2\,),\\ &= \mathrm{Tr}((\widetilde{F}^*\widetilde{F})^2) + 2\mathrm{Tr}(\,((H-\widetilde{F})\widetilde{F}^*)^*((H-\widetilde{F})\widetilde{F}^*)\,),\\ &\qquad + \mathrm{Tr}(\,((H-\widetilde{F})^*(H-\widetilde{F}))^2\,),\\ &\geq \mathrm{Tr}((\widetilde{F}^*\widetilde{F})^2),\end{aligned}$$

with equality if and only if $H - \widetilde{F} = 0$. Thus, the minimal frame potential is achieved by letting $H = \widetilde{F}$, that is, by choosing the arbitrary dual as the canonical one. □

Having computed the unique dual of minimal energy, we now characterize those sequences for which the sum of their energy with their dual energy is minimal.

Proposition 6 *The frames $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$ for which,*

$$\mathrm{FP}(\{f_m\}_{m=1}^M) + \mathrm{FP}(\{(F^*F)^{-1}f_m\}_{m=1}^M)$$

is minimized are precisely the Parseval frames for $\mathbb{H}_N$.

PROOF. Let F be the analysis operator of $\{f_m\}_{m=1}^M$, and let $\{\lambda_n\}_{n=1}^N$ be the eigenvalues of the corresponding frame operator F^*F. The frame operator of the canonical dual $\{(F^*F)^{-1}f_m\}_{m=1}^M$ is then $(F^*F)^{-1}$, with eigenvalues $\{1/\lambda_n\}_{n=1}^N$. By the Lemma 4, we may express the potentials in terms of these eigenvalues,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) + \mathrm{FP}(\{(F^*F)^{-1}f_m\}_{m=1}^M) = \sum_{n=1}^N \lambda_n^2 + \sum_{n=1}^N \frac{1}{\lambda_n^2}. \tag{13}$$

For any n, the minimal value of $\lambda_n^2 + 1/\lambda_n^2$ is achieved by letting $\lambda_n = 1$. Thus, (13) is bounded below by $2N$, and this bound is obtained provided $\lambda_n = 1$ for all n. In other words, the lower bound is achieved if and only if $F^*F = I$. As such frames always exist for any $M \geq N$, the lower bound is a minimal value, and is obtained if and only if $\{f_m\}_{m=1}^M$ is a Parseval frame for $\mathbb{H}_N$.

These results are but two of several that demonstrate the value of studying the frame potential as a quantity unto itself. However, the main purpose of this paper is the development of the physical interpretation that was introduced in the previous section. Therefore, we now return to this intuition, and continue to make the connections between finite tight frames and the optimal arrangements of points under the influence of the frame force.

Inspired by the way in which electrons on a sphere will naturally seek their optimal distribution, we, in the previous section, defined a force that causes points on spheres to seek an optimal degree of orthogonality. We then computed the total potential energy contained within such a system to be the frame potential (10), and intuitively indicated how the minimizers of this potential are "optimally orthogonal" sets.

We now begin the formal process of characterizing these minimizers. Though the physical motivation was presented in the context of real Euclidean spaces, we may generalize the concepts so that the results apply to arbitrary finite-dimensional real or complex Hilbert spaces.

To pose the problem rigorously, given an N-dimensional Hilbert space $\mathbb{H}_N$, and $a \geq 0$, let $S(a)$ denote the generalized sphere of radius a, namely $S(a) = \{f \in \mathbb{H}_N : \|f\| = a\}$. Furthermore, given any positive sequence $\{a_m\}_{m=1}^M$, let $S(a_1, \ldots, a_M)$ be the Cartesian product of the corresponding sequence of spheres: $S(a_1, \ldots, a_M) = S(a_1) \times \cdots \times S(a_M)$.

We seek a characterization of the minimizers of the frame potential, regarded as the function $\mathrm{FP} : S(a_1, \ldots, a_M) \to \mathbb{R}$. Providing a complete characterization requires much effort, and as such, is not presented until the next section.

For now, we content ourselves by finding lower bounds for this restricted frame potential. The argument is essentially the same as that of [3]. Nevertheless, we present the proof, as our result is slightly more general, and is crucial to motivating the work in the next section.

Proposition 7 *Let $\mathbb{H}_N$ be any N-dimensional Hilbert space, and let $\{a_m\}_{m=1}^M$ be any finite sequence of nonnegative numbers. If $M \leq N$, the values of the restricted frame potential $\mathrm{FP} : S(a_1, \ldots, a_M) \to \mathbb{R}$ are bounded below by,*

$$\sum_{m=1}^{M} a_m^4 \leq \mathrm{FP}(\{f_m\}_{m=1}^M),$$

where the lower bound is achieved if and only if $\{f_m\}_{m=1}^M \in S(a_1, \ldots, a_M)$ is an orthogonal sequence. If $M \geq N$, the values are bounded below by,

$$\frac{1}{N}\left[\sum_{m=1}^{M} a_m^2\right]^2 \leq \mathrm{FP}(\{f_m\}_{m=1}^M),$$

where the lower bound is achieved if and only if $\{f_m\}_{m=1}^M \in S(a_1, \ldots, a_M)$ is a tight frame for $\mathbb{H}_N$.

PROOF. If $M \leq N$, then,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{m'=1}^M |\langle f_m, f_{m'} \rangle|^2 \geq \sum_{m=1}^M |\langle f_m, f_m \rangle|^2 = \sum_{m=1}^M a_m^4,$$

with equality if and only if the "off-diagonal" terms in the summation are zero. Thus, we have equality if and only if $\langle f_m, f_{m'} \rangle = 0$ for all $m \neq m'$, that is, if and only if $\{f_m\}_{m=1}^M$ is an orthogonal sequence.

Note that we have not used the assumption $M \leq N$. Indeed, the value in question is a valid lower bound for the frame potential regardless of the size of M. However, if $M > N$, there is no orthogonal sequence $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$, to yield this bound, and an improved bound is given by the second statement.

For $M \geq N$, take any $\{f_m\}_{m=1}^M \in S(a_1, \ldots, a_M)$ with corresponding analysis operator F. Letting $\{\lambda_n\}_{n=1}^N$ be the eigenvalues of F^*F, the trace of the frame operator is,

$$\begin{aligned}\sum_{n=1}^N \lambda_n = \mathrm{Tr}(F^*F) = \sum_{n=1}^N \langle F^*F e_n, e_n \rangle &= \sum_{n=1}^N \langle \sum_{m=1}^M \langle e_n, f_m \rangle f_m, e_n \rangle, \\ &= \sum_{m=1}^M \sum_{n=1}^N |\langle f_m, e_n \rangle|^2 = \sum_{m=1}^M \|f_m\|^2 = \sum_{m=1}^M a_m^2.\end{aligned}$$

Meanwhile Lemma 4 states that,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \mathrm{Tr}((F^*F)^2) = \sum_{n=1}^N \lambda_n^2.$$

A lower bound for the frame potential may therefore be obtained by finding the point on the plane $\{\{\lambda_n\}_{n=1}^N : \sum_{n=1}^N \lambda_n = \sum_{m=1}^M a_m^2\}$ which lies closest to the origin. This classical problem is solved using Lagrange multipliers, with the coordinates of the optimal point found to have equal value,

$$\lambda_1 = \cdots = \lambda_N = \frac{1}{N} \sum_{m=1}^M a_m^2,$$

leading to the bound on the frame potential,

$$\mathrm{FP}(\{f_m\}_{m=1}^M) \geq \sum_{n=1}^N \left[\frac{1}{N} \sum_{m=1}^M a_m^2 \right]^2 = \frac{1}{N} \left[\sum_{m=1}^M a_m^2 \right]^2.$$

Note this lower bound is achieved if any only if all of the eigenvalues of the frame operator are constant, that is, if and only if the frame operator is a constant multiple of the identity, which is equivalent to the sequence being a tight frame. □

We take care to note that Proposition 7 is a statement about lower bounds as opposed to minimal values. Of course, as orthogonal sequences of arbitrary lengths $\{a_m\}_{m=1}^{M}$ always exist provided $M \leq N$, the first bound is actually achieved, and is therefore a minimum.

However, while Proposition 7 guarantees that tight frames would be the minimizers of the frame potential, provided they exist, it does not guarantee their existence in the first place. Indeed, it is impossible for a tight frame to exist if the requisite lengths violate the fundamental inequality (5). This begs the questions: "What is the minimal value of the frame potential when the fundamental inequality is violated?" and "Provided the fundamental inequality holds, is the existence of a tight frame guaranteed?"

We answer these questions in the following section. Inspired by the approach of Benedetto and Fickus, we find the global minimizers of our frame potential by first characterizing the local minimizers. We now state their main result.

Theorem 8 (Benedetto & Fickus [3]) *Given positive integers M and N, consider the frame potential restricted to M copies of the unit sphere,* $\mathrm{FP} : S(1, \ldots, 1) \to \mathbb{R}$. *For this function,*

(1) every local minimizer is also a global minimizer,
(2) if $M \leq N$, the minimum value is N, and the minimizers are precisely the orthonormal sequences in $\mathbb{H}_N$,
(3) if $M \geq N$, the minimum value is M^2/N, and the minimizers are precisely the unit-norm tight frames for $\mathbb{H}_N$.

Thus, in the special case when all the spheres are radius 1, this theorem shows that the physical generalization of orthonormal bases is equivalent to their algebraic generalization. That is, that the minimizers of the frame potential correspond to tight frames, provided $M \geq N$.

By removing the restriction upon the radii, we are able to not only generalize this result, but also exploit the generality to characterize the lengths of a tight frame's elements.

5 The Physical Interpretation of the Fundamental Inequality

In the previous sections, we established how the intuitive concept of the frame force lead naturally to the definition of the frame potential. We subsequently made a case for the study of the frame potential in its own right, though in point of fact, none of those results truly exploited the physical interpretation of frames.

We consequently shift our focus in this section towards addressing some intriguing results that clarify the intuition, and, more importantly, use the physical rationale as a means of obtaining new results in frame theory. Our greatest success lies in the characterization of the lengths of the elements of finite tight frames, a result which superficially appears unrelated to the frame force, but is nevertheless only fully explained in that context.

To this end, we study how the theory of frames is connected to the application of several standard notions of classical mechanics to the frame force.

5.1 Points at equilibrium

Intuitively, a system of particles is in equilibrium under a force when no small perturbation of the system results in lesser potential energy. In this respect, the sequences of points in equilibrium under the frame force are precisely the local minimizers of the frame potential. Having restricted the movement of our points to spheres of fixed yet arbitrary radii, we now characterize the local minimizers of the frame potential on the corresponding restricted domain.

To be precise, though the frame potential

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^M \sum_{n=1}^M |\langle f_m, f_n\rangle|^2,$$

may be defined for any sequence $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$, we instead restrict the domain to be $S(a_1, \ldots, a_M)$, as defined in the previous section,

$$S(a_1, \ldots, a_M) \equiv S(a_1) \times \cdots S(a_M), \quad S(a_m) = \{f \in \mathbb{H}_N : \|f\| = a_m\}.$$

We endeavor to characterize the local minimizers of $\mathrm{FP} : S(a_1, \ldots, a_M) \to \mathbb{R}$.

We note that on the surface, finding a nontrivial characterization of the minimizers seems like an impossible task, since the parallel problem corresponding to the more intuitive Coulomb force has never been solved. However, as we dig a little deeper, we will see that the frame potential is mathematically much easier to understand than the Coulomb potential. Even more remarkably, the qualitative character of the local minimizers is completely determined by the distribution of the requisite lengths $\{a_m\}$.

We begin our formal investigation with a trivial result that concerns nothing more than sequences of real numbers.

Proposition 9 *Given any sequence $\{c_m\}_{m=1}^M \subset \mathbb{R}$ with $c_1 \geq \cdots \geq c_M \geq 0$, and any $N \leq M$, there is a unique index N_0 with $1 \leq N_0 \leq N$, such that the* *inequality*

$$(N - n)c_n > \sum_{m=n+1}^{M} c_m \tag{14}$$

holds for $1 \leq n < N_0$*, while the opposite inequality*

$$(N - n)c_n \leq \sum_{m=n+1}^{M} c_m \tag{15}$$

holds for $N_0 \leq n \leq N$*.*

PROOF. Let $\mathcal{I}$ be the set of indices n such that (15) holds, namely $\mathcal{I} = \{n : (N-n)c_n \leq \sum_{m=n+1}^{M} c_n\}$. Clearly $\mathcal{I} \neq \emptyset$, since $N \in \mathcal{I}$. Also, if $n \in \mathcal{I}$, then $n+1 \in \mathcal{I}$, since,

$$\begin{aligned}[N - (n+1)]c_{n+1} &= -c_{n+1} + (N-n)c_{n+1}, \\ &\leq -c_{n+1} + (N-n)c_n, \\ &\leq -c_{n+1} + \textstyle\sum_{m=n+1}^{M} c_m, \\ &= \textstyle\sum_{m=n+2}^{M} c_m.\end{aligned}$$

Thus, N_0 is uniquely defined as the minimum index in $\mathcal{I}$. □

Heuristically speaking, this result provides a means of measuring how evenly distributed the magnitudes of a sequence of scalars are. To be precise, given a nonnegative decreasing sequence $\{c_m\}_{m=1}^{M}$, one compares the individual terms c_m to the "average" [8] of the subsequent terms $\sum_{n=m+1}^{M} c_n/(N-m)$. Thus, the index N_0 provided by the previous result corresponds to the precise point in the sequence where the terms cease to be larger than the smaller terms on "average." Generally speaking, one expects the index N_0 to be small if the magnitudes of the scalars $\{c_m\}$ are somewhat evenly distributed. On the other hand, when the values $\{c_m\}$ are widely varying, the index N_0 is typically large.

Returning to the search for a characterization of the local minimizers of the frame potential, we recall the physical intuition that originally inspired this problem. In the context of the weighted frame force, a vector f_m on the sphere $S(a_m)$ is identified as being a particle of mass a_m^2 located at the "projected" vector $f_m/\|f_m\|$ on the unit sphere. Given any sequence $\{a_m\}_{m=1}^{M}$ with $a_1 \geq \cdots \geq a_m > 0$, we apply Proposition 9 to the corresponding sequences of masses $\{a_m^2\}_{m=1}^{M}$. In so doing, we divide the masses into two categories, namely those that are stronger than their lesser masses on average, and those that are not. The specific characterization of the local minimizers is given in terms of these two categories.

[8] Of course, this is not a true mean average unless $N = M$.

Theorem 10 *Given a sequence* $\{a_m\}_{m=1}^M \subset \mathbb{R}$ *such that* $a_1 \geq \cdots \geq a_m > 0$*, and any* $N \leq M$*, let* N_0 *denote the smallest index* n *for which ,*

$$(N-n)a_n^2 \leq \sum_{m=n+1}^{M} a_m^2,$$

holds (cf. Proposition 9). Then, any local minimizer of the frame potential $\mathrm{FP} : S(a_1, \ldots, a_M) \to \mathbb{R}$ *is of the form*

$$\{f_m\}_{m=1}^M = \{f_m\}_{m=1}^{N_0-1} \cup \{f_m\}_{m=N_0}^M,$$

where $\{f_m\}_{m=1}^{N_0-1}$ *is an orthogonal set for whose orthogonal complement* $\{f_m\}_{m=N_0}^M$ *forms a tight frame.*

The proof of Theorem 10 heavily relies upon a deep intuitive understanding of the frame potential. Thus, we recommend a careful study of the proof to anyone interested in understanding the subtleties of the frame force. We nevertheless relegate the proof of Theorem 10 to the appendix, due to the length and technical nature of the argument.

In any case, the strength of the result of Theorem 10 is well worth the time and energy required for the proof. Indeed, the remaining significant results of this paper are near immediate consequences of Theorem 10. However, we reserve our further comments about the intuitive interpretation of Theorem 10 until after we have stated its corollaries.

5.2 Optimal energy

Generally speaking, there is no single, well-defined "optimal" means of distributing points on spheres. That is, even in the low-dimensional real case of distributing points on the unit sphere $S(1) \subset \mathbb{R}^3$, there are many competing notions of what it means to be optimally distributed. Each one of these notions is worthy of study in their own right, with the differing notions of equality arising from different applications.

With respect to Coulomb's Law, the "optimal" means of distributing M electrons on a conductive spherical shell corresponds to a minimization of the potential energy

$$\sum_{m=1}^{M} \sum_{n \neq m} \frac{1}{\|x_m - x_n\|}.$$

Of course, any global minimizer of this potential is necessarily a local minimizer, so that the optimal arrangement is necessarily in equilibrium.

However, not all arrangements of points in equilibrium under Coulomb's Law are necessarily optimal. For an intriguing example of this phenomenon, it is known that the twenty vertices one obtains by inscribing a dodecahedron inside the unit sphere are in equilibrium under Coulomb's law. Nevertheless, it is also known that there are other arrangements of twenty electrons on the sphere that possess a lesser potential value.

Indeed, when dealing with an arbitrary force, one should not foolhardily presume that a characterization of sets in equilibrium should in any way lead to a subsequent characterization of optimal arrangements.

We were therefore quite surprised to discover that this is precisely what we could do when dealing with the frame force. In fact, every arrangement of points which is in equilibrium under the frame force is necessarily an optimal arrangement. That is, every local minimizer of the frame potential is also a global minimizer. Consequently, the characterization of Theorem 10 applies to all minimizers, as summarized in the following result.

Corollary 11 *Given a sequence $\{a_m\}_{m=1}^M \subset \mathbb{R}$ with $a_1 \geq \cdots \geq a_M > 0$, and any $N \leq M$, let N_0 denote the smallest index n such that,*

$$(N-n)a_n^2 \leq \sum_{m=n+1}^{M} a_n^2,$$

holds. Then, for the frame potential $\mathrm{FP} : S(a_1, \ldots, a_M) \rightarrow \mathbb{R}$,

(1) the minimal value is $\sum_{m=1}^{N_0-1} a_m^4 + \frac{1}{N-N_0+1}\left(\sum_{m=N_0}^{M} a_m^2\right)^2$,
(2) any local minimizer is also a global minimizer,
(3) the minimizers are precisely those sequences where $\{f_m\}_{m=1}^{N_0-1}$ is an orthogonal set for whose orthogonal complement $\{f_m\}_{m=N_0}^{M}$ forms a tight frame.

PROOF. To begin, we compute the frame potential for any sequence of the form $\{f_m\}_{m=1}^{N_0-1} \cup \{f_m\}_{m=N_0}^{M}$, where $\{f_m\}_{m=1}^{N_0-1}$ is an orthogonal sequence for whose orthogonal complement the sequence $\{f_m\}_{m=N_0}^{M}$ is a tight frame. Thus, for $m = 1, \ldots, N_0 - 1$, $\langle f_m, f_n \rangle = 0$ for $n \neq m$, and so,

$$\begin{aligned}\mathrm{FP}(\{f_m\}_{m=1}^{M}) &= \sum_{m=1}^{N_0-1} |\langle f_m, f_m \rangle|^2 + \sum_{m=N_0}^{M} \sum_{n=N_0}^{M} |\langle f_m, f_n \rangle|^2, \\ &= \sum_{m=1}^{N_0-1} a_m^4 + \mathrm{FP}(\{f_m\}_{m=N_0}^{M}).\end{aligned}$$

In accordance with Corollary 2, the fact that $\{f_m\}_{m=N_0}^{M}$ is a tight frame for the $(N - N_0 + 1)$-dimensional orthogonal complement implies that

$$\mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^{N_0-1} a_m^4 + \frac{\left(\sum_{m=N_0}^M a_m^2\right)^2}{N - N_0 + 1}. \tag{16}$$

In light of Theorem 10, we observe that any local minimizer of the frame potential is necessarily of this form, and consequently attains the potential value given in (16).

Next, we note that $S(a_1, \ldots, a_M)$ is compact, being a Cartesian product of spheres. The continuity of the frame potential upon this compact domain then guarantees that a global minimizer exists. As any global minimizer is also a local minimizer, the value given in (16) is indeed the global minimum, yielding the first claim.

Since all local minimizers attain that same value, the second claim is demonstrated.

To address the third claim, note that any sequence of such form has already been shown to attain the minimal value, while Theorem 10 yields the converse implication. □

To provide an intuitive context for this result, recall that the frame potential of a sequence $\{f_m\}_{m=1}^M$ with $\|f_m\| = a_m$, corresponds to the potential energy of a physical system. To be precise, the frame potential is the potential energy for a system of M particles under the weighted frame force,

$$\mathrm{WFF} : [0, \infty) \times [0, \infty) \times S \times S \to \mathbb{H}_N,$$
$$\mathrm{WFF}(w_1, w_2, g_1, g_2) = w_1 w_2 \langle g_1, g_2 \rangle (g_1 - g_2),$$

where the m particle is of mass $w_m = a_m^2$ and is located at the "projected" vector $g_m = f_m / \|f_m\|$.

As stated earlier, the inner product term within the expression for the frame force yields a force which is repulsive when the angle between two points is acute, and attractive when the angle is obtuse. When, as in Corollary 11, we minimize the frame potential, this intuition corresponds to the points seeking to make all other points orthogonal to themselves.

However, when $N \leq M$, that is, when the number of points exceeds the number of available dimensions, the points will begin to compete with each other for the available resources. In other words, though each point wants a single dimension for itself, the gap between supply and demand will compel at least some of the points to share.

Of course, the appearance of the weights in the expression of the frame force implies that the more massive points, that is, the points that correspond to frame elements of greater norm, are able to exert a greater push than the lesser points. It is therefore conceivable, though by no means obvious, that a single point much heavier than all of the other points could have enough strength so as to force all others into orthogonality.

Amazingly, Corollary 11 not only verifies that this phenomenon actually occurs, but also provides a precise quantitative characterization of the requisite disparity in the masses of the points. Specifically, if $N_0 > 1$, then by definition

$$a_1^2 > \frac{\sum_{m=2}^{M} a_m^2}{N-1}, \tag{17}$$

and any minimizer of the frame potential will necessarily have f_1 orthogonal to all other elements. As then $\{f_m\}_{m=2}^{M}$ spans an $(N-1)$-dimensional space, the inequality in (17) is naturally interpreted as saying that the point corresponding to f_1 has a greater ratio of mass to dimension spanned than all the remaining points combined. In effect, f_1 is so powerful so as to take its fill, and let the remaining points squabble over its leftovers.

If in addition we have $N_0 > 2$, then

$$a_2^2 > \frac{\sum_{m=3}^{M} a_m^2}{N-2},$$

and the preceeding scenario is repeated, albeit on a smaller scale. Note at this stage f_2 is no longer competing with f_1, as that fight has already been lost. Instead, the salient issue is whether f_2 is indeed more powerful on average than the remaining points $\{f_m\}_{m=3}^{M}$.

And so, to determine the nature of the minimizers of the frame potential, one first applies Proposition 9 to the sequence of masses $\{a_m^2\}_{m=1}^{M}$, as explicitly stated in Corollary 11. The index N_0 then separates the elements of the local minimizer $\{f_m\}_{m=1}^{M}$ into two groups, namely the points which are more powerful than their subordinates on average $\{f_m\}_{m=1}^{N_0-1}$, and the points which are weaker $\{f_m\}_{m=N_0}^{M}$. The strong points $\{f_m\}_{m=1}^{N_0-1}$ push all others out of their dimensions, yielding an orthogonal set. The weaker points $\{f_m\}_{m=N_0}^{M}$ are then forced to share the remaining $(N - N_0 + 1)$-dimensional space amongst themselves. By nevertheless seeking to be as orthogonal to each other as possible, the weaker points form a tight frame for their span, namely the orthogonal complement of the stronger points.

Thus, the elements of optimal frames are very interesting political creatures, in that their success in securing resources is determined by either attempting to completely subdue the weaker points, or, when this is impossible, forming an alliance with those same points so as to overcome a more powerful adversary. As such, these optimal frames are quite opportunistic.

As a natural consequence of these ideas, one is attracted to considering the special case of Corollary 11 when $N_0 = 1$, i.e. when the masses are distributed evenly enough so that no point is more powerful than the rest on average. In this case, every minimizer of the frame potential is a tight frame, and as such there is an entire realm of tight frames whose existence is now guaranteed. We now discuss this case in detail.

5.3 A balance of power

Up to this point, our work concerning a physical theory for frames has been focused upon understanding the physical behavior of a system of points under the influence of the frame force.

Though this is all well and good, we must acknowledge that the theory up to this point is somewhat unsatisfying. Indeed, we introduced the frame force as a tool to help us understand tight frames. However, up to this point we have been placing more emphasis upon the tool than upon the theory that the tool is supposed to be clarifying.

We therefore present our final result with satisfaction, as the statement may be understood by anyone with even a passing knowledge of frames, and superficially appears to have no relation to the physical theory. However, the justification of the statement, both formal and heuristic, depends upon a keen understanding of the frame force.

Corollary 12 *Given an N-dimensional Hilbert space $\mathbb{H}_N$ and a sequence of positive scalars $\{a_m\}_{m=1}^M$, there exists a tight frame $\{f_m\}_{m=1}^M$ for $\mathbb{H}_N$ of lengths $\|f_m\| = a_m$ for all $m = 1, \ldots, M$ if and only if,*

$$\max_{m=1,\ldots,M} a_m^2 \leq \frac{1}{N} \sum_{m=1}^{M} a_m^2, \tag{18}$$

or equivalently, if and only if $\sqrt{N}\|a_m\|_\infty \leq \|a_m\|_2$.

PROOF. Though the critical details of the following argument may also be found in the proof of Theorem 10 in the appendix, we reiterate the main ideas for the sake of clarity. To begin, recall that the argument detailing the necessity of the fundamental inequality (18) is found in Corollary 2.

To demonstrate its sufficiency, take any sequence $\{a_m\}_{m=1}^M$ that satisfies (18).

Without loss of generality, we may assume that $\{a_m\}_{m=1}^M$ is arranged in decreasing order. As the frame potential $\mathrm{FP} : S(a_1, \ldots, a_M) \to \mathbb{R}$ is a continuous function over a compact set, it possesses a global, and hence local, minimizer $\{f_m\}_{m=1}^M \in S(a_1, \ldots, a_M)$. By Theorem 10, we therefore have that

$$\{f_m\}_{m=1}^M = \{f_m\}_{m=1}^{N_0-1} \cup \{f_m\}_{m=N_0}^M,$$

where $\{f_m\}_{m=1}^{N_0-1}$ is an orthogonal set for whose orthogonal complement $\{f_m\}_{m=N_0}^M$ is a tight frame, and where N_0 is the minimum index n such that

$$(N-n)a_n^2 \leq \sum_{m=n+1}^M a_m^2.$$

Since we are assuming,

$$Na_1^2 = N(\max a_m)^2 \leq \sum_{m=1}^M a_m^2,$$

we have,

$$(N-1)a_1^2 \leq \sum_{m=2}^M a_m^2.$$

Thus $N_0 = 1$, and so $\{f_m\}_{m=1}^M$ is a tight frame for the orthogonal complement of the empty set, namely a tight frame for $\mathbb{H}_N$. $\square$

This result justifies our labeling of (18) as the *fundamental inequality on the lengths of a tight frame's elements*, or simply as the *fundamental inequality* when the context is understood. Having this definition, we take a moment to restate the particular form of Corollary 11 when $N_0 = 1$.

Corollary 13 *Consider the frame potential restricted to spheres whose radii satisfy the fundamental inequality. Then,*

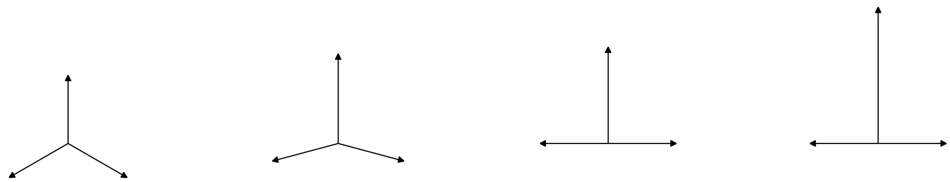
(1) the minimal value is $(\sum_{m=1}^M a_m^2)^2/N$,
(2) any local minimizer is a global minimizer,
(3) the minimizers are precisely the tight frames for $\mathbb{H}_N$ *of those lengths.*

We also note that for any $N \leq M$, the fundamental inequality will always hold for constant sequences. In particular, for the unit-norm case $a_m = 1$ for all m, our results then reduce to the main results of Benedetto and Fickus [3].

To conclude, we present several ways in which the fundamental inequality may be understood and interpreted. As usual, a simple low-dimensional example may be used to understand frame phenomena in general.

In particular, to understand the qualitative change between sequences satisfying the fundamental inequality and those that do not, we consider a perturbed

Fig. 4. As the one element grows in power, others are forced into orthogonality. In the last image, the fundamental inequality is violated, and the resulting sequence is no longer a tight frame.



version of the so-called Mercedes-Benz UNTF of three elements for $\mathbb{R}^2$. That is, we begin with three equally spaced points on the unit circle, and study how the tight frame evolves as the length of a single point is increased, while the other lengths are held constant. Or, in the context of the parallel intuition of the weighted frame force, we consider how the equilibrium of a set of three points of equal mass evolves as the mass of a single point is increased.

That is, we consider the parametrized family of tight frames

$$\left\{ \begin{bmatrix} 0 \\ \sqrt{-2\cos 2\theta} \end{bmatrix}, \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$$

as θ decreases from $2\pi/3$ to $\pi/2$. The first three images of Figure 4 show the resulting arrangements for $\theta = 2\pi/3,\ 7\pi/12,$ and $\pi/2$ respectively. In effect, as the first element grows in strength, the other two elements are increasingly pulled into orthogonality, so as to preserve equilibirum.

However, $\theta = \pi/2$ is the critical parameter, as this is the last value for which the arrangement forms a tight frame. Indeed, our very definition becomes invalid for $\theta < \pi/2$. At this point, our elements are of lengths $\{\sqrt{2},\ 1,\ 1\}$, which is precisely the point at which the Fundamental Inequality becomes equality.

Beyond this point, we continue to increase the length of the first element, and track the evolution of the equilibrium. However, as soon as the length of the first element exceeds $\sqrt{2}$, the second and third elements cease to move, as it is not possible to become "more orthogonal" than already being orthogonal.

The last image of Figure 4 demonstrates the equilibrium state of the vectors of lengths $\{2,\ 1,\ 1\}$. We note that the fundamental inequality is violated, namely that the application of Proposition 9 to $\{4,\ 1,\ 1\}$ with $N = 2$ yields $N_0 = 2$. And, in accordance with Corollary 11 with $N_0 = 2$, the resulting arrangement is not a tight frame, but rather a single vector along with the remaining two forming a tight frame for the remaining one-dimensional orthogonal complement. Though this final sequence is still a minimizer of the frame potential and consequently as "orthogonal as possible," the sequence is not a tight frame due to the inequity in the lengths.

In fact, perhaps the moral of this entire story is that tight frames, at their very core, are creatures of balance. This perspective is reaffirmed by our earlier results about effective frame forces given in Proposition 3. Namely that, while each point generates a force field whose strength grows according to its weight, the total collection of weights must be distributed evenly enough so that the net effect of all points combined is zero.

Thus, in order for a tight frame to exist, even the largest component of the force field, namely that created by the most massive point, must be able to be cancelled out by the other points, working together if necessary. When the fundamental inequality is violated, the weights are unbalanced and this cancellation is impossible, regardless of the way in which the remaining points are arranged.

At an even more basic level, an intuitive justification for the existence of something like the fundamental inequality can be made from the very definition of a tight frame. That is, letting $\mathrm{Proj}_{f_m} f$ denote the orthogonal projection of f onto an axis spanned by f_m, we have that $\{f_m\}_{m=1}^M$ is a tight frame for $\mathbb{H}_N$ if and only if

$$\sum_{m=1}^{M} a_m^2 \mathrm{Proj}_{f_m} f = \sum_{m=1}^{M} \langle f, f_m \rangle f_m = Af$$

for all $f \in \mathbb{H}_N$. However, when one element becomes disproportionately large compared to the rest, the corresponding mass term skewers the entire sum. Geometrically, the action of the frame operator then stretches that axis in a way that cannot be compensated by the remaining points. The frame operator behaves more like a projection onto an axis as opposed to a scaled version of the identity.

5.4 The fundamental inequality in wireless systems

As we mentioned in the introduction, Viswanath and Anantharam encountered our "fundamental inequality" while investigating the capacity region in synchronous Code-Division Multiple Access (CDMA) systems.

In a CDMA system, there are M users[9] who share the available spectrum. The sharing is achieved by "scrambling" M-dimensional *user vectors* into smaller, N-dimensional vectors. In terms of frame theory, this scrambling corresponds to the application of a synthesis operator S corresponding to M distinct N-dimensional *signature* vectors of length $\sqrt{N}$. Noise-corrupted versions of these synthesized vectors arrive at a receiver, where the signature vectors are used to help extract the original user vectors.

[9] We are transcribing the notation from [35] to match our own.

Viswanath and Anantharam showed that the design of the optimal signature matrix S depends upon the *powers* $\{p_m\}_{m=1}^M$ of the individual users. In particular, they divided the users into two classes: those that are *oversized* and those that are not, by applying the idea of Proposition 9 to $\{p_m\}_{m=1}^M$. While the oversized users are assigned orthogonal channels for their personal use, the remaining users have their signature vectors designed so as to be Welch Bound Equality (WBE) sequences. As the Welch Bound inequality corresponds to the bounding of the frame potential from below, the WBE sequences correspond to tight frames.

When no user is oversized, that is, when the fundamental inequality is satisfied, Viswanath and Anantharam show the optimal signature sequences S must satisfy $SDS^* = p_{\text{tot}}I$, where D is a diagonal matrix whose entries are the powers $\{p_m\}_{m=1}^M$, and where $p_{\text{tot}} = \sum_{m=1}^M p_m$.

By letting $F = D^{1/2}S^*/\sqrt{N}$, this problem reduces to finding an $M \times N$ matrix F whose mth row is of norm $\sqrt{p_m}$, and such that $F^*F = (p_{\text{tot}}/N)I$. That is, their problem reduces to finding a tight frame for $\mathbb{H}_N$ of lengths $\{\sqrt{p_m}\}_{m=1}^M$. While Viswanath and Anantharam gave one solution to this problem, using an explicit construction [10] , we have characterized all solutions to this problem using a physical interpretation of frame theory.

The fact that the same results were independently obtained in this radically different setting is immensely gratifying for us, reaffirming our belief that our physical interpretation may be used to motivate, prove and explain new results in frame theory, and elsewhere.

6 Conclusion

Benedetto and Fickus showed that both orthonormal bases and unit-norm tight frames arise as minimizers of the same quantity: the frame potential. In so doing, they opened a door unto a new, intuitive approach for the understanding of finite frames. In this paper, we built upon their work by both extending their notions to a more general setting, and by dealing with many subtle issues that they overlooked by restricting themselves to the unit-norm setting.

In so doing, we further justified the perception of tight frames as maximally orthogonal sequences. We found a "fundamental inequality" which characterizes the lengths of a tight frame's elements, and, when this inequality was violated, computed those sequences which were as close to being tight as possible. In addition, we were able to interpret this inequality physically, being

[10] Which should be further investigated as a technique of tight frame construction.

a statement of competition over limited resources. Finally, we discussed how both our theory and interpretation were discovered independently in a very different setting, further validating the intrinsic value of these concepts.

A Proof of Theorem 10

Our argument roughly parallels the proof of Theorem 7.4 of [3], in which the authors restricted themselves to sequences of normalized vectors. Indeed, a few of their arguments may be followed verbatim even when the norms are arbitrary. However, most of our conclusions will require a significant modification of their techniques.

To begin, let $\{f_m\}_{m=1}^M$ be a local minimizer of,

$$\mathrm{FP} : S(a_1, \dots, a_M) \to \mathbb{R}, \mathrm{FP}(\{f_m\}_{m=1}^M) = \sum_{m=1}^{M} \sum_{n=1}^{M} |\langle f_m, f_n \rangle|^2,$$

with corresponding analysis operator F and frame operator F^*F. Since F^*F is positive definite, we may arrange the distinct eigenvalues $\{\lambda_j\}_{j=1}^J$ of the frame operator such that $\lambda_1 > \cdots > \lambda_J \geq 0$. In addition, since F^*F is self-adjoint, the corresponding eigenspaces $\{E_j\}_{j=1}^J$ are mutually orthogonal.

Much of the following argument will relate the elements of the minimizing sequence $\{f_m\}_{m=1}^M$ to the eigenspaces $\{E_j\}_{j=1}^J$. Towards this end, we let,

$$\mathcal{I}_j = \{m : f_m \in E_j\} \subseteq \{1, \dots, M\},$$

for each $j = 1, \dots, J$.

Having fixed our notation, we now present a brief outline of the proof. We may decompose the entire argument into seven claims:

1. Each f_m is an eigenvector for F^*F.
2. For $\lambda_j > 0$, $\{f_m\}_{m\in\mathcal{I}_j}$ is a λ_j-tight frame for E_j.
3. For $j < J$, $\{f_m\}_{m\in\mathcal{I}_j}$ is linearly independent.
4. For $j < J$, $\{f_m\}_{m\in\mathcal{I}_j}$ is orthogonal, and $\lambda_j = a_m^2$.
5. $\lambda_J = \sum_{m\in\mathcal{I}_J} a_m^2/(N - M + |\mathcal{I}_J|)$.
6. $\{N_0, \dots, M\} \subseteq \mathcal{I}_J$.
7. $\{N_0, \dots, M\} = \mathcal{I}_J$.

We take care to note that not all of these results are of equal significance. In fact, the proof of the third claim is much more difficult than the others, being a technical result that nevertheless requires a deep intuitive understanding.

However, the outline does serve to delineate the radically varying techniques used throughout the proof. And as we shall demonstrate, each claim is built upon the previous claims, with each contributing a little more knowledge about the minimizer. This layered approach will allow us to incrementally determine the necessary structure.

Claim 1: Each f_m is an eigenvector for F^*F.

The argument is precisely that of Theorem 7.3 of [3], to which we refer any interested reader. We nevertheless reiterate the basic argument, so as to provide a context for the remainder of the proof. The fundamental idea is that any minimizer of a function of M variables is necessarily composed of M minimizers of functions of one variable.

In particular, for any index $m' = 1, \ldots, M$, we restrict the domain of the frame potential by fixing all but the m'th argument as the coordinates of the minimizer $\{f_m\}_{m \neq m'}$. In other words, we view the frame potential as the function,

$$\mathrm{FP} : S(a_{m'}) \to \mathbb{R}, \quad \mathrm{FP}(f) = 2 \sum_{m \neq m'} |\langle f, f_m \rangle|^2 + a_{m'}^2 + \mathrm{FP}(\{f_m\}_{m \neq m'}).$$

Thus $f = f_{m'}$ is a local minimizer of the real-valued function FP, whose domain is defined by the condition $\|f\|^2 = a_{m'}^2$. Consequently, for $f = f_{m'}$ the well-known Lagrange equations of multivariable calculus must be satisfied. Namely, there exists some scalar c such that,

$$\nabla \mathrm{FP}(f) = c \nabla \|f\|^2,$$

for $f = f_{m'}$. A careful simplification of this equality reveals that $F^*F f_{m'} = (1 + \frac{1}{2}c) f_{m'}$, namely that $f_{m'}$ is an eigenvector of F^*F. Since m' is arbitrary, we have our first claim.

Note that we may now partition our minimizer $\{f_m\}_{m=1}^M$ according to the mutually orthogonal eigenspaces of F^*F, namely,

$$\{f_m\}_{m=1}^M = \bigcup_{j=1}^J \{f_m\}_{m \in \mathcal{I}_j}.$$

Claim 2: For $\lambda_j > 0$, $\{f_m\}_{m \in \mathcal{I}_j}$ is a λ_j-tight frame for E_j.

Take any j with $\lambda_j > 0$, and let F_j be the analysis operator corresponding to the sequence $\{f_m\}_{m \in \mathcal{I}_j}$, regarded as a subset of E_j. Given any $f \in E_j$, we have,

$$\lambda f = F^*Ff = \sum_{m=1}^{M} \langle f, f_m \rangle f_m.$$

However, for $m \notin \mathcal{I}_j$, f_m belongs to an eigenspace of F^*F distinct from E_j, and consequently $\langle f, f_m \rangle = 0$. Therefore,

$$\lambda f = \sum_{m=1}^{M} \langle f, f_m \rangle f_m = \sum_{m \in \mathcal{I}_j} \langle f, f_m \rangle f_m = F_j{}^* F_j f.$$

As $f \in E_j$ is arbitrary, the frame operator $F_j{}^*F_j : E_j \to E_j$ satisfies $F_j{}^*F_j = \lambda_j I$, and the second claim is proven.

Our minimizer $\{f_m\}_{m=1}^M$ is now partitioned as a collection of mutually orthogonal tight frames, bearing in mind the possible exception that occurs when $\lambda_J = 0$.

Claim 3: For $j < J$, $\{f_m\}_{m \in \mathcal{I}_j}$ is linearly independent.

We prove by contradiction, i.e. we assume that for some $j < J$, the collection $\{f_m\}_{m \in \mathcal{I}_j}$ is linearly dependent, and show that $\{f_m\}_{m=1}^M$ is not a local minimizer of the frame potential.

Specifically, given $0 < \varepsilon < 1$ we construct $\{g_m(\varepsilon)\}_{m=1}^M \in S(a_m, \ldots, a_M)$ satisfying both,

$$\lim_{\varepsilon \to 0} \{g_m(\varepsilon)\}_{m=1}^M = \{f_m\}_{m=1}^M,$$

in the standard topology of $S(a_m, \ldots, a_M)$, and,

$$\mathrm{FP}(\{g_m(\varepsilon)\}_{m=1}^M) < \mathrm{FP}(\{f_m\}_{m=1}^M),$$

for sufficiently small ε. That is, we provide sequences $\{g_m(\varepsilon)\}_{m=1}^M$ arbitrarily close to $\{f_m\}_{m=1}^M$ that possess a strictly lesser potential value. The conclusion that $\{f_m\}_{m=1}^M$ is not a local minimizer of the frame potential is then immediate.

To begin, let h be any unit eigenvector in the "lowest" eigenspace, i.e. let $h \in E_J$ with $\|h\| = 1$. Next, we note that since $\{f_m\}_{m \in \mathcal{I}_j}$ is linearly dependent, there exists complex scalars $\{z_m\}_{m \in \mathcal{I}_j}$ not all 0 such that,

$$\sum_{m \in \mathcal{I}_j} \overline{z_m} a_m f_m = 0.$$

By rescaling if necessary, we may assume without loss of generality that $|z_m| \leq 1/2$ for all $m \in \mathcal{I}_j$.

For any $0 < \varepsilon < 1$, we define $\{g_m(\varepsilon)\}_{m=1}^M$,

$$g_m(\varepsilon) = \begin{cases} (1-\varepsilon^2|z_m|^2)^{1/2} f_m + \varepsilon a_m z_m h, & \text{for } m \in \mathcal{I}_j, \\ f_m, & \text{for } m \notin \mathcal{I}_j. \end{cases}$$

We first note that since $\varepsilon^2|z_m|^2 \leq 1$, then $\{g_m(\varepsilon)\}_{m=1}^M$ is well-defined. It is also clear that $\lim\{g_m(\varepsilon)\}_{m=1}^M = \{f_m\}_{m=1}^M$.

Furthermore, since any $f_m \in E_j$ is necessarily orthogonal to $h \in E_J$, then,

$$\|g_m(\varepsilon)\|^2 = \|(1-\varepsilon^2|z_m|^2)^{1/2} f_m + \varepsilon z_m a_m h\|^2 \\ = (1-\varepsilon^2|z_m|^2)\|f_m\|^2 + \varepsilon^2|z_m|^2 a_m^2 \|h^2\| = a_m^2,$$

for any $m \in \mathcal{I}_j$. On the other hand, if $m \notin \mathcal{I}_j$, then we still have that $\|g_m(\varepsilon)\| = \|f_m\| = a_m$. Consequently, we indeed have that $\{g_m(\varepsilon)\}_{m=1}^M \in S(a_m, \ldots, a_M)$.

Therefore, all that is required to complete the proof of the third claim is the somewhat tedious demonstration that $\mathrm{FP}(\{g_m(\varepsilon)\}_{m=1}^M) < \mathrm{FP}(\{f_m\}_{m=1}^M)$ for all sufficiently small ε. To aid with the notation, we usually abbreviate $g_m(\varepsilon)$ by g_m from this point onward.

As $\{g_m\}_{m=1}^M$ was constructed by perturbing only a portion of $\{f_m\}_{m=1}^M$, while leaving the rest unchanged, we accordingly arrange the terms of $\mathrm{FP}(\{g_m\}_{m=1}^M)$ into three groups,

$$\mathrm{FP}(\{g_m\}_{m=1}^M) = \sum_{m\notin\mathcal{I}_j}\sum_{m\notin\mathcal{I}_j} |\langle g_m, g_n\rangle|^2, \tag{A.1}$$

$$+ 2\sum_{m\in\mathcal{I}_j}\sum_{n\notin\mathcal{I}_j} |\langle g_m, g_n\rangle|^2, \tag{A.2}$$

$$+ \sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j} |\langle g_m, g_n\rangle|^2. \tag{A.3}$$

For the first group of terms (A.1), there is no difference between f and g,

$$\sum_{m\notin\mathcal{I}_j}\sum_{m\notin\mathcal{I}_j} |\langle g_m, g_n\rangle|^2 = \sum_{m\notin\mathcal{I}_j}\sum_{m\notin\mathcal{I}_j} |\langle f_m, f_n\rangle|^2.$$

For the second group of terms (A.2), one term in each inner product has been perturbed while the other remains the same. Since $f_m \in \mathcal{I}_j$ is orthogonal to $f_n \notin \mathcal{I}_j$, then (A.2) becomes

$$2\sum_{m\in\mathcal{I}_j}\sum_{n\notin\mathcal{I}_j} |\langle (1-\varepsilon^2|z_m|^2)^{1/2} f_m + \varepsilon a_m z_m h, f_n\rangle|^2, \\ = 2\sum_{m\in\mathcal{I}_j}\sum_{n\notin\mathcal{I}_j} \varepsilon^2 a_m^2 |z_m|^2 |\langle h, f_n\rangle|^2 = 2\varepsilon^2 \sum_{m\in\mathcal{I}_j} a_m^2|z_m|^2 \sum_{n\in\mathcal{I}_j} |\langle h, f_n\rangle|^2.$$

To simplify this expression, we note that $h \in E_J$ is orthogonal to f_n with $n \in \mathcal{I}_j$, and thus,

$$\begin{aligned}\lambda_J = \langle \lambda_J h, h\rangle = \langle F^*Fh, h\rangle &= \sum_{n=1}^{M} |\langle h, f_n\rangle|^2, \\ &= \sum_{n\in\mathcal{I}_j} |\langle h, f_n\rangle|^2 + \sum_{n\notin\mathcal{I}_j} |\langle h, f_n\rangle|^2 = \sum_{n\in\mathcal{I}_j} |\langle h, f_n\rangle|^2.\end{aligned}$$

To summarize, we have that the terms of (A.2) reduce to,

$$2\varepsilon^2\lambda_J \sum_{m\in\mathcal{I}_j} a_m^2|z_m|^2.$$

For the third group of terms (A.3), the story becomes quite complicated, as both arguments of the inner product are perturbed simultaneously. We begin with a direct expansion of $|\langle g_m, g_n\rangle|^2$ for $m, n \in \mathcal{I}_j$, taking advantage of the fact that $\langle f_m, h\rangle = 0 = \langle f_n, h\rangle$ since $f_m, f_n \in E_j$, while $h \in E_J$,

$$\begin{aligned}|\langle g_m, g_n\rangle|^2 &= |\langle \sqrt{1-\varepsilon^2|z_m|^2} f_m + \varepsilon a_m z_m h, \sqrt{1-\varepsilon^2|z_n|^2} f_n + \varepsilon a_n z_n h\rangle|^2, \\ &= |\sqrt{(1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)}\langle f_m, f_n\rangle + \varepsilon^2 a_m a_n z_m \overline{z_n}|^2, \\ &= (1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)|\langle f_m, f_n\rangle|^2, \\ &\quad + 2\Re\sqrt{(1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)}\langle f_m, f_n\rangle \varepsilon^2 a_m a_n \overline{z_m} z_n, \\ &\quad + a_m^2 a_n^2 |z_m|^2 |z_n^2| \varepsilon^4, \\ &= \{1 - (|z_m|^2 + |z_n|^2)\varepsilon^2 + |z_m|^2|z_n|^2\varepsilon^4\}|\langle f_m, f_n\rangle|^2, \\ &\quad + 2\varepsilon^2\{1 + \sqrt{(1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)} - 1\}\Re\langle a_m\overline{z_m} f_m, a_n\overline{z_n} f_n\rangle, \\ &\quad + a_m^2 a_n^2 |z_m|^2 |z_n^2| \varepsilon^4.\end{aligned}$$

To continue, we arrange the terms in the expansion,

$$\begin{aligned}|\langle g_m, g_n\rangle|^2 &= |\langle f_m, f_n\rangle|^2 - \varepsilon^2(|z_m|^2 + |z_n|^2)|\langle f_m, f_n\rangle|^2, \\ &\quad + 2\varepsilon^2\Re\langle a_m\overline{z_m} f_m, a_n\overline{z_n} f_n\rangle, && \text{(A.4)} \\ &\quad + \varepsilon^4|z_m|^2|z_n|^2|\langle f_m, f_n\rangle|^2, && \text{(A.5)} \\ &\quad + 2\varepsilon^2[\sqrt{(1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)} - 1]\Re\langle a_m\overline{z_m} f_m, a_n\overline{z_n} f_n\rangle, && \text{(A.6)} \\ &\quad + \varepsilon^4 a_m^2 a_n^2 |z_m|^2 |z_n^2|. && \text{(A.7)}\end{aligned}$$

As we only need to understand the value of $|\langle g_m(\varepsilon), g_n(\varepsilon)\rangle|^2$ for small values of ϵ, it suffices to rougly estimate the higher order terms. In particular, the coefficients of ε^4 in (A.5) and (A.7) are bounded, and consequently both of these terms may be absorbed into an ubiquitous $O(\varepsilon^4)$ term.

Furthermore, as $|z_m| \leq 1/2$ for all $m = 1, \dots, M$, a Taylor's Theorem expansion reveals the existence of a uniform constant C such that,

$$\sqrt{(1-\varepsilon^2|z_m|^2)(1-\varepsilon^2|z_n|^2)}-1 \leq C\varepsilon^2,$$

for all sufficiently small ε. The presence of an additional ε^2 in (A.6) then gives the entire terms may also be absorbed into an $O(\varepsilon^4)$ term. Indeed, the entire purpose of adding and subtracting 1 to the square root in the expansion of $|\langle g_m, g_n\rangle|^2$ was to draw off the dominant term (A.4).

Consequently, we may compute (A.3) as

$$\begin{aligned}\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle g_m, g_n\rangle|^2 = &\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 + O(\varepsilon^4),\\ &-\varepsilon^2\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}(|z_m|^2+|z_n|^2)|\langle f_m, f_n\rangle|^2, && \text{(A.8)}\\ &+2\varepsilon^2\Re\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}\langle a_m\overline{z_m}f_m, a_n\overline{z_n}f_n\rangle. && \text{(A.9)}\end{aligned}$$

To simplify (A.8), we note by symmetry that,

$$\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}(|z_m|^2+|z_n|^2)|\langle f_m, f_n\rangle|^2 = 2\sum_{m\in\mathcal{I}_j}|z_m|^2\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2.$$

Now, the second claim guarantees that $\{f_n\}_{n\in\mathcal{I}_j}$ is a λ_j-tight frame for E_j. And, for $m\in\mathcal{I}_j$, $f_m\in E_j$ and so,

$$\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 = \lambda_j\|f_m\|^2 = \lambda_j a_m^2.$$

Thus, (A.8) reduces to,

$$= -2\varepsilon^2\lambda_j\sum_{m\in\mathcal{I}_j}a_m^2|z_m|^2.$$

To simply (A.9), we observe that

$$\Re\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}\langle a_m\overline{z_m}f_m, a_n\overline{z_n}f_n\rangle = \Re\langle\sum_{m\in\mathcal{I}_j}a_m\overline{z_m}f_m, \sum_{n\in\mathcal{I}_j}a_n\overline{z_n}f_n\rangle.$$

We now recall that the scalars $\{z_m\}_{m\in\mathcal{I}_j}$ we originally chosen to satisfy

$$\sum_{m\in\mathcal{I}_j}a_m\overline{z_m}f_m = 0,$$

and consequently (A.9) is 0. In fact, our desire to eliminate this term from the expansion of the frame potential is the sole reason $\{g_m\}_{m=1}^M$ was defined in this manner in the first place. It is by no means an overstatement to say that the entire proof of this theorem, as well as the major results of this paper, depend critically upon this subtle fact.

Returning again to (A.3), we have,

$$\sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle g_m, g_n\rangle|^2 = \sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 - 2\varepsilon^2\lambda_j\sum_{m\in\mathcal{I}_j}a_m^2|z_m|^2 + O(\varepsilon^4),$$

which yields,

$$\begin{aligned}\mathrm{FP}(\{g_m(\varepsilon)\}_{m=1}^M) &= \sum_{m\notin\mathcal{I}_j}\sum_{m\notin\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 + 2\varepsilon^2\lambda_J\sum_{m\in\mathcal{I}_j}a_m^2|z_m|^2,\\ &+ \sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 - 2\varepsilon^2\lambda_j\sum_{m\in\mathcal{I}_j}a_m^2|z_m|^2 + O(\varepsilon^4),\end{aligned}$$

when combined with our previous work on (A.1) and (A.2).

To simplify even further, we note that $\langle f_m, f_n\rangle = 0$ for $m \in \mathcal{I}_j$ and $n \notin \mathcal{I}_j$, and therefore,

$$\begin{aligned}\mathrm{FP}(\{f_m\}_{m=1}^M) &= \sum_{m=1}^{M}\sum_{n=1}^{N}|\langle f_m, f_n\rangle|^2,\\ &= \sum_{m\notin\mathcal{I}_j}\sum_{m\notin\mathcal{I}_j}|\langle f_m, f_n\rangle|^2 + \sum_{m\in\mathcal{I}_j}\sum_{n\in\mathcal{I}_j}|\langle f_m, f_n\rangle|^2.\end{aligned}$$

Combining the remaining common terms yields our final, Taylor-like expansion of the frame potential of our perturbed points,

$$\mathrm{FP}(\{g_m(\varepsilon)\}_{m=1}^M) = \mathrm{FP}(\{f_m\}_{m=1}^M) + 2\varepsilon^2(\lambda_J - \lambda_j)\sum_{m\in\mathcal{I}_j}a_m^2|z_m|^2 + O(\varepsilon^4).$$

Now, by assumption $\lambda_J < \lambda_j$, and so $(\lambda_J - \lambda_j) < 0$. Furthermore, as the linear dependence constants $\{z_m\}_{m\in\mathcal{I}_j}$ are not all 0, we have that the coeffcient of ε^2 is strictly negative. As the ε^2 term dominates the $O(\varepsilon^4)$ term as $\varepsilon \to 0$, we indeed have that,

$$\mathrm{FP}(\{g_m(\varepsilon)\}_{m=1}^M) < \mathrm{FP}(\{f_m\}_{m=1}^M),$$

for all sufficiently small ε, a contradiction of the fact that $\{f_m\}_{m=1}M$ is a local minimizer of the frame potential. Our initial assumption, namely that $\{f_m\}_{m\in\mathcal{I}_j}$ is linearly dependent, must be incorrect, and so the third claim is proven.

Claim 4: For $j < J$, $\{f_m\}_{m\in\mathcal{I}_j}$ is orthogonal, and $\lambda_j = a_m^2$.

Since $j < J$, then $\lambda_j > \lambda_j \geq 0$, and the second claim states that $\{f_m\}_{m\in\mathcal{I}_j}$ is a λ_j-tight frame for E_j. Letting F_j be the corresponding analysis operator, we have that ${F_j}^*F_j = \lambda_j I$.

However, the third claim states that $\{f_m\}_{m\in\mathcal{I}_j}$ is linearly independent. As tight frames must also be spanning sets, $\{f_m\}_{m\in\mathcal{I}_j}$ is a basis for E_j. But, the basis elements $\{f_m\}_{m\in\mathcal{I}_j}$ form the columns of ${F_j}^*$, and consequently this matrix must be square.

And for square matrices, ${F_j}^*F_j = \lambda_j I$ automatically implies that $F_j{F_j}^* = \lambda_j I$. Consequently the columns of ${F_j}^*$ are orthogonal of uniform length $\sqrt{\lambda_j}$. Again, realizing that the columns of ${F_j}^*$ are formed by $\{f_m\}_{m\in\mathcal{I}_j}$, the fourth claim is demonstrated.

At this point, we may now partition the elements of the minimizer $\{f_m\}_{m=1}^M$ into two sets, namely,

$$\{f_m\}_{m=1}^M = \bigcup_{j=1}^J \{f_m\}_{m\in\mathcal{I}_j} = \{f_m\}_{m\in\mathcal{I}_J^C} \cup \{f_m\}_{m\in\mathcal{I}_J},$$

where $\{f_m\}_{m\in\mathcal{I}_J^C}$ is an orthogonal collection whose orthogonal complement contains $\{f_m\}_{m\in\mathcal{I}_J}$.

Claim 5: $\lambda_J = \sum_{m\in\mathcal{I}_J} a_m^2/(N - M + |\mathcal{I}_J|)$.

We first prove that $\lambda_J > 0$ by means of contradiction.

Though the first claim guarantees that each element of $\{f_m\}_{m=1}^M$ is indeed an eigenvector of F^*F, we may not immediately conclude that every eigenvalue of F^*F corresponds to some element f_m. Indeed, for any $m \in \mathcal{I}_j$,

$$\begin{aligned} a_m^2\lambda_j = \langle \lambda_j f_m, f_m\rangle = \langle F^*F f_m, f_m\rangle = \|F f_m\|^2, \\ = \sum_{n=1}^M |\langle f_m, f_n\rangle|^2 \geq |\langle f_m, f_m\rangle|^2 = a_m^4, \end{aligned} \quad \text{(A.10)}$$

and so $\lambda_j \geq a_m^2 > 0$.

Thus, if $\lambda_J = 0$, then no elements of $\{f_m\}_{m=1}^M$ live with the lowest eigenspace E_J, i.e. $\mathcal{I}_J = \emptyset$. Consequently, the fourth claim guarantees that $\{f_m\}_{m=1}^M \subset \mathbb{H}_N$ is an orthogonal set. Since $N \leq M$ by assumption [11] , we have that $\{f_m\}_{m=1}^M$ is an orthogonal basis for $\mathbb{H}_N$, and consequently that $N = M$. Therefore F^*F is a diagonal matrix whose diagonal entries are the scalars $\{a_m^2\}_{m=1}^M > 0$. However, since the diagonal entries of F^*F are also the eigenvalues, we consequently cannot have an eigenvalue λ_J of value 0, a contradiction.

Knowing that $\lambda_J > 0$, we invoke the second claim to yield that $\{f_m\}_{m\in\mathcal{I}_J}$ is a

[11] One may create a slightly more general version of Theorem 10, by not assuming that $N \leq M$. In that case, for $M < N$ any local minimzer of the frame potential must be an orthogonal set.

λ_J-tight frame for E_J, yielding,

$$\lambda_J = \frac{\sum_{m\in\mathcal{I}_J} a_m^2}{\dim E_J},$$

in accordance with Corollary 2. To conclude the proof of this claim, recall that for $j < J$, $\{f_m\}_{m\in\mathcal{I}_j}$ is a basis for E_j. Consequently, for $j < J$, $|\mathcal{I}_j| = \dim E_j$, and,

$$M = \sum_{j=1}^{J} |\mathcal{I}_j| = |\mathcal{I}_J| + \sum_{j=1}^{J-1} |\mathcal{I}_j|, = |\mathcal{I}_J| + \sum_{j=1}^{J-1} \dim E_j = |\mathcal{I}_J| + N - \dim E_J.$$

Thus, $\dim E_J = N - M + |\mathcal{I}_J|$, yielding the fifth claim.

We now have the minimizer $\{f_m\}_{m=1}^M$ partitioned into two sets, namely

$$\{f_m\}_{m=1}^M = \bigcup_{j=1}^{J} \{f_m\}_{m\in\mathcal{I}_j} = \{f_m\}_{m\in\mathcal{I}_J^C} \cup \{f_m\}_{m\in\mathcal{I}_J},$$

where $\{f_m\}_{m\in\mathcal{I}_J^C}$ is an orthogonal collection for whose orthogonal complement, namely E_J, the collection $\{f_m\}_{m\in\mathcal{I}_J}$ forms a tight frame. What remains to be shown is the precise manner in which the indices are partitioned. In particular, the frame elements of sufficient length should correspond to indices in $\mathcal{I}_J^C$, while the smaller frame elements should correspond to $\mathcal{I}_J$.

Claim 6: $\{N_0, \ldots, M\} \subseteq I_J$.

We show the equivalent inclusion $\mathcal{I}_J^C \subseteq \{1, \ldots, N_0 - 1\}$. Given any $n \in \mathcal{I}_J^C$, let j be the index of the corresponding eigenvalue, i.e. $f_n \in E_{\lambda_j}$. Recall from the forth and fifth claims that,

$$\frac{\sum_{m\in\mathcal{I}_J} a_m^2}{N - M + |\mathcal{I}_J|} = \lambda_J < \lambda_j = a_n^2.$$

This, coupled with the fact that $\{a_m^2\}_{m=1}^M$ is decreasing yields,

$$|\mathcal{I}_J \cap \{1, \ldots, n\}| a_n^2 + \sum_{m=n+1}^{M} a_m^2 \le \sum_{\substack{m=1,\\ m\in\mathcal{I}_J}}^{n} a_m^2 + \sum_{m=n+1}^{M} a_m^2,$$

$$= \sum_{m\in\mathcal{I}_J} a_m^2 + \sum_{\substack{m=n+1,\\ m\in\mathcal{I}_J^C}}^{M} a_m^2 < (N - M + |\mathcal{I}_J|)\, a_n^2 + |\mathcal{I}_J^C \cap \{n+1, \ldots, M\}| a_n^2.$$

To simplify, we first note that,

$$|\mathcal{I}_J| = |\mathcal{I}_J \cap \{1, \ldots, n\}| + |\mathcal{I}_J \cap \{n+1, \ldots, M\}|.$$

Adding $|\mathcal{I}_J^C \cap \{n+1, \ldots, M\}|$ to this equation yields,

$$\begin{aligned} |\mathcal{I}_J| + |\mathcal{I}_J^C \cap \{n+1, \ldots, M\}| &= |\mathcal{I}_J \cap \{1, \ldots, n\}| + |\{n+1, \ldots, M\}|, \\ &= |\mathcal{I}_J \cap \{1, \ldots, n\}| + (M-n). \end{aligned}$$

Returning to our inequality, we then have

$$|\mathcal{I}_J \cap \{1, \ldots, n\}| a_n^2 + \sum_{m=n+1}^{M} a_m^2, < (N - M + |\mathcal{I}_J \cap \{1, \ldots, n\}| + (M-n))\, a_n^2,$$

which reduces to,

$$\sum_{m=n+1}^{M} a_m^2 < (N-n) a_n^2.$$

Since N_0 is by definition the minimal value n for which

$$(N-n) a_n^2 \leq \sum_{m=n+1}^{M} a_m^2,$$

we must have $n < N_0$, i.e. $n \in \{1, \ldots, N_0 - 1\}$. Since $n \in \mathcal{I}_J^C$ was arbitrary, $\mathcal{I}_J^C \subseteq \{1, \ldots, N_0 - 1\}$, and the sixth claim is demonstrated. We consequently have that all of the "smaller" frame elements $\{f_m\}_{m=N_0}^{M}$ exist within the "lowest" eigenspace E_J. What remains to be shown is that none of the larger frame elements live there there as well.

Claim 7: $\{N_0, \ldots, M\} = I_J$.

Having the sixth claim, we need only to show that $\{1, \ldots, N_0 - 1\} \cap \mathcal{I}_J = \emptyset$. By means of contradiction, assume $\{1, \ldots, N_0 - 1\} \cap \mathcal{I}_J \neq \emptyset$, with minimal index n_0 and maximal index n_1.

Taking (A.10) along with the fifth claim we have,

$$a_{n_0}^2 \leq \lambda_J = \frac{\sum_{m \in \mathcal{I}_J} a_m^2}{N - M + |\mathcal{I}_J|}.$$

Multiplying by $N - M + |\mathcal{I}_J|$, and subtracting $a_{n_0}^2$ yields,

$$[(N - M + |\mathcal{I}_J|) - 1] a_{n_0}^2 \leq \sum_{m \in \mathcal{I}_J,\ m > n_0} a_m^2.$$

In other words, we have that the first element of $\{a_m^2\}_{m \in \mathcal{I}_J}$ is less than the remaining elements on "average." Thus, when applied to the sequence $\{a_m^2\}_{m \in \mathcal{I}_J}$ of $|\mathcal{I}_J|$ elements, Proposition 9 guarantees a similar inequality for all subsequent points in the sequence. In particular, for the index n_1, we have,

$$[(N - M + |\mathcal{I}_J|) - |\{1, \ldots, n_1\} \cap \mathcal{I}_J|] a_{n_1}^2 \leq \sum_{m \in \mathcal{I}_J,\ m > n_1} a_m^2.$$

To simplify this expression, we note that the sixth claim along with the fact that n_1 is the maximal element of $\{1, \ldots, N_0 - 1\} \cap \mathcal{I}_J$ imply,

$$\{n_1 + 1, \ldots, M\} \cap \mathcal{I}_J = \{N_0, \ldots, M\}.$$

Thus, we have that both,

$$\sum_{m \in \mathcal{I}_J m > n_1} a_m^2 = \sum_{m=N_0}^{N} a_m^2,$$

and furthermore that,

$$|\mathcal{I}_J| - |\{1, \ldots, n_1\} \cap \mathcal{I}_J| = |\{n_1 + 1, \ldots, M\} \cap \mathcal{I}_J| = |\{N_0, \ldots, M\}| = M - N_0 + 1.$$

It follows that,

$$(N - M + |\mathcal{I}_J|) - |\{1, \ldots, n_1\} \cap \mathcal{I}_J| = N - (N_0 - 1).$$

Since $n_1 \leq N_0 - 1$, then $a_{N_0 - 1} \leq a_{n_1}$, and so,

$$N - (N_0 - 1) a_{N_0 - 1}^2 \leq N - (N_0 - 1) a_{n_1}^2 \leq \sum_{m=N_0}^{N} a_m^2,$$

i.e., $n = N_0$ is an index which satisfies

$$a_n^2 \leq \sum_{m=n+1}^{N} a_m^2.$$

But this is a contradiction, since by definition N_0 is the minimal such index. Thus $\{1, \ldots, N_0 - 1\} \cap \mathcal{I}_J = \emptyset$, and so the seventh claim is demonstrated.

Consequently,

$$\{f_m\}_{m=1}^{M} = \{f_m\}_{m \in \mathcal{I}_J^C} \cup \{f_m\}_{m \in \mathcal{I}_J} = \{f_m\}_{m=1}^{N_0 - 1} \cup \{f_m\}_{m=N_0}^{M},$$

where $\{f_m\}_{m=1}^{N_0 - 1}$ is an orthogonal set, for whose orthogonal complement the remaining vectors $\{f_m\}_{m=N_0}^{M}$ forms a tight frame, giving the result. $\square$

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