

**PROBLEMS ON PAVING AND
THE KADISON-SINGER PROBLEM: FOR THE AIM WEB
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ABSTRACT. For a background and up to date information about paving see the posted note: *Paving and the Kadison-Singer Problem*. Please send problems which should be posted in this section to Pete Casazza at pete@math.missouri.edu.

1. NOTATION

Notation 1.1. Given an orthonormal basis $\{e_i\}_{i \in I}$ for a Hilbert space \mathbb{H} , for any $A \subset I$ we denote by P_A the **diagonal projection** whose matrix has entries all zero except for the (i, i) -entries for $i \in A$ which are all one. For a matrix $A = (a_{ij})_{i,j=1}^n$ set $\delta_p = \max\{|a_{ii}| : i = 1, 2, \dots, n\}$. A **diagonal symmetry** is a diagonal matrix $(a_{ij})_{i,j=1}^n$ with $|a_{ii}| = 1$ for $1 \leq i \leq n$ and $a_{ij} = 0$ otherwise.

Notation 1.2. Given an orthonormal basis $\{e_i\}_{i \in I}$ for a Hilbert space \mathbb{H} , a **diagonal decomposition of the identity** (k -d.d. for short) is a family of diagonal projections with disjoint ranges and

$$\sum_{i=1}^k P_i = I.$$

Notation 1.3. If A is an $n \times n$ matrix, define

$$\alpha_k(A) = \min\{\max \|P_i A P_i\| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}$$

and if A is an infinite matrix bounded in the operator norm, define

$$\alpha_k(A) = \inf\{\max \|P_i A P_i\| : \{P_i\}_{i=1}^k \text{ is a } k\text{-d.d.}\}.$$

If $A \neq 0$, in both cases define

$$\tilde{\alpha}_k(A) = \frac{\alpha_k(A)}{\|A\|}.$$

2. THE PAVING CONJECTURE

Anderson's Paving Conjecture (PC) [2]

For every $\epsilon > 0$, there is some “universal” positive integer $k = k(\epsilon)$ so that for every zero-diagonal finite matrix A , there exists a k -d.d. $\{P_i\}_{i=1}^k$ so that for all $i = 1, 2, \dots, k$ we have

$$(2.1) \quad \|P_i A P_i\| \leq \epsilon \|A\|.$$

When we have the inequality in Equation 2.1 for a matrix A , we say that A is **(k, ϵ) -pavable**.

Given a fixed matrix A , if for every $\epsilon < 1$ there is a $k \in \mathbb{N}$ so that A is (k, ϵ) -pavable then we say that A is **pavable**. If we have a class of matrices, we say this **class is pavable** (respectively, **(k, ϵ) -pavable**) if every member of the class is pavable (respectively, $(k\epsilon)$ -pavable).

Paving for an arbitrary matrix A means paving for $A - E(A)$ where $E(A)$ denotes the diagonal of A with respect to a fixed basis. A simple iteration argument shows that PC is equivalent to the existence of a universal k working for just one fixed $\epsilon < 1$.

There are a number of restricted classes of matrices for which paveability with respect to any fixed basis is equivalent to PC:

1. Self-adjoint matrices [5, 6].
2. Unitary operators [5, 6].
3. Positive operators [5, 6].
4. Invertible operators (or invertible operators with zero diagonal) [5, 6]
5. Orthogonal projections [5, 6]
6. Gram matrices [5, 6].
7. Lower (respectively upper) triangular matrices [7].

3. PAVING CONJECTURES EQUIVALENT TO PC

Conjecture 3.1. *For every zero-diagonal matrix A (finite or infinite) there is a k and $\epsilon < 1$ so that A is (k, ϵ) -pavable. I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$ for some k .*

The equivalence of Conjecture 3.1 and PC is a simple diagonal process. Note that Conjecture 3.1 does not require the universality of k and the bound below one depends on A .

Conjecture 3.2. *There is a universal k so that for all zero-diagonal matrices A (finite and infinite), there is a $\epsilon < 1$ so that A is (k, ϵ) -pavable. (I.e., For every zero-diagonal matrix A we have $\tilde{\alpha}_k(A) < 1$.*

Note: The rest of the conjectures in this section require universal constants k and $\epsilon < 1$ which are independent of $n \in \mathbb{N}$ and the matrix.

Conjecture 3.3. [4] *There exists an $\epsilon < 1$ and a natural number k so that all orthogonal projections A on ℓ_2^{2n} with $1/2$'s on the diagonal are (k, ϵ) -pavable.*

Note that Conjecture 3.3 does not require zero-diagonal. In [4] it is shown that Conjecture 3.3 fails for $k = 2$. I.e. There is no $\epsilon < 1$ so that this class of projections is $(2, \epsilon)$ -pavable.

Conjecture 3.4. [4] *There is a universal k and an $\epsilon < 1$ so that every norm one self-adjoint operator U with zero diagonal and satisfying $U^2 = I$ is (k, ϵ) -pavable.*

Conjecture 3.5. [9] *There exist universal constants $0 < \delta, \epsilon < 1$ and $k \in \mathbb{N}$ so that all orthogonal projections P on ℓ_2^n with $\delta(P) \leq \delta$ are (k, ϵ) -pavable.*

Weaver also posed an equal norm version of this conjecture.

Conjecture 3.6. [9] *There exist universal constants $0 < \delta, \sqrt{\delta} \leq \epsilon < 1$ and $k \in \mathbb{N}$ so that all orthogonal projections P on ℓ_2^n with $\delta(P) \leq \delta$ and $\|Pe_i\| = \|Pe_j\|$ for all $i, j = 1, 2, \dots, n$ are (k, ϵ) -pavable.*

4. RELATED PROBLEMS

Problem 4.1. *If PC is true, then are the upper triangular invertibles on $\ell_2(\mathbb{N})$ path connected? (Larson, Paulsen, Orr, Weiss, Zhang).*

The problem of the connectedness of the upper triangular invertible matrices is well-known in the non-self-adjoint operator community.

Problem 4.2. *Are there analogs for Problem 4.1 in other classes of operators such as Laurent, Toeplitz, analytic toeplitz, etc.? (Weiss).*

Problem 4.3. *Are the Laurent operators paveable? (Bourgain-Tzafriri [3], Halpern, Kaftal and Weiss [8]).*

Problem 4.4. *Are all Laurent operators with H^∞ -symbol paveable? (Paulsen, Weiss).*

Problem 4.5. *Is PC equivalent to PC for some k -d.d. $\{P_i\}_{i=1}^k$ where*

$$|\text{rank } P_i - \text{rank } P_j| \leq 1, \text{ for all } 1 \leq i, j \leq k?$$
(Casazza, Edidin, Weiss).

5. THE AKEMANN-ANDERSON CONJECTURE

Akemann and Anderson [1] posed a conjecture which would imply a positive solution to KS.

Conjecture B ([1], 7.1.3) There exists $\gamma, \epsilon > 0$ (and independent of n) such that for any projection P with $\delta_p < \gamma$ there is a diagonal symmetry S such that $\|PSP\| < 1 - \epsilon$.

Conjecture B is still open despite considerable effort having been expended on it. Weaver [10] states that a counterexample to Conjecture B would probably lead to a negative solution to the Paving Conjecture.

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