

Frames containing a Riesz basis and preservation of this property under perturbations.

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Abstract

Aldroubi has shown how one can construct any frame $\{g_i\}_{i=1}^{\infty}$ starting with one frame $\{f_i\}_{i=1}^{\infty}$, using a bounded operator U on the space of square summable sequences $l^2(N)$. We study the overcompleteness of the frames in terms of properties of U . We also discuss perturbation of frames in the sense that two frames are “close” if a certain operator is compact. In this way we obtain an equivalence relation with the property that frames in the same equivalence class have the same overcompleteness. On the other hand we show that perturbation in the Paley-Wiener sense does not have this property.

Finally we construct a frame which is norm-bounded below, but which does not contain a Riesz basis. The construction is based on the delicate difference between the unconditional convergence of the frame representation, and the fact that a convergent series in the frame elements need not converge unconditionally.

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1 Introduction.

The introduction of frames for a Hilbert space \mathcal{H} goes back to the paper [DS] from 1952, where they are used in nonharmonic Fourier analysis. A frame is a family $\{f_i\}_{i \in I}$ of elements in \mathcal{H} which can be considered as an “overcomplete basis”: every element in \mathcal{H} can be written as a linear combination of the frame elements f_i , with square integrable coefficients, which do not need to be unique. A natural theoretical question (which is also important for applications, e.g., representation of an operator using a basis) is how far frames are away from bases, i.e., one may ask questions like

- 1) does a frame contain a Riesz basis?
- 2) which conditions imply that a frame consists of a Riesz basis plus finitely many elements ?
- 3) what happens with the overcompleteness if the frame elements are perturbed?

The reason for the interest in Riesz bases and not just bases is that frames and Riesz bases are closely related: a Riesz bases is just a frame $\{f_i\}_{i=1}^{\infty}$, where the elements are ω -independent, i.e.,

$$\sum_{i \in I} c_i f_i = 0, \{c_i\}_{i=1}^{\infty} \in \ell^2(I) \Rightarrow c_i = 0, \forall i \in I.$$

Some answers has been found by Holub [H] , who concentrates on the second question. Here we go one step further, in that we are mainly interested in frames which just contain a Riesz basis. For such frames one defines the excess as the number of elements one should take away to obtain a Riesz basis.

In the first part of the paper we apply a result of Albroubi [A] , explaining how one can map a frame onto another using a bounded operator U on ℓ^2 . Our results concern the relation between the frames involved and properties of U . Independent of that we construct a norm-bounded frame not containing a Riesz basis.

In section 3 we concentrate on the third question. We introduce the concept “compact perturbation”. This leads to an equivalence relation on the set of frames, with the property that frames in the same equivalence class have the same overcompleteness properties; this means, that if a frame contains a Riesz basis then all members in the class contain a Riesz basis, and all those frames have the same excess.

Finally we show that perturbation in the Paley-Wiener sense [C3] do not have this pleasant property.

2 Frames containing a Riesz basis.

Let \mathcal{H} be a separable Hilbert space. A family $\{f_i\}_{i \in I}$ is called a *frame* for \mathcal{H} if

$$\exists A, B > 0 : A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \forall f \in \mathcal{H}.$$

A and B are called *frame bounds*. The frame is *tight* if we can choose $A = B$. A *Riesz basis* is a family of elements which is the image of an orthonormal basis by a bounded invertible operator. Frequently we will use an equivalent characterization [Y]: $\{f_i\}_{i \in I}$ is a Riesz basis if there exist numbers $A, B > 0$ such that

$$(1) \quad A \sum |c_i|^2 \leq \|\sum c_i f_i\|^2 \leq B \sum |c_i|^2,$$

for all finite sequences $\{c_i\}$.

Also, a basis $\{f_i\}_{i \in I}$ is a Riesz basis if and only if it is unconditional (meaning that if $\sum c_i f_i$ converges for some square summable coefficients $\{c_i\}$, then it actually converges unconditionally) and $0 < \inf_i \|f_i\| \leq \sup_i \|f_i\| < \infty$.

There is a close connection between frames and Riesz bases:

$$\{f_i\}_{i \in I} \text{ is a Riesz basis} \Leftrightarrow [\{f_i\}_{i \in I} \text{ is a frame and } \sum c_i f_i = 0, \{c_i\} \in \ell^2 \Rightarrow c_i = 0, \forall i.]$$

In words: a Riesz basis is a frame, where the elements are ω -independent. If $\{f_i\}_{i=1}^\infty$ is a Riesz basis, then the numbers A, B appearing in (1) are actually frame bounds.

If $\{f_i\}_{i \in I}$ is a frame (or if only the upper frame condition is satisfied) then we define the *pre-frame operator* as an operator from the space of square summable sequences with index set I into \mathcal{H} :

$$T : \ell^2(I) \rightarrow \mathcal{H}, \quad T\{c_i\} := \sum_{i \in I} c_i f_i.$$

The operator T is bounded. Composing T with its adjoint

$$T^* : \mathcal{H} \rightarrow \ell^2(I), \quad T^* f = \{\langle f, f_i \rangle\}_{i \in I}$$

we get the *frame operator*

$$S = TT^* : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf := \sum_{i \in I} \langle f, f_i \rangle f_i,$$

which is a bounded and invertible operator. This immediately leads to the *frame decomposition*; every $f \in \mathcal{H}$ can be written as

$$f = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i,$$

where the series converges unconditionally. So a frame has a property similar to a basis: every element in \mathcal{H} can be written as a linear combination of the frame elements. For more information about basic properties of frames we refer to the original paper [DS] and the research tutorial [HW].

For convenience, we will index our frames by the natural numbers in the sequel. The main difference between a frame $\{f_i\}_{i=1}^{\infty}$ and a basis is that a frame can be overcomplete, so it might happen that $f \in \mathcal{H}$ has a representation $f = \sum_{i=1}^{\infty} c_i f_i$ for some coefficients c_i which are different from the *frame coefficients* $\langle f, S^{-1}f_i \rangle$. In applications one might wish not to have “too much redundancy”. In that spirit Holub [H] discusses *near-Riesz bases*, i.e. frames $\{f_i\}_{i=1}^{\infty}$ consisting of a Riesz basis $\{f_i\}_{i \in N-\sigma}$ plus finitely many elements $\{f_i\}_{i \in \sigma}$. The number of elements in σ is called the *excess*. Let us denote the kernel of the operator T by N_T . If $\{f_i\}_{i=1}^{\infty}$ is a frame, then

$\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis $\Leftrightarrow N_T$ has finite dimension $\Leftrightarrow \{f_i\}_{i=1}^{\infty}$ is unconditional.

The first of the above biimplications is due to Holub [H], who also proves the second under the assumption that the frame is norm-bounded below. The generalization above is proved by the authors in [CC].

If the conditions above are satisfied, then the excess is equal to $\dim(N_T)$.

If $\dim(N_T) = \infty$, two things can happen: $\{f_i\}_{i=1}^{\infty}$ consists of a Riesz basis plus infinitely many elements (in which case we will say that $\{f_i\}_{i=1}^{\infty}$ has infinite excess) or $\{f_i\}_{i=1}^{\infty}$ does not contain a Riesz basis at all. In the present paper we concentrate on frames which contain a Riesz basis. Every frame can be mapped onto such a frame (in fact, onto an arbitrary frame) using a construction of Aldroubi [A], which we now shortly describe.

Let $\{f_i\}_{i=1}^{\infty}$ be a frame and $U : \ell^2(\mathbf{N}) \rightarrow \ell^2(\mathbf{N})$ a bounded operator. Let

$\{u_{i,j}\}_{i,j \in \mathbf{N}}$ be the matrix for U with respect to some basis. Define the family $\{g_i\}_{i=1}^\infty \in \mathcal{H}$ by

$$g_i = \sum_{j=1}^{\infty} u_{i,j} f_j.$$

By an abuse of notation we will sometimes write $\{g_i\}_{i=1}^\infty = U\{f_i\}_{i=1}^\infty$. A result of Aldroubi (differently formulated) states that

$$\{g_i\}_{i=1}^\infty \text{ is a frame} \Leftrightarrow \exists \gamma > 0 : \|UT^*f\| \geq \gamma \cdot \|T^*f\|, \quad \forall f \in \mathcal{H}.$$

It is important that *every* frame $\{g_i\}_{i=1}^\infty$ can be generated in this way, i.e., given the frame $\{g_i\}_{i=1}^\infty$ we just have to find the operator U mapping $\{f_i\}_{i=1}^\infty$ to $\{g_i\}_{i=1}^\infty$. In connection with Aldroubi's construction there are (at least) two natural questions related to Holub's work: how is the excess of $\{g_i\}_{i=1}^\infty$ related to that of $\{f_i\}_{i=1}^\infty$, and which conditions imply that $\{g_i\}_{i=1}^\infty$ actually is a Riesz basis? We shall give answers to both questions in this section.

The definition of $\{g_i\}_{i=1}^\infty$ immediately shows that

$$\{\langle g_i, f \rangle\} = U\{\langle f_i, f \rangle\}, \quad \forall f \in \mathcal{H};$$

this is true whether or not $\{g_i\}_{i=1}^\infty$ builds a frame. The formula leads to an expression for the pre-frame operator associated with $\{g_i\}_{i=1}^\infty$. We let U^T denote the transpose of U and \bar{U} be the operator corresponding to the matrix where all entries in the matrix of U are complex conjugated. It is easy to prove that

$$\sum_{i=1}^{\infty} c_i g_i = TU^T\{c_i\}_{i=1}^\infty, \quad \forall \{c_i\}_{i=1}^\infty \in \ell^2(\mathbf{N}).$$

So if $\{g_i\}_{i=1}^\infty$ contains a Riesz basis, then its excess is equal to $\dim(N_{TU^T})$. For the calculation of this number we need a lemma, the proof of which we leave to the reader. Corresponding to an operator V we denote its range by R_V .

Lemma 1: *Let X, Y be vector spaces and $V : X \rightarrow Y$ a linear mapping. Given a subspace $Z \subseteq Y$, define $V^{-1}(Z) := \{x \in X \mid Vx \in Z\}$. Then*

$$\dim(V^{-1}(Z)) = \dim(Z \cap R_V) + \dim(N_V).$$

Theorem 2: $\dim(N_{TU^T}) = \dim(R_{U^T} \cap N_T) + \dim(R_{\bar{U}}^\perp)$.

Proof: Theorem 2 is an easy consequence of Lemma 1 and the calculation

$$\{\{c_i\}_{i=1}^\infty \mid TU^T\{c_i\}_{i=1}^\infty = 0\} = \{\{c_i\}_{i=1}^\infty \mid U^T\{c_i\}_{i=1}^\infty \in N_T\} = (U^T)^{-1}(N_T).$$

So if $\{g_i\}_{i=1}^\infty$ actually is a frame containing a Riesz basis, then Theorem 2 gives a recipe for calculation of the excess. In particular, if $\{f_i\}_{i=1}^\infty$ is a near-Riesz basis and R_U has finite codimension, then also $\{g_i\}_{i=1}^\infty$ is a near-Riesz basis. Concerning Riesz bases we have another result, which can be proved by the interested reader:

Proposition 3: $\{g_i\}_{i=1}^\infty$ is a Riesz basis $\Leftrightarrow \bar{U} : R_{T^*} \rightarrow \ell^2(N)$ is surjective.

More generally one may wish that the frame at least contain a Riesz basis. As shown in [C2] it is the case for a *Riesz frame*, which is a frame with the property that every subfamily is a frame for its closed linear span, with a common lower bound.

It is easy to construct a frame which does not contain a Riesz basis if one allows a subsequence of the frame elements to converge against 0 in norm. We now present an example showing that the same can be the case for a frame which is norm-bounded below. Our approach is complementary to recent work of Seip [Se], who proves that there exist frames of complex exponentials for $L^2(-\pi, \pi)$ which do not contain a Riesz basis. While Seip relies on the theory for sampling and interpolation our approach is more elementary, just using functional analysis. Furthermore our construction puts focus on a different point, namely the difference between convergence and unconditional convergence of an expansion in the frame elements.

Proposition 4: *There exists a tight frame for \mathcal{H} , which is norm-bounded below, but which does not contain a Riesz basis.*

The proof needs several lemmas, so let us shortly explain the basic idea. As we have seen, $\sum_{i \in I} c_i f_i$ converges unconditionally for every set of frame coefficients $\{c_i\}$. But nothing guarantees that convergence of $\sum_{i \in I} c_i f_i$ implies unconditional convergence for general coefficients $\{c_i\}$. Our proof consists in a construction of a frame where no total subset is unconditional, and hence not a Riesz basis. Technically the first step is to decompose \mathcal{H} into a direct sum of finite dimensional subspaces of increasing dimension. The idea behind the proof might be useful in other situations as well.

Lemma 5: Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for a finite dimensional space \mathcal{H}_n . Define

$$f_j = e_j - \frac{1}{n} \sum_{i=1}^n e_i \quad \text{for } j = 1, \dots, n$$

$$f_{n+1} = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i.$$

Then

$$\sum_{j=1}^{n+1} |\langle f, f_j \rangle|^2 = \|f\|^2, \forall f \in \mathcal{H}_n.$$

Proof: Given $f \in \mathcal{H}_n$, write $f = \sum_{i=1}^n a_i e_i$, $a_i = \langle f, e_i \rangle$. If we let P denote the orthogonal projection onto the unit vector $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i$, then

$$Pf = \frac{1}{n} \langle f, \sum_{i=1}^n e_i \rangle \sum_{i=1}^n e_i = \frac{\sum_{i=1}^n a_i}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i.$$

Therefore

$$\|Pf\|^2 = \frac{|\sum_{i=1}^n a_i|^2}{n} = |\langle f, f_{n+1} \rangle|^2.$$

Also

$$\begin{aligned} \|(I - P)f\|^2 &= \|f - Pf\|^2 = \left\| \sum_{i=1}^n a_i e_i - \frac{\sum_{j=1}^n a_j}{n} \sum_{i=1}^n e_i \right\|^2 \\ &= \left\| \sum_{i=1}^n \left(a_i - \frac{\sum_{j=1}^n a_j}{n} \right) e_i \right\|^2 = \sum_{i=1}^n \left| a_i - \frac{\sum_{j=1}^n a_j}{n} \right|^2 = \sum_{i=1}^n |\langle f, f_i \rangle|^2. \end{aligned}$$

Putting the two results together we obtain

$$\|f\|^2 = \|Pf\|^2 + \|(I - P)f\|^2 = \sum_{i=1}^{n+1} |\langle f, f_i \rangle|^2$$

and the proof is complete. **Q.E.D.**

Given a sequence $\{g_i\}_{i \in I} \subseteq \mathcal{H}$ its *unconditional basis constant* is defined as the number

$$\sup \left\{ \left\| \sum_{i \in I} \sigma_i c_i g_i \right\| \mid \left\| \sum_{i \in I} c_i g_i \right\| = 1 \text{ and } \sigma_i = \pm 1, \forall i \right\}.$$

As shown in [Si], a total family $\{g_i\}_{i \in I}$ consisting of non-zero elements is an unconditional basis for \mathcal{H} if and only if it has finite unconditional basis constant.

Lemma 6: *Define $\{f_1, \dots, f_{n+1}\}$ as in Lemma 5. Any subset of $\{f_1, f_2, \dots, f_{n+1}\}$ which spans \mathcal{H}_n has unconditional basis constant greater than or equal to $\sqrt{n-1} - 1$.*

Proof: Since $\sum_{i=1}^n f_i = 0$, any subset of $\{f_1, \dots, f_{n+1}\}$ which spans \mathcal{H}_n must contain $n-1$ elements from $\{f_1, \dots, f_n\}$ plus f_{n+1} . By the symmetric construction it is enough to consider the family $\{f_1, \dots, f_{n-1}, f_{n+1}\}$. We have

$$\begin{aligned} \|\sum_{i=1}^{n-1} f_i\| &= \|\sum_{i=1}^{n-1} e_i - \frac{n-1}{n} \sum_{i=1}^n e_i\| = \|(1 - \frac{n-1}{n}) \sum_{i=1}^{n-1} e_i - \frac{n-1}{n} e_n\| \\ &= \|\frac{1}{n} \sum_{i=1}^{n-1} e_i - \frac{n-1}{n} e_n\| = \sqrt{\frac{n-1}{n^2} + \frac{(n-1)^2}{n^2}} = \frac{1}{n} \sqrt{n(n-1)} \leq 1. \end{aligned}$$

Now consider $\|\sum_{i=1}^{n-1} (-1)^n f_i\|$; if n is odd this number is equal to $\|\sum_{i=1}^{n-1} (-1)^n e_i\| = \sqrt{n-1}$, and if n is even it is equal to

$$\|\sum_{i=1}^{n-1} (-1)^i e_i - \frac{1}{n} \sum_{i=1}^n e_i\| \geq \|\sum_{i=1}^{n-1} (-1)^i e_i\| - \|\frac{1}{n} \sum_{i=1}^n e_i\| \geq \sqrt{n-1} - \frac{\sqrt{n}}{n} \geq \sqrt{n-1} - 1.$$

That is, in all cases,

$$\|\sum_{i=1}^{n-1} (-1)^n f_i\| \geq \sqrt{n-1} - 1.$$

Combining this with the norm estimate $\|\sum_{i=1}^{n-1} f_i\| \leq 1$ it follows that the unconditional basis constant of $\{f_1, \dots, f_{n-1}\}$ is greater than or equal to $\sqrt{n-1} - 1$, so clearly the same is true for $\{f_1, \dots, f_{n-1}, f_{n+1}\}$. **Q.E.D.**

Now we are ready to do the construction for Proposition 4. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for \mathcal{H} and define

$$\mathcal{H}_n := \text{span}\{e_{\frac{(n-1)n}{2}+1}, e_{\frac{(n-1)n}{2}+2}, \dots, e_{\frac{(n-1)n}{2}+n}\}.$$

So $\mathcal{H}_1 = \text{span}\{e_1\}$, $\mathcal{H}_2 = \text{span}\{e_2, e_3\}$, $\mathcal{H}_3 = \text{span}\{e_4, e_5, e_6\}$,

By construction,

$$\mathcal{H} = \left(\sum_{n=1}^{\infty} \bigoplus \mathcal{H}_n\right)_{\mathcal{H}}.$$

That is, $g \in \mathcal{H} \Leftrightarrow g = \sum_{n=1}^{\infty} g_n, g_n \in \mathcal{H}_n$, and $\|g\|^2 = \sum_{n=1}^{\infty} \|g_n\|^2$. We refer to [LT] for details about such decompositions.

For each space \mathcal{H}_n we construct the sequence $\{f_i^n\}_{i=1}^{n+1}$ as in Lemma 5, starting with the orthonormal basis $\{e_{\frac{(n-1)n}{2}+1}, \dots, e_{\frac{(n-1)n}{2}+n}\}$. Specifically, given $n \in N$,

$$f_i^n = e_{\frac{(n-1)n}{2}+i} - \frac{1}{n} \sum_{j=1}^n e_{\frac{(n-1)n}{2}+j}, \quad 1 \leq i \leq n$$

$$f_{n+1}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^n e_{\frac{(n-1)n}{2}+j}.$$

Lemma 7: $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ is a frame for \mathcal{H} , with bounds $A = B = 1$.

Proof: Write $g \in \mathcal{H}$ as $g = \sum_{n=1}^{\infty} g_n, g_n \in \mathcal{H}_n$. Given $n \in N$ it is clear that

$$\langle g, f_i^n \rangle = \langle g_n, f_i^n \rangle \quad \text{for } i = 1, \dots, n+1.$$

From this calculation it follows that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{n+1} |\langle g, f_i^n \rangle|^2 = \sum_{n=1}^{\infty} \sum_{i=1}^{n+1} |\langle g_n, f_i^n \rangle|^2 = \sum_{n=1}^{\infty} \|g_n\|^2 = \|g\|^2,$$

where we have used Lemma 5. **Q.E.D.**

Lemma 8: No subsequence of $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ is a Riesz basis for \mathcal{H} .

Proof: Any subsequence of $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ which spans \mathcal{H} must contain n elements from $\{f_i^n\}_{i=1}^{n+1}$ and so by Lemma 6, its unconditional basis constant is greater than or equal to $\sqrt{n-1} - 1$ for every n . That is, the unconditional basis constant is infinite, hence the subsequence can not be an unconditional basis for \mathcal{H} . **Q.E.D.**

Lemma 7 and Lemma 8 proves Proposition 4. It would be interesting to determine whether Proposition 4 still holds if one only considers classes of frames with a special structure, for example Weyl-Heisenberg frames, wavelet frames, or frames consisting of translates of a single function.

Remark: Corresponding to a subfamily $\{f_i\}_{i=1}^n$ of a frame $\{f_i\}_{i=1}^\infty$ we define the frame operator by $S_n : \text{span}\{f_i\}_{i=1}^n \rightarrow \text{span}\{f_i\}_{i=1}^n$, $S_n f = \sum_{i=1}^n \langle f, f_i \rangle f_i$. The orthogonal projection of \mathcal{H} onto $\text{span}\{f_i\}_{i=1}^n$ is given by

$$P_n f = \sum_{i=1}^n \langle f, S_n^{-1} f_i \rangle f_i$$

. According to [C1, C2], we say that the *projection method* works if

$$\langle f, S_n^{-1} f_i \rangle \rightarrow \langle f, S^{-1} f_i \rangle \quad \text{for } n \rightarrow \infty, \forall f \in \mathcal{H}, \forall i \in N.$$

The ‘‘block structure’’ of the frame $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ constructed here shows that the projection method can be used. As shown in [C2] the method can also be used for every Riesz basis. The more general questions whether a frame contains a Riesz basis and whether the projection method works, do not seem to be strongly related.

3 Excess preserving perturbation.

At several places in the following we need results for perturbation of frames and Riesz bases. We denote the frames by $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty$, usually with the convention that $\{f_i\}_{i=1}^\infty$ is the frame we begin with, and $\{g_i\}_{i=1}^\infty$ is the perturbed family. Common for all the result is that they can be formulated using the *perturbation operator* K mapping a sequence $\{c_i\}$ of numbers to $\sum c_i(f_i - g_i)$.

Theorem 9: Let $\{f_i\}_{i=1}^\infty, \{g_i\}_{i=1}^\infty \subseteq \mathcal{H}$.

a) If $\{f_i\}_{i=1}^\infty$ is a frame for \mathcal{H} and K is compact as an operator from $\ell^2(N)$ into \mathcal{H} , then $\{g_i\}_{i=1}^\infty$ is a frame for its closed linear span.

b) Suppose $\{f_i\}_{i=1}^\infty$ is a frame for \mathcal{H} with bounds A, B . If there exist numbers $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and

$$\|\sum c_i(f_i - g_i)\| \leq \lambda \cdot \|\sum c_i f_i\| + \mu \sqrt{\sum |c_i|^2}$$

for all finite sequences $\{c_i\}$, then $\{g_i\}_{i=1}^\infty$ is a frame for \mathcal{H} with bounds $A(1 - (\lambda + \frac{\mu}{\sqrt{A}}))^2, B(1 + \lambda + \frac{\mu}{\sqrt{B}})^2$.

c) If $\{f_i\}_{i=1}^\infty$ is a Riesz basis for $\overline{\text{span}}\{f_i\}_{i=1}^\infty$ and the perturbation condition in b) is satisfied, then $\{g_i\}_{i=1}^\infty$ is a Riesz basis for $\overline{\text{span}}\{g_i\}_{i=1}^\infty$.

For the proofs we refer to [C2, C3, CH]. As an easy consequence of a) we have the following

Corollary: *If $\{f_i\}_{i=1}^\infty$ is a frame and $\sigma \subseteq N$ is finite, then $\{f_i\}_{i \in N-\sigma}$ is a frame for $\overline{\text{span}}\{f_i\}_{i \in N-\sigma}$.*

Our next result connects Theorem 9 with the question about overcompleteness of the involved frames:

Theorem 10: *Suppose that $\{f_i\}_{i=1}^\infty$ is a frame containing a Riesz basis, that $\{g_i\}_{i=1}^\infty$ is total, and that K is compact as a mapping from $l^2(N)$ into \mathcal{H} . Then $\{g_i\}_{i=1}^\infty$ is a frame for \mathcal{H} containing a Riesz basis, and the frames $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ have the same excess.*

Proof: First assume that $\{f_i\}_{i=1}^\infty$ has finite excess equal to n . By changing the index set we may write $\{f_i\}_{i=1}^\infty = \{f_i\}_{i=1}^n \cup \{f_i\}_{i=n+1}^\infty$, where $\{f_i\}_{i=n+1}^\infty$ is a Riesz basis for \mathcal{H} . Let A be a lower frame bound for $\{f_i\}_{i=n+1}^\infty$ and choose $\mu < \sqrt{A}$. By compactness there exists a number $m > n$ such that

$$\left\| \sum_{i=m+1}^{\infty} c_i (f_i - g_i) \right\| \leq \mu \sqrt{\sum_{i=m+1}^{\infty} |c_i|^2}$$

for all sets of sequences $\{c_i\} \subseteq l^2(N)$. So by the remark after Theorem 9, $\{g_i\}_{i=m+1}^\infty$ is a Riesz basis for $\overline{\text{span}}\{g_i\}_{i=m+1}^\infty$. If we define the operator T on \mathcal{H} by

$$Tf_i = f_i, \quad n < i \leq m, \quad Tf_i = g_i, \quad i \geq m+1,$$

(extended by linearity) then we have an invertible operator on \mathcal{H} . The argument is that every $f \in \mathcal{H}$ has a representation $f = \sum_{i=n+1}^{\infty} c_i f_i$, leading to

$$\|(I - T)f\| = \left\| \sum_{i=m+1}^{\infty} c_i (f_i - g_i) \right\|$$

$$\leq \mu \sqrt{\sum_{i=m+1}^{\infty} |c_i|^2} \leq \frac{\mu}{\sqrt{A}} \left\| \sum_{i=m+1}^{\infty} c_i f_i \right\| \leq \frac{\mu}{\sqrt{A}} \cdot \|f\|.$$

As a consequence,

$$\text{codim}(\overline{\text{span}}\{g_i\}_{i=m+1}^{\infty}) = \text{codim}(\overline{\text{span}}\{f_i\}_{i=m+1}^{\infty}) = m - n.$$

Take $m - n$ independent elements $\{g_{i_k}\}_{k=1}^{m-n}$ outside $\overline{\text{span}}\{g_i\}_{i=m+1}^{\infty}$. Then $\{g_{i_k}\}_{k=1}^{m-n} \cup \{g_i\}_{i=m+1}^{\infty}$ is a frame for $\overline{\text{span}}\{\{g_{i_k}\}_{k=1}^{m-n} \cup \{g_i\}_{i=m+1}^{\infty}\} = \mathcal{H}$, since only finitely many elements have been taken away from the frame $\{g_i\}_{i=1}^{\infty}$. If now $\sum_{k=1}^{m-n} c_k g_{i_k} + \sum_{i=m+1}^{\infty} c_i g_i = 0$, then all coefficients are zero; first,

$$\sum_{k=1}^{m-n} c_k g_{i_k} = - \sum_{i=m+1}^{\infty} c_i g_i = 0$$

(if the sums were not equal to zero we could delete an element g_{i_k} and still have a frame for \mathcal{H} contradicting the fact that $\text{codim}(\overline{\text{span}}\{g_i\}_{i=m+1}^{\infty}) = m - n$) and since $\{g_{i_k}\}_{k=1}^{m-n}$ is an independent set and $\{g_i\}_{i=m+1}^{\infty}$ a Riesz basis, all coefficients must be zero. So $\{g_{i_k}\}_{k=1}^{m-n} \cup \{g_i\}_{i=m+1}^{\infty}$ is a Riesz basis, i.e., $\{g_i\}_{i=1}^{\infty}$ also has excess n .

Now suppose that $\{f_i\}_{i=1}^{\infty}$ has infinite excess. Let $\{f_i\}_{i \in I}$ be a subset which is a Riesz basis. Then the corresponding set $\{g_i\}_{i \in I}$ spans a space of finite codimension, i.e., $\text{codim}(\overline{\text{span}}\{g_i\}_{i \in I}) < \infty$. This follows by the same compactness argument as we used in the finite excess case, which shows that there exist finitely many $f_i, i \in I$ with the property that if we take them away then we obtain a family which spans a space with the same codimension as the corresponding space of g_i 's. Now take a finite family $\{g_i\}_{i \in J}$ such that $\{g_i\}_{i \in I \cup J}$ is total. Since $\{f_i\}_{i \in I \cup J}$ is a frame with finite excess, the finite excess result gives that $\{g_i\}_{i \in I \cup J}$ is a frame containing a Riesz basis, implying that $\{g_i\}_{i=1}^{\infty}$ has infinite excess. **Q.E.D.**

We can express the result in the following way: define an equivalence relation \sim on the set of frames for \mathcal{H} by

$$\{f_i\}_{i=1}^{\infty} \sim \{g_i\}_{i=1}^{\infty} \Leftrightarrow K \text{ is compact as an operator from } l^2(N) \text{ into } \mathcal{H}.$$

The equivalence relation partitions the set of frames into equivalence classes. If a frame contains a Riesz basis, then every frame in its equivalence class

contains a Riesz basis, and the frames have the same excess.

Let us go back to Theorem 10. If $\{g_i\}_{i=1}^\infty$ is not total, we still know (from Theorem 9) that $\{g_i\}_{i=1}^\infty$ is a frame for its closed span. By checking the proof of Theorem 10 we obtain

Corollary: *Suppose that $\{f_i\}_{i=1}^\infty$ is a frame containing a Riesz basis and that K is compact as a mapping from $\ell^2(N)$ into \mathcal{H} . Then $\{g_i\}_{i=1}^\infty$ is a frame for its closed linear span, and it contains a Riesz basis for this space. The excess referring to $\overline{\text{span}}\{g_i\}_{i=1}^\infty$ is equal to the excess of $\{f_i\}_{i=1}^\infty$ referring to \mathcal{H} , plus the dimension of the orthogonal complement of $\overline{\text{span}}\{g_i\}_{i=1}^\infty$ in \mathcal{H} .*

Now we want to study the excess property of perturbations in the sense of Theorem 9 b). We need a result, which might be interesting in itself. To motivate it, consider a near-Riesz basis $\{f_i\}_{i=1}^\infty$ containing a Riesz basis $\{f_i\}_{i \in I}$. Unfortunately, the lower bound for $\{f_i\}_{i \in I}$ can be arbitrarily small compared to the lower bound A of $\{f_i\}_{i=1}^\infty$. Our result states, that if we are willing to delete sufficiently (still finitely) many elements, then we can obtain a family which is a Riesz basis for its closed span, and which has a lower bound so close to A as we want:

Proposition 11: *Let $\{f_i\}_{i=1}^\infty$ be a near-Riesz basis with lower bound A . Given $\epsilon > 0$, there exists a finite set $J \subseteq N$ such that $\{f_i\}_{i \in N-J}$ is a Riesz basis for its closed span, with lower bound $A - \epsilon$.*

Proof: As in the proof of Theorem 10, write $\{f_i\}_{i=1}^\infty = \{f_i\}_{i=1}^n \cup \{f_i\}_{i=n+1}^\infty$, where $\{f_i\}_{i=n+1}^\infty$ is a Riesz basis for \mathcal{H} . Let $d(\cdot, \cdot)$ denote the distance inside \mathcal{H} (i.e., $d(f, E) = \inf_{g \in E} \|f - g\|$ for $f \in \mathcal{H}, E \subseteq \mathcal{H}$) and choose a number $m > n$ such that

$$d(f_j, \text{span}\{f_i\}_{i=n+1}^m) < \sqrt{\frac{\epsilon}{n}}, \quad j = 1, \dots, n.$$

We want to show that $\{f_i\}_{i=m+1}^\infty$ is a Riesz basis for its closed span, with lower bound $A - \epsilon$. Let P denote the orthogonal projection onto $\overline{\text{span}}\{f_i\}_{i=n+1}^m$. Since $\|\sum c_i f_i\| \geq \|\sum c_i (I - P)f_i\|$ for all sequences, it suffices to show that $\{(I - P)f_i\}_{i=m+1}^\infty$ satisfies the lower Riesz basis condition with bound $A - \epsilon$.

Let $f \in (I - P)\mathcal{H}$. Then

$$\begin{aligned} \sum_{i=m+1}^{\infty} |\langle f, (I - P)f_i \rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, (I - P)f_i \rangle|^2 - \sum_{i=1}^n |\langle f, (I - P)f_i \rangle|^2 \\ &\geq A\|f\|^2 - \sum_{i=1}^n \|f\|^2 \cdot \|(I - P)f_i\|^2 \geq (A - \epsilon)\|f\|^2. \end{aligned}$$

Now we only have to show that $\{(I - P)f_i\}_{i=m+1}^{\infty}$ is ω -independent. But if $\sum_{i=m+1}^{\infty} c_i(I - P)f_i = 0$, then $\sum_{i=m+1}^{\infty} c_i f_i = P \sum_{i=m+1}^{\infty} c_i f_i$, implying that both sides are equal to zero, since $P \sum_{i=m+1}^{\infty} c_i f_i \in \overline{\text{span}\{f_i\}_{i=n+1}^m}$ and $\{f_i\}_{i=n+1}^{\infty}$ is independent. Therefore $c_i = 0$ for all i . **Q.E.D.**

Theorem 12: Let $\{f_i\}_{i=1}^{\infty}$ be a frame for \mathcal{H} with bounds A, B . Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and assume that there exist $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and

$$\|\sum c_i(f_i - g_i)\| \leq \lambda \cdot \|\sum c_i f_i\| + \mu \cdot \sqrt{\sum |c_i|^2}$$

for all finite sequences $\{c_i\}$. Then

$$\{f_i\}_{i=1}^{\infty} \text{ is a near-Riesz basis} \Leftrightarrow \{g_i\}_{i=1}^{\infty} \text{ is a near-Riesz basis,}$$

in which case $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess.

Proof: First assume that $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis with excess n . Let m be chosen as in the proof of Proposition 11, corresponding to an ϵ satisfying the condition $\lambda + \frac{\mu}{\sqrt{A-\epsilon}} < 1$. Let Q denote the orthogonal projection onto $\overline{\text{span}\{f_i\}_{i=m+1}^{\infty}}$. Then every element $f \in \mathcal{H}$ can be written $f = (I - Q)f + Qf = (I - Q)f + \sum_{i=m+1}^{\infty} c_i f_i$, for some coefficients c_i . Now define an operator $T : \mathcal{H} \rightarrow \mathcal{H}$ by

$$Tf = f, \quad f \in \overline{\text{span}\{f_i\}_{i=m+1}^{\infty}}^{\perp}, \quad Tf_i = g_i, \quad i \geq m + 1.$$

T is bounded. Given $f \in \mathcal{H}$ we choose a representation as above. Then

$$\|(I - T)f\| = \left\| \sum_{i=m+1}^{\infty} c_i(f_i - g_i) \right\| \leq \lambda \cdot \left\| \sum_{i=m+1}^{\infty} c_i f_i \right\| + \mu \cdot \sqrt{\sum_{i=m+1}^{\infty} |c_i|^2}$$

$$\leq (\lambda + \frac{\mu}{\sqrt{A-\epsilon}}) \|\sum_{i=m+1}^{\infty} c_i f_i\| = (\lambda + \frac{\mu}{\sqrt{A-\epsilon}}) \|Qf\| \leq (\lambda + \frac{\mu}{\sqrt{A-\epsilon}}) \|f\|.$$

It follows, that T is an isomorphism of \mathcal{H} onto \mathcal{H} . So $\{g_i\}_{i=m+1}^{\infty}$ is a Riesz basis for its closed span, and

$$\dim(\overline{\text{span}}\{g_i\}_{i=m+1}^{\infty})^{\perp} = \dim(\overline{\text{span}}\{f_i\}_{i=m+1}^{\infty})^{\perp}.$$

As a consequence, $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess.

Now assume that $\{g_i\}_{i=1}^{\infty}$ is a near-Riesz basis. By reindexing we may again assume that $\{g_i\}_{i=n+1}^{\infty}$ is a Riesz basis for \mathcal{H} . Define a bounded operator $W : \mathcal{H} \rightarrow \mathcal{H}$ by $Wf := \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle g_i$. Then as in the original proof from [C3] one proves that W is an isomorphism of \mathcal{H} onto \mathcal{H} . If we define $W_n : \mathcal{H} \rightarrow \mathcal{H}$ by $W_n f = \sum_{i=n+1}^{\infty} \langle f, S^{-1}f_i \rangle g_i$, then this operator has a range with finite codimension in \mathcal{H} , which we will write as

$$\text{codim}_{\mathcal{H}}(R_{W_n}) < \infty.$$

Now let $\{e_i\}_{i=1}^{\infty}$ be the natural basis for $l^2(N)$, i.e., e_i is the sequence with 1 in the i 'th entry, otherwise 0. There exists a bounded invertible operator $V : \mathcal{H} \rightarrow \overline{\text{span}}\{e_i\}_{i=n+1}^{\infty}$ such that $Vg_i = e_i$ for $i \geq n+1$, and clearly

$$\text{codim}_{\overline{\text{span}}\{e_i\}_{i=n+1}^{\infty}}(R_{VW_n}) < \infty.$$

Observe that $VW_n f = \sum_{i=n+1}^{\infty} \langle f, S^{-1}f_i \rangle e_i = \{\langle f, S^{-1}f_i \rangle\}_{i=n+1}^{\infty}$. So

$$(VW_n)^* \{c_i\} = \sum_{i=n+1}^{\infty} c_i S^{-1}f_i = S^{-1} \sum_{i=n+1}^{\infty} c_i f_i.$$

Since $R_{VW_n}^{\perp} = N_{(VW_n)^*}$ has finite dimension, also $\{c_i\}_{i=n+1}^{\infty} \mapsto \sum_{i=n+1}^{\infty} c_i f_i$ has a finite dimensional kernel. Therefore

$$T : l^2(N) \rightarrow \mathcal{H}, \quad T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$$

has a finite dimensional kernel, and now the theorem of Holub implies that $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis. By the first part of the Theorem the two frames $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ now have the same excess, and the proof is complete.

Q.E.D.

Unfortunately, the requirement that $\{f_i\}_{i=1}^\infty$ has finite excess is needed in Theorem 12. In fact we are able to construct examples, where $\{f_i\}$ is a tight frame with infinite excess and $\{g_i\}$ does not contain a Riesz basis, but where the perturbation condition is satisfied. Let us shortly describe how one can do this. Define $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ as in Lemma 7. Given $\epsilon > 0$, let

$$g_i^n = e_{\frac{(n-1)n}{2}+i} - \frac{1-\epsilon}{n} \sum_{j=1}^n e_{\frac{(n-1)n}{2}+j}, \quad 1 \leq i \leq n$$

$$g_{n+1}^n = \frac{1}{\sqrt{n}} \sum_{j=1}^n e_{\frac{(n-1)n}{2}+j}.$$

Now, given a sequence $\{c_i^n\}$ we have

$$\begin{aligned} \left\| \sum c_i^n (f_i^n - g_i^n) \right\| &= \epsilon \cdot \left\| \sum_{n=1}^{\infty} \left[\sum_{i=1}^n c_i^n \right] \frac{1}{n} \sum_{j=1}^n e_{\frac{(n-1)n}{2}+j} \right\| \\ &\leq \epsilon \sqrt{\sum_{n=1}^{\infty} \left| \sum_{i=1}^n c_i^n \frac{1}{\sqrt{n}} \right|^2} \leq \epsilon \sqrt{\sum |c_i^n|^2}. \end{aligned} \quad (1)$$

By Lemma 5, $\{f_i^n\}_{i=1, n=1}^{n, \infty}$ is a frame with bounds 1. If we choose $\epsilon < 1$, then the perturbation condition is satisfied with $\lambda = 0, \mu = \epsilon$, implying that $\{g_i^n\}_{i=1, n=1}^{n+1, \infty}$ is a frame with bounds $(1-\epsilon)^2, (1+\epsilon)^2$.

Claim: $\{g_i^n\}_{i=1, n=1}^{n, \infty}$ is a Riesz basis for \mathcal{H} .

We only need to prove that $\{g_i^n\}_{i=1, n=1}^{n, \infty}$ satisfies the lower Riesz basis condition. Given a sequence $\{c_i^n\}$ we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \sum_{i=1}^n c_i^n g_i^n \right\| &\geq \left\| \sum_{n=1}^{\infty} \sum_{i=1}^n c_i^n e_{\frac{(n-1)n}{2}+i} \right\| - (1-\epsilon) \left\| \sum_{n=1}^{\infty} \left(\sum_{i=1}^n c_i^n \right) \frac{1}{n} \sum_{i=1}^n e_{\frac{(n-1)n}{2}+i} \right\| \\ &\geq \sqrt{\sum |c_i^n|^2} - (1-\epsilon) \sqrt{\sum_{n=1}^{\infty} \left| \sum_{i=1}^n c_i^n \frac{1}{\sqrt{n}} \right|^2} \geq \epsilon \sqrt{\sum |c_i^n|^2}. \end{aligned}$$

So actually we have an example where $\{f_i^n\}_{i=1, n=1}^{n+1, \infty}$ does not contain a Riesz basis but the perturbed family does. To obtain the example we were looking for, we use that $\{g_i^n\}$ has the lower bound $(1 - \epsilon)^2$. By (1) above we can consider $\{f_i^n\}$ as a perturbation of $\{g_i^n\}$ if $\frac{\epsilon}{1-\epsilon} < 1$, i.e., if $\epsilon < \frac{1}{2}$. So we get our example by choosing $\epsilon < 1/2$ and switching the roles of $\{f_i^n\}$ and $\{g_i^n\}$.

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