# A Low Complexity Replacement Scheme for Erased Frame Coefficients

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## ABSTRACT

One key property of frames is their resilience against erasures due to the possibility of generating stable, yet over-complete expansions. Blind reconstruction is one common methodology to reconstruct a signal when frame coefficients have been erased. In this paper we introduce several novel low complexity replacement schemes which can be applied to the set of faulty frame coefficients before blind reconstruction is performed, thus serving as a preconditioning of the received set of frame coefficients. One main idea is that frame coefficients associated with frame vectors close to the one erased should have approximately the same value as the lost one. It is shown that injecting such low complexity replacement schemes into blind reconstruction significantly reduce the worst-case reconstruction error. We then apply our results to the circle frames. If we allow linear combinations of different neighboring coefficients for the reconstruction of missing coefficients, we can even obtain perfect reconstruction for the circle frames under certain weak conditions on the set of erasures.

**Keywords:** Frames, circle frames, fusion frames, erasures

## 1. INTRODUCTION

The theory of frames is a nowadays very well established field studying redundancy both as a mathematical concept and as a methodology for signal processing. One key property of frames is their resilience against erasures due to the possibility of generating stable, yet over-complete expansions. Over the last years, there has been extensive work demonstrating and analyzing the impact of erased frame coefficients on signal reconstruction (see[4, 6, 12–14]).

A common approach is to find the (average and worst case) reconstruction error when the erased coefficients are set equal to zero and the reconstruction operator is fixed. Then one tries to construct frames which produce the least error for this case. Since typically a frame is fairly over-complete as compared to the number of erasures, the remaining frame vectors will span the space leaving open the possibility of perfect reconstruction. However, this fact may be of limited use in practice, since it would require computing a new inverse frame operator for the remaining (non-erased) subset of the frame vectors each time; the computational cost of this method is generally too high for practicality and the ensuing time delays are often unacceptable.

Another problem is that the effects of erasures and of quantization are often not comparable when using blind reconstruction. Recently, a combined method for dealing with errors caused by quantization of frame coefficients and by erasures in subsequent transmission was introduced in Bodmann, et al.[3] The idea is to embed check bits in the quantization of frame coefficients, causing a possible, but controlled quantizer overload.

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The novel contribution of this paper is an 'easy to implement' replacement scheme for erasures which is very low cost and already gives a significant improvement over the standard method of blind reconstruction which amounts to setting erased coefficients to zero. Thus in the trade-off between algorithmic complexity and accuracy of reconstruction, our scheme provides a surprising balance between those extreme cases.

## 2. REVIEW OF FRAMES

Throughout this paper we let $\mathcal{H}$ denote a $d$-dimensional Hilbert space. In this finite-dimensional situation, we call $\mathcal{F} = \{f_i\}_{i=1}^N$ a *frame* for $\mathcal{H}$, if it is a spanning set. A frame can not only be regarded as a mere concept for providing redundant expansions, but also as an analysis tool. In fact, it allows the analysis of data by studying the associated *frame coefficients* $\{\langle x, f_i\rangle\}_{i=1}^N$, wherefore the map $V : \mathcal{H} \to \ell_2(\{1, 2, \dots, N\})$ defined by

$$V : \mathcal{H} \to \ell_2(\{1, 2, \dots, N\}), \quad x \mapsto \{\langle x, f_i\rangle\}_{i=1}^N$$

is termed the *analysis operator*. The adjoint $V^*$ of the analysis operator is typically referred to as the *synthesis operator* and satisfies $V^*((c_i)_{i=1}^N) = \sum_{i=1}^N c_i f_i$. The main operator associated with a frame, which provides a stable reconstruction process, is the *frame operator*

$$S = V^*V : \mathcal{H} \to \mathcal{H}, \quad x \mapsto \sum_{i=1}^N \langle x, f_i\rangle f_i,$$

which is a positive, self-adjoint invertible operator on $\mathcal{H}$. In general, $S$ allows reconstruction of a signal $x \in \mathcal{H}$ from its frame coefficients through the reconstruction formula

$$x = \sum_{i=1}^N \langle x, f_i\rangle S^{-1} f_i.$$

Several special cases of frames are particularly interesting for applications. A frame $\mathcal{F} = \{f_i\}_{i=1}^N$ is called an *$A$-tight frame*, if its associated frame operator $S$ satisfies $S = AI$, i.e., is some multiple $A > 0$ of the identity $I$ on $\mathcal{H}$. In this case, the reconstruction does not require inversion of an operator anymore, and becomes

$$x = \frac{1}{A} \sum_{i=1}^N \langle x, f_i\rangle f_i \quad \text{for all } x \in \mathcal{H}.$$

If $A = 1$, then $\mathcal{F}$ is called a *Parseval frame*. We will see later that for the problem we study in this paper, the consideration of Parseval frames is sufficient.

## 3. A REPLACEMENT SCHEME

We now introduce a simple algorithm that, in a similar fashion as sigma-delta quantization, compensates for erasures. The crucial idea is to replace each erased frame coefficient of a frame $\{f_1, f_2, \dots, f_N\} \subseteq \mathcal{H}$, say, with the value of the preceding frame coefficient, if this hasn't been erased, too. Otherwise we set it equal to zero. More precisely, if two consecutive coefficients are erased, the replacement scheme amounts to replacing the first one by the value of its predecessor, but the second one remains zero. After this simple modification of the received frame coefficients, we reconstruct using the original frame operator.

Intuitively, frames should do well if frame coefficients do not change much from one to the next. This is the case as we will see, *if the ordering is carefully chosen.* Thus, to this end, we now assume that the ordering of the frame vectors is fixed. Moreover, throughout the paper, we identify indices cyclically, so $f_0 \equiv f_N$ and $f_{N+1} \equiv f_1$, unless otherwise noted. The replacement scheme is now described in Figure 1.

For a precise analysis of this algorithm, we first need to formalize its modifications of the frame coefficients in a more manageable manner. For this, given a set $J \subseteq \{1, 2, \dots, N\}$ we introduce the matrix $F_J \in M(N, \mathbb{R})$ by

$$(F_J)_{i,j} = \begin{cases} 1 & : \ j = i-1 \text{ and } i \in J \text{ and } i-1 \notin J, \\ 1 & : \ j = i \text{ and } i \notin J, \\ 0 & : \ \text{else.} \end{cases} \tag{1}$$

FRS (Frame Coefficient Replacement Scheme)

**Parameters:**

- Ordered frame $\{f_1, f_2, \ldots, f_N\} \subseteq \mathcal{H}$.
- Associated frame coefficients $\{c_1, c_2, \ldots, c_N\} \subseteq \mathbb{R}$.
- Indices of erased frame coefficients $J \subseteq \{1, 2, \ldots, N\}$.

**Algorithm:**

1) For $i = 1, \ldots, N$ do
2) If $i \in J$ and $i - 1 \notin J$ then
3) Set $c_i = c_{i-1}$.
4) else if $i \in J$ and $i - 1 \in J$ then
5) Set $c_i = 0$.
6) end.
7) end.

**Output:**

- Corrected frame coefficients $\{c_1, c_2, \ldots, c_N\}$.

**Figure 1.** The FRS Algorithm for preconditioning of the sequence of received incomplete frame coefficients before reconstruction.

This allows us to write the action of the algorithm in the following way.

Lemma 3.1. *Let $\{f_1, f_2, \ldots, f_N\}$ be a frame for $\mathcal{H}$, let $\{c_1, c_2, \ldots, c_N\} \subseteq \mathbb{R}$ be associated frame coefficients, and let $J \subseteq \{1, 2, \ldots, N\}$ denote a set of indices of erased frame coefficients. Then the output of* (FRS) *can be written as*

$$F_J(c_1, c_2, \ldots, c_N)^T,$$

*where $F_J$ is defined in* (1).

*Proof.* This follows immediately from the definition of (FRS) and $F_J$. □

We are now in a position to formulate a measure for accuracy of reconstruction applying (FRS) followed by the 'original' reconstruction using the inverse frame operator. We first focus on the class of $A$-tight frames and will later in this section discuss the generalization to arbitrary frames.

Definition 3.2. *Let $\{f_1, f_2, \ldots f_N\}$ be an $A$-tight frame for $\mathcal{H}$ with analysis operator denoted by $V$. Then the* worst-case reconstruction error *resulting from erasure of coefficients indexed by $J \subset \{1, 2, \ldots N\}$ using the preconditioning algorithm* (FRS) *is defined by the operator norm*

$$er_{(\mathrm{FRS})}(V, J) = \|\frac{1}{A}V^* F_J V - I\|,$$

*where $F_J$ is defined in* (1).

*For notational convenience, we also introduce*

$$D_J = \sum_{j \in J} D_j, \quad \text{where } D_j = E_{j,j-1} - E_{j,j},$$

*with the matrices* $\{E_{i,j}\}_{i,j=1}^N$ *being the canonical matrix units, which yields the reformulation*

$$er_{(\text{FRS})}(V,J) = \|\frac{1}{A}V^*D_JV\|.$$

Before delving into the analysis of the worst case reconstruction error for (FRS), we first observe that it is sufficient to consider the special case of a Parseval frame in the sense that for the general case only a multiplicative controllable factor comes into play. For this, we will consider a frame and its associated canonical Parseval frame, which is the provably closest Parseval frame in $\ell_2$ norm (see Casazza and Kutyniok[8]). We can even show this claim for a more elaborate algorithm than (FRS) by assuming that we replace lost coefficients with some linear combination of their neighbors, and not only the preceding one.

PROPOSITION 3.3. *Let* $\{f_1, f_2, \dots f_N\}$ *be a frame for* $\mathcal{H}$ *with analysis operator denoted by* $V$ *and frame operator* $S$ *satisfies* $S \le BI$*, and let* $\epsilon > 0$ *be the error bound for the associated canonical Parseval frame, i.e.,*

$$\epsilon = \max_{i=1,\dots,N} \|f_i - S^{1/2}f_i\|.$$

*Let* $J \subset \{1,2,\dots N\}$ *be the set of erased indices and let* $F \in M(n,\mathbb{R})$ *be some replacement matrix (could be more general that* $F_J$*) satisfying the constraint that* $F_{i,j}=0$ *if* $i,j \in J$*. Then*

$$\|V^*FV - I\| \le B\cdot\epsilon\cdot\|(S^{-1/2}V)^*F(S^{-1/2}V) - I\|.$$

*Proof.* Let $x \in \mathcal{H}$. Then

$$\begin{aligned}
\|V^*FVx - x\| &= \|\sum_{i=1}^N \langle x, f_i\rangle f_i - \sum_{j=1}^N a_{ij}\langle x, f_j\rangle f_i\| \\
&= \|S^{1/2}\sum_{i=1}^N \langle S^{1/2}x, S^{-1/2}f_i\rangle S^{-1/2}f_i - \sum_{j=1}^N a_{ij}\langle S^{1/2}x, S^{-1/2}f_j\rangle S^{-1/2}f_i\| \\
&\le \|S^{1/2}\|\cdot\|\sum_{i=1}^N \langle S^{1/2}x, S^{-1/2}f_i\rangle S^{-1/2}f_i - \sum_{j=1}^N a_{ij}\langle S^{1/2}x, S^{-1/2}f_j\rangle S^{-1/2}f_i\| \\
&\le \|S^{1/2}\|\cdot\epsilon\cdot\|S^{1/2}x\| \\
&\le \|S^{1/2}\|^2\cdot\epsilon\cdot\|x\| \\
&= B\cdot\epsilon\cdot\|x\|.
\end{aligned}$$

This finishes the proof. □

This result allows us to focus entirely on the class of tight frames by using Proposition 3.3 to extend the resulting error bounds to arbitrary frames.

In addition, for now, we focus on non-consecutive erasures, because in practice erasures typically occur randomly and rarely, and thus, the probability of having neighboring erasures is relatively small. This allows us to derive a first error bound for the worst-case reconstruction error.

THEOREM 3.4. *Let* $\{f_1, f_2, \dots f_N\}$ *be an* $A$*-tight frame for* $\mathcal{H}$ *with analysis operator denoted by* $V$*. Let* $J$ *be an index subset of* $\{1,2,\dots,N\}$ *which does not contain two consecutive numbers (modulo* $N$*), then*

$$er_{(\text{FRS})}(V,J) \le \frac{1}{A}\max_{1\le i\le N}\|f_i - f_{i-1}\|.$$

*Proof.* Using the identities $\|T\| = \|T^*T\|^{1/2}$ and $\|T^*T\| = \|TT^*\|$ for any operator $T$ as well as the estimate $VV^* \leq I$, we obtain

$$\|V^*D_JV\| = (\|V^*D_J^*VV^*D_JV\|)^{1/2} \leq \|V^*D_J^*D_JV\|^{1/2} = \|D_JVV^*D_J^*\|^{1/2}.$$

Next we compute this expression for the singleton $J = \{j\}$ by

$$\|V^*D_jV\| = \|(E_{j,j-1} - E_{j,j})VV^*(E_{j-1,j} - E_{j,j})\|^{1/2} = \|f_j - f_{j-1}\|.$$

This allows us to derive the following bound for $er_{(\mathrm{FRS})}(V, J)$ always keeping in mind that we are in the situation of non-consecutive erasures:

$$er_{(\mathrm{FRS})}(V, J) = \|\frac{1}{A}V^*D_JV\| \leq \max_{1\leq i\leq N} \frac{1}{A}\|V^*D_jV\| = \frac{1}{A} \max_{1\leq i\leq N} \|f_i - f_{i-1}\|.$$

□

This result shows that in order to minimize the worst-case reconstruction error under our replacement scheme, the ordering of the frame elements should be chosen such that

$$\max_{1\leq i\leq N} \|f_i - f_{i-1}\| \tag{2}$$

is minimized. However, there does not exist a practical algorithm at this time to achieve this, not even in a approximative sense. Hence we are now concerned with considering frames $\mathcal{F} = \{f_i\}_{i=1}^N$ for which the quantity (2) is as small as possible with the ordering given by the construction of the frame.

## 4. CIRCLE FRAMES

### 4.1. Definition and Notations

As a class of frames, for which the quantity (2) is 'reasonably small', we consider the so-called circle frames. *Circle frames* were defined by Bodmann and Paulsen[4] aiming to generate 'good' frames for sigma-delta quantization which are also localized. We start by stating the definition of these frames.

Definition 4.1. *In $\mathbb{R}^2$ an $M = 2N$-element circular frame $\{f_j\}_{j=1}^M$ is defined as the unit norm vectors*

$$f_j = (\cos(2\pi j/M), \sin(2\pi j/M)).$$

*Let now $d \geq 2$ be arbitrary. Given an orthonormal basis $\{e_n\}_{n=1}^d$ for $\mathbb{R}^d$, the associated $dM$-circle frame is defined as the union of the $M$-element circular frames for the two-dimensional subspaces $W_i = \{e_i, e_{i+1}\}$, $1 \leq i \leq d-1$ and $W_d = \{e_d, e_1\}$.*

### 4.2. Error Bound for (FRS)

The following remark becomes essential when applying our replacement scheme in light of (2).

Remark 4.2. *In Bodmann and Paulsen,[4] it was shown that an $M = 2N$-element circular frame is $N$-tight. Moreover, a $dM$-circle frame also forms an $N$-tight frame, and can be ordered as $\{f_j\}_{j=1}^{dM}$, where*

$$\|f_j - f_{j-1}\| \leq \frac{2\pi}{M}, \quad j = 1, \ldots, dM.$$

*This ordering even keeps the groups of $M$-vectors from each two dimensional subspace in their same order.*

Applying this remark to Theorem 3.4 immediately provides us with a superior estimate for the worst-case reconstruction error:

Theorem 4.3. *Let $V$ be the analysis operator of a $dM$-circle frame for $\mathbb{R}^d$, which we assume to be ordered as in Remark 4.2. Let $J$ be an index subset of $\{1, 2, \ldots, dM\}$ which does not contain two consecutive numbers (modulo $dM$), then*

$$er_{(\mathrm{FRS})}(V, J) \leq \frac{\pi}{N^2}.$$

## 4.3. A Distributed Version of (FRS)

However, we might not be able to process the complete frame in one step, but have first only access to each subspace separately – as in the theory of fusion frames by Casazza, Kutyniok, et al..[7,10,11] Thus we now introduce a distributed version of (FRS).

In Figure 2, we first introduce the distributed replacement scheme based on (FRS).

DFRS (Distributed Frame Coefficient Replacement Scheme)

**Parameters:**

- Even positive integer $M = 2N$.
- $dM$-circle frame $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_d$, where each $\mathcal{F}_i = \{f_{ij}\}_{j=1}^M$ is an $M$-element circular frame for the subspace $W_i$.
- Associated frame coefficients $\{c_{ij}\}_{i=1,\ldots,d,j=1,\ldots,M}$.
- Indices of erased frame coefficients $J_i \subseteq \{1, 2, \ldots, M\}$, $i = 1, \ldots, d$.

**Algorithm:**

1) For $i = 1, \ldots, M$ do
2) Apply (FRS) to $\mathcal{F}_i$, $\{c_{ij}\}_{j=1,\ldots,M}$, and $J_i$.
3) end.

**Output:**

- Corrected frame coefficients $\{c_{ij}\}_{i=1,\ldots,d,j=1,\ldots,M}$.

**Figure 2.** The DFRS Algorithm for distributed preconditioning of the sequence of received incomplete $dM$-circle frame coefficients before reconstruction.

For the sake of brevity, we will define the matrix $F_J^{(D)}$ such that the output of (DFRS) can be written as

$$F_J^{(D)}(\{c_{ij}\}_{i=1,\ldots,d,j=1,\ldots,M})^T. \tag{3}$$

We will then study the following worst-case reconstruction error related to this scheme.

Definition 4.4. *Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_d$ be a $dM = d2N$-circle frame as in* (DFRS) *with analysis operator of $\mathcal{F}$ denoted by $V$ and of $\mathcal{F}_i$ denoted by $V_i$, $i = 1, \ldots, d$. Then the* worst-case reconstruction error *resulting from erasure of coefficients indexed by $J_i \subseteq \{1, 2, \ldots, M\}$, $i = 1, \ldots, d$ under the preconditioning algorithm* (DFRS), *is defined by the operator norm*

$$er_{(\mathrm{DFRS})}(V, J) = \|\frac{1}{N} V^* F_J^{(D)} V - I\|,$$

*where $F_J^{(D)}$ is defined in* (3)*.*

We need one preliminary lemma to compute the worst-case reconstruction error for the reconstruction in one of the subspaces and to deduce the contribution they make to $er_{(\mathrm{FRS})}(V, J)$. For this, for each $i = 1, 2, \ldots, d$, let $P_i$ be the orthogonal projection of $\mathbb{R}^d$ onto $W_i$.

Lemma 4.5. *Retaining the previously introduced notion $(P_i)_{i=1}^d$, we have*

$$\|\sum_{i=1}^d P_i(f)\|^2 = 2\sum_{i=1}^d \|P_i(f)\|^2 \quad \textit{for all } f \in \mathbb{R}^d.$$

*Proof.* On the one hand, we have

$$\|\sum_{i=1}^{d} P_i(f)\|^2 = \|\sum_{i=1}^{d}(\langle f, e_i\rangle e_i + \langle f, e_{i+1}\rangle e_{i+1})\|^2 = \|2\sum_{i=1}^{d}\langle f, e_{i+1}\rangle e_{i+1}\|^2 = 4\|f\|^2. \tag{4}$$

On the other hand,

$$\sum_{i=1}^{d}\|P_i(f)\|^2 = \sum_{i=1}^{d}\|\langle f, e_i\rangle e_i + \langle f, e_{i+1}\rangle e_{i+1}\|^2 = \sum_{i=1}^{N}(|\langle f, e_i\rangle|^2 + |\langle f, e_{i+1}\rangle|^2) = 2\|f\|^2. \tag{5}$$

Combining (4) and (5) yields the claim. □

This lemma now allows us to derive an estimate for the worst-case reconstruction error of a $dM$-circle frame for $\mathbb{R}^d$ using (DFRS) in terms of the worst-case reconstruction error of the $M$-element circular frames it contains.

THEOREM 4.6. *Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_d$ be a $dM = d2N$-circle frame as in* (DFRS) *with analysis operator of $\mathcal{F}$ denoted by $V$ and of $\mathcal{F}_i$ denoted by $V_i$, $i = 1, \ldots, d$. Let $J_i \subseteq \{1, 2, \ldots, M\}$, $i = 1, \ldots, d$ index erasures, then*

$$er_{(\mathrm{DFRS})}(V, J) \leq \sqrt{2}\, max\ \{er_{(\mathrm{FRS})}(V_1, J_1), er_{(\mathrm{FRS})}(V_2, J_2), \ldots, er_{(\mathrm{FRS})}(V_d, J_d)\}. \tag{6}$$

*Moreover, if none of the sets $J_i$ contains consecutive indices, then*

$$er_{(\mathrm{DFRS})}(V, J) \leq \frac{\sqrt{2}}{2N}. \tag{7}$$

*Proof.* Let $f \in \mathbb{R}^d$. First, we observe that the reconstruction error for the $P_i f$'s satisfies

$$\|\frac{1}{N}V_i^* F_{J_i} V_i P_i f - P_i f\| \leq er_{(\mathrm{FRS})}(V_i, J_i)\|P_i f\|\,.$$

Using Lemma 4.5, this implies

$$\|\frac{1}{N}V^* F_J^{(D)} V f - f\| \leq \sum_{i=1}^{d} er_{(\mathrm{FRS})}(V_i, J_i)^2\|P_i f\|^2 = \max_{1\leq i\leq d} er_{(\mathrm{FRS})}(V_i, J_i)^2 \sum_{i=1}^{d}\|P_i f\|^2 = 2\max_{1\leq i\leq d} er_{(\mathrm{FRS})}(V_i, J_i)^2\|f\|^2\,.$$

The estimate for the operator norm now results from choosing a normalized vector $f$,

$$er_{(\mathrm{DFRS})}(V, J) = \|\frac{1}{N}V^* F_J^{(D)} V - I\| \leq \sqrt{2}\max_{1\leq i\leq d} er_{(\mathrm{FRS})}(V_i, J_i)\,.$$

For the moreover-part, it follows from (6) that

$$er_{(\mathrm{DFRS})}(V, J) \leq \sqrt{2} er_{(\mathrm{FRS})}(V_1, J_1).$$

Since setting $\theta = 2\pi/M$,

$$er_{(\mathrm{FRS})}(V_1, J_1) = \frac{1}{N}\|f_{12} - f_{11}\| = \frac{1}{N}\sqrt{\frac{\sin(\theta)^2}{\sin(\frac{\theta}{2})^2}} \leq \frac{1}{2N},$$

we proved (7). □

AFRS (Averaging Frame Coefficient Replacement Scheme)

**Parameters:**

- Ordered frame $\{f_1, f_2, \ldots, f_N\} \subseteq \mathcal{H}$.
- Associated frame coefficients $\{c_1, c_2, \ldots, c_N\} \subseteq \mathbb{R}$.
- Indices of erased frame coefficients $J \subseteq \{1, 2, \ldots, N\}$.

**Algorithm:**

1) For $i = 1, \ldots, N$ do
2) If $i \in J$ and $i-1 \notin J$ and $i+1 \notin J$ then
3) Set $c_i = (c_{i-1} + c_{i+1})/\|f_{i-1} + f_{i+1}\|$.
4) else if $i \in J$ and $i-1 \in J$ or $i \in J$ and $i+1 \in J$ then
5) Set $c_i = 0$.
6) end.
7) end.

**Output:**

- Corrected frame coefficients $\{c_1, c_2, \ldots, c_N\}$.

**Figure 3.** The AFRS Algorithm for preconditioning of the sequence of received incomplete frame coefficients before reconstruction.

### 4.4. An Averaging Replacement Scheme

In this subsection we will introduce yet another replacement scheme for circle frames, this time using an averaging strategy, which surprisingly yields *perfect reconstruction* for 'most' erasures. In fact, if a single coefficient is lost (or a whole family of coefficients with no two consecutive ones is lost), the neighboring coefficients will be *averaged* to fill in the missing one(s), which allows to reconstruct the signal perfectly.

This replacement scheme based on averaging is described in Figure 3. We formulate the replacement scheme for an arbitrary frame, since the specific structure of a circle frame is not required for it to be applicable.

Setting $F_J^{(A)} \in M(N, \mathbb{R})$ to be

$$(F_J^{(A)})_{i,j} = \begin{cases} 1/\|f_{i-1} + f_{i+1}\| & : \; j = i-1 \text{ and } i \in J \text{ and } i-1, i+1 \notin J, \\ 1/\|f_{i-1} + f_{i+1}\| & : \; j = i+1 \text{ and } i \in J \text{ and } i-1, i+1 \notin J, \\ 1 & : \; j = i \text{ and } i \notin J, \\ 0 & : \; \text{else.} \end{cases} \tag{8}$$

As in the cases before, this again allows us to write the output of (AFRS) as

$$F_J^{(A)}(c_1, c_2, \ldots, c_N)^T.$$

We will now study the following worst-case reconstruction error related to this scheme.

Definition 4.7. *Let $\{f_1, f_2, \ldots, f_{dM}\}$ be a $dM$-circle frame with associated analysis operator denoted by $V$. Then the* worst-case reconstruction error *resulting from erasure of coefficients indexed by $J \subset \{1, 2, \ldots dM\}$,* *preconditioned by* (AFRS)*, is defined by the operator norm*

$$er_{(\mathrm{AFRS})}(V,J) = \|\frac{1}{A}V^* F_J^{(A)} V - I\|,$$

*where* $F_J^{(A)}$ *is defined in* (8).

Then, surprisingly, we obtain perfect reconstruction if the erasures are not consecutive.

THEOREM 4.8. *Let* $\{f_1, f_2, \dots, f_{dM}\}$ *be a* $dM$*-circle frame with associated analysis operator denoted by* $V$*. Let* $J$ *be an index subset of* $\{1, 2, \dots, dM\}$ *whichout consecutive indices (modulo* $dM$*). Then*

$$er_{(\mathrm{AFRS})}(V,J) = 0.$$

*Proof.* Since we know the arrangement of vectors in a circle-frame, we can rebuild a given vector by a linear combination of its immediate neighbors.The vectors in a given circle have equal angles between them. Hence we know that the direction of the vector $f_i$ is $f_{i-1} + f_{i+1}$. To normalize this direction, we divide by the norm, $\|f_{i-1} + f_{i+1}\|$. It follows that

$$f_i = \frac{f_{i-1} + f_{i+1}}{\|f_{i-1} + f_{i+1}\|}.$$

From this, the claim follows immediately. □

The surprise is that this theorem proves perfect reconstruction even if about half the coefficients are lost (provided no endpoints are lost and no consecutive ones). The algorithm is astonishingly simple, yet the favorable structure of the frame ensures superior behavior for 'filling' in the missing frame coefficients.

## 5. CONCLUSION AND OUTLOOK

There are many variations of the above replacement scheme that could be pursued. Here, we have focused on erasures of non-adjacent coefficients. We could take into account what happens if we have two consecutive erasures in a Parseval frame $\{f_i\}_{i=1}^N$. In this case, we have several options. First, if we lose the coefficients $\langle f, f_i\rangle$, $\langle f, f_{i+1}\rangle$, we might replace each with $\langle f, f_{i-1}\rangle$ or we might replace $\langle f, f_i\rangle$ with $\langle f, f_{i-1}\rangle$ and $\langle f, f_{i+1}\rangle$ with $\langle f, f_{i+2}\rangle$. More generally, if we have $s$-consecutive erasures one could apply a large number of variations of this replacement scheme. While the error estimates we have computed here would be straightforward, but somewhat tedious to calculate, in practice the most relevant quantity would be an erasure-averaged error for an appropriate distribution of erasures.

Since the circular and semi-circular frames are divided into spanning sets for two dimensional Hilbert spaces, technically, any two vectors from the frame will give perfect reconstruction. The problem here is that trying to do reconstruction with a tiny fraction of the coefficients - in the case of a signal - results in the "noise" and quantization error overpowering the signal and overpowering the reconstruction error from our scheme. Similarly, in the case of the above frames, they can be divided into $M/2$ pairs of orthonormal sets and these could be used to give perfect reconstruction but with the same problems as above.

As previously mentioned, the above variations are computationally expensive. However, if *burst errors*, or consecutive erasures are to be expected for some reason, it makes sense to find a middle ground between the perfect reconstruction scheme and the simple replacement scheme. For example, if the primary causes of vector erasure tend to affect consecutive vectors, we can use an erasure model where erasures of neighboring coefficients are strongly correlated, but those of non-neighboring ones are almost independent.

Finally, we could consider an analogous replacement strategy for fusion frames, which have already been studied in the context of erasures (see[2, 7, 9–11, 17]). Adapting our strategy requires choosing a basis in each subspace and replacing all coordinates of an erased vector in a subspace by the coordinates from a "neighboring" subspace. We leave the details for further study.

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