Riesz frames and approximation of the frame coefficients.

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Abstract

A frame is a family $\{f_i\}_{i=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ with the property that every element in $\mathcal{H}$ can be written as a (infinite) linear combination of the frame elements. Frame theory describes how one can choose the corresponding coefficients, which are called frame coefficients. From the mathematical point of view this is gratifying, but for applications it is a problem that the calculation requires inversion of an operator on $\mathcal{H}$.
The projection method is introduced in order to avoid this problem. The basic idea is to consider finite subfamilies $\{f_i\}_{i=1}^{n}$ of the frame and the orthogonal projection P_n onto its span. For $f \in \mathcal{H}, \mathcal{P}\backslash\{$ has a representation as a linear combination of $f_i, i = 1, 2, ..n$, and the corresponding coefficients can be calculated using finite dimensional methods. We find conditions implying that those coefficients converge to the correct frame coefficients as $n \to \infty$, in which case we have avoided the inversion problem. In the same spirit we approximate the solution to a moment problem. It turns out, that the class of "well-behaving frames" are identical for the two problems we consider.

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1 The projection method.

We begin with some definitions. Let $\mathcal{H}$ be a separable Hilbert space. All index sets are assumed to be countable.

Definitions: 1) A family $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ if $\{f_i\}_{i\in I}$ is total and there exist numbers $A, B > 0$ such that

$$A \sum_{i\in I} |c_i|^2 \leq || \sum_{i\in I} c_i f_i||^2 \leq B \sum_{i\in I} |c_i|^2$$

for all sequences $\{c_i\}_{i\in I} \in \ell^2(I)$.

2) A family $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$ if

$$\exists A, B > 0 : A||f||^2 \leq \sum_{i\in I} | < f, f_i > |^2 \leq B||f||^2, \ \ \forall f \in \mathcal{H}.$$

A, B are called *frame bounds.*

3) $\{f_i\}_{i\in I}$ is a *frame sequence* if $\{f_i\}_{i\in I}$ is a frame for its closed span.

For notational convenience we index our frames by the natural numbers N in the sequel. It will be clear that our results and proofs are completely general and that the case of a different index set can be handled by an adaption of the notation.

We begin by reminding the reader that the important thing with frames is that a frame $\{f_i\}_{i=1}^{\infty}$ gives rise to a decomposition of the underlying space $\mathcal{H}$: if we define the *frame operator* by

$$S : \mathcal{H} \to \mathcal{H}, \ \ \mathcal{S}\{ = \sum_{\rangle=\infty}^{\infty} < \{, \{_\rangle > \{_\rangle,$$

then S is bounded and invertible, and

$$f = SS^{-1}f = \sum_{i=1}^{\infty} < f, S^{-1}f_i > f_i, \ \ \forall f \in \mathcal{H}.$$

It is shown in [C3] that the optimal frame bounds (i.e., maximal lower bound and minimal upper bound) are $\frac{1}{||S^{-1}||}$ and $||S||$, respectively.

$\{< f, S^{-1}f_i >\}$ are called the *frame coefficients.* From the point of view of applications the problem with calculation of the frame coefficients is to invert S. An idea to overcome this difficulty is to "truncate" the problem. So we look at finite subfamilies $\{f_i\}_{i=1}^n, n \in N$ and define the space $\mathcal{H}_n := span\{f_i\}_{i=1}^n$ and the corresponding frame operator

$$S_n : \mathcal{H}_n \to \mathcal{H}_n, \quad S_n f = \sum_{i=1}^{n} < f, f_i > f_i.$$

One can show that the orthogonal projection of $f \in \mathcal{H}$ onto $\mathcal{H}_n$ is given by

$$P_n f = \sum_{i=1}^{n} < f, S_n^{-1} f_i > f_i = \sum_{i=1}^{n} < f, f_i > S_n^{-1} f_i.$$

Since $P_n f \to f$ as $n \to \infty$, one can hope that the corresponding coefficients converges to the frame coefficients for f, i.e., that

$$(1) \quad < f, S_n^{-1} f_i > \to < f, S^{-1} f_i > \text{ as } n \to \infty, \ \ \forall i \in N, \ \forall f \in \mathcal{H}.$$

If (1) is satisfied we say that the *projection method* works. In this case the frame coefficients can be approximated as close as we want using finite dimensional methods, i.e.,linear algebra, since S_n is an operator on the finite dimensional space $\mathcal{H}_n$. For a discussion of the projection method we refer to [C1]; in particular it is shown there, that the projection method works if and only if

$$\forall j \in N \ \exists c_j : \quad ||S_n^{-1} f_j|| \leq c_j \ \text{ for } \ n \geq j.$$

If the projection method works the next natural question is how fast the convergence in (1) is. For example, one might wish that the set of coefficients $\{< f, S_n^{-1} f_i >\}_{i=1}^n$ converges to the set of frame coefficients in the ℓ^2-sense, i.e., that

$$(2) \sum_{i=1}^{n} | < f, S_n^{-1} f_i > - < f, S^{-1} f_i > |^2 + \sum_{i=n+1}^{\infty} | < f, S^{-1} f_i > |^2 \to 0, \ \forall f \in \mathcal{H}.$$

We say that the *strong projection method* works if (2) is satisfied. Note again that this condition depends on the indexing of the elements. The second term $\sum_{i=n+1}^{\infty} | < f, S^{-1} f_i > |^2 \to 0$ for every frame, $\forall f \in \mathcal{H}$, so we only need to show that $\sum_{i=1}^{n} | < f, S_n^{-1} f_i > - < f, S^{-1} f_i > |^2 \to 0, \ \forall f \in \mathcal{H}$.

Our results concern some special frames, which we introduce now. A family $\{f_i\}_{i=1}^{\infty}$ is called a *Riesz frame* if every subfamily $\{f_i\}_{i=1}^{\infty}, I \subseteq N$, is a frame sequence, with bounds A, B which are common for all those frames. Clearly one only has to check the existence of a common lower bound, and it is easy to check that $\{f_i\}_{i=1}^{\infty}$ is a Riesz frame if this condition is satisfied for all finite index sets $I \subseteq N$.
$\{f_i\}_{i=1}^{\infty}$ is a *conditional Riesz frame* if there are common bounds for all the frame sequences $\{f_i\}_{i=1}^{n}, n \in N$. In terms of the operators S_n this is equivalent to the condition $sup_n ||S_n^{-1}||_{\mathcal{H}_n} < \infty$. Observe that this notion depends on the indexing of the frame elements. For example, if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis, then $\{e_i, \frac{1}{i} e_i\}_{i=1}^{\infty}$ is a conditional Riesz frame,but $\{\frac{1}{i} e_i, e_i\}_{i=1}^{\infty}$ is not.

It is clear how to extend the definitions and methods to the case of a frame indexed by an arbitrarily countable index set I: take a family $\{I_n\}_{n=1}^{\infty}$ of finite subsets of I such that

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \nearrow I$$

and replace $\{f_i\}_{i=1}^{n}$ by $\{f_i\}_{i \in I_n}$. All the following results are true for general index sets with this modification.

Our results are most conveniently formulated in operator terminology. Let $T, \{T_n\}_{n=1}^{\infty}$ be operators from $\mathcal{H}$ into the Hilbert space $(\mathcal{K}, < \cdot, \cdot >_{\mathcal{K}})$. We say that $T_n \to T$ in the *strong operator topology* if $T_n f \to Tf, \forall f \in \mathcal{H}$, and that $T_n \to T$ in the *weak operator topology* if $< g, T_n f >_{\mathcal{K}} \to < g, Tf >_{\mathcal{K}}, \forall f \in \mathcal{H}, \forall \} \in \mathcal{K}$. For basic properties of these topologies we refer to Conway [C].

Theorem 1: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame. Then the following are equivalent:*

(a) The strong projection method works.

(b) $\{f_i\}_{i=1}^{\infty}$ is a conditional Riesz frame.

(c) $S_n^{-1} P_n \to S^{-1}$ in the strong operator topology .

(d) $S_n^{-1} P_n \to S^{-1}$ in the weak operator topology.

(e) $lim_{n\to\infty}\sum_{i=n+1}^{\infty}| < S_n^{-1}P_nf, f_i > |^2 = 0,\ \forall f \in \mathcal{H}$.

Proof: We begin with some general observations. We consider $S_n^{-1}P_n$ as an operator on $\mathcal{H}$, and at several places we use that $||S_n^{-1}||_{\mathcal{H}_n} = ||S_n^{-1}P_n||,\ \forall n$. For every $f \in \mathcal{H}$ we have

$$< S_n^{-1}P_nf, f > = < S_n^{-1}\sum_{i=1}^{n} < f, S_n^{-1}f_i > f_i, f > = \sum_{i=1}^{n} | < f, S_n^{-1}f_i > |^2.$$

Also, since $S_n^{-1}P_n$ is a positive operator there are positive (and hence self adjoint) operators $T_n : \mathcal{H} \to \mathcal{H}$ such that $S_n^{-1}P_n = T_n^2$.
Let A, B be frame bounds for $\{f_i\}_{i=1}^{\infty}$. Then $\{S^{-1}f_i\}_{i=1}^{\infty}$ is a frame with bounds $1/A, 1/B$.

$(a) \Rightarrow (b)$: We prove the contrapositive, so assume $sup_n\|S_n^{-1}\||_{\mathcal{H}_n} = \infty$. Since $\|S_n^{-1}P_n\| \leq \|T_n\|^2$, it follows that $sup_n\|T_n\| = \infty$. By the principle of uniform boundedness, there is an $f \in \mathcal{H}$ with $\|f\| = 1$ and $sup_n||T_nf|| = \infty$, implying that

$$sup_n < S_n^{-1}P_nf, f > = sup_n < T_n^2f, f > = sup_n < T_nf, T_nf > = sup_n\|T_nf\|^2 = \infty.$$

Now, for this f,

$$(\sum_{i=1}^{n} | < f, S^{-1}f_i > - < f, S_n^{-1}f_i > |^2)^{1/2} \geq$$

$$(\sum_{i=1}^{n} | < f, S_n^{-1}f_i > |^2)^{1/2} - (\sum_{i=1}^{n} | < f, S^{-1}f_i > |^2)^{1/2} \geq \sqrt{< S_n^{-1}P_nf, f >} - \frac{1}{\sqrt{A}},$$

which does not go to 0 as $n \to \infty$. Hence, the strong projection method does not work.

$(b) \Rightarrow (c)$: If $\{f_i\}_{i=1}^{\infty}$ is a conditional Riesz frame and $f \in \mathcal{H}$, then

$$||S_n^{-1}P_nf - S^{-1}f|| = ||S_n^{-1}P_nf - P_nS^{-1}f + P_nS^{-1}f - S^{-1}f||$$

$$\leq ||S_n^{-1}P_nf - S_n^{-1}P_nS_nP_nS^{-1}f|| + ||P_nS^{-1}f - S^{-1}f||$$

$$\leq ||S_n^{-1}P_n|| \cdot ||f - S_nP_nS^{-1}f|| + ||P_nS^{-1}f - S^{-1}f|| \to 0 \text{ for } n \to \infty,$$

since $\{||S_n^{-1}P_n||\}_{n=1}^{\infty}$ is bounded and $S_nP_n \to S,\ P_n \to I$, both in the strong operator topology.

$(e) \Rightarrow (c) \Rightarrow (a)$ is proven in [C3, Theorem 4.5].

$(c) \Rightarrow (d)$: obvious.

$(d) \Rightarrow (b)$: Let $f \in \mathcal{H}$. Then

$$||T_nf||^2 =< T_nf, T_nf >=< S_n^{-1}P_nf, f > \to < S^{-1}f, f > \text{ for } n \to \infty.$$

Therefore the family $\{T_n\}_{i=1}^{\infty}$ is pointwise bounded on $\mathcal{H}$, and so by the principle of uniform boundedness, the operators are uniformly bounded. Hence,

$$sup_n\|S_n^{-1}\|_{\mathcal{H}_n} \leq sup_n\|T_n\|^2 < \infty.$$

$(c) \Rightarrow (e)$: Suppose (c) is satisfied. Then, for any $f \in \mathcal{H}$,

$$(\sum_{i=n+1}^{\infty} | < S_n^{-1}P_nf, f_i > |^2)^{1/2} \leq$$

$$(\sum_{i=n+1}^{\infty} | < (S^{-1} - S_n^{-1}P_n)f, f_i > |^2)^{1/2} + (\sum_{i=n+1}^{\infty} | < S^{-1}f, f_i > |^2)^{1/2} \leq$$

$$\sqrt{B}\|(S^{-1} - S_n^{-1}P_n)f\| + (\sum_{i=n+1}^{\infty} | < S^{-1}f, f_i > |^2)^{1/2} \to 0 \text{ for } n \to \infty.$$

Q.E.D.

Note in particular the equivalence between (c) and (d)! Condition (c) is also interesting from the point of view of applications. It is well known among frame experts that it is difficult to calculate the dual frame $\{S^{-1}f_i\}_{i=1}^{\infty}$ of a given frame, and (c) shows that we can find arbitrarily good approximations using finite dimensional methods if $\{f_i\}_{i=1}^{\infty}$ is a conditional Riesz frame, since then $S_n^{-1}f_i \to S^{-1}f_i,\ \forall i$. Also, (d) shows that the condition for the strong projection method to work can be formulated in a very similar way as the condition for the projection method to work: the projection method works iff

$$< f, S_n^{-1} f_i > \to < f, S^{-1} f_i >, \ \forall i, \ \forall f \in \mathcal{H}$$

and the strong projection method works iff

$$< f, S_n^{-1} P_n g > \to < f, S^{-1} g >, \ \forall f, g \in \mathcal{H}.$$

Corollary 2: *If $\{f_i\}_{i=1}^{\infty}$ is a Riesz frame, then the strong projection method works for every subfamily $\{f_i\}_{i \in I}, I \subseteq N$, for every indexing of the frame elements.*

One could expect the opposite of Corollary 2 to be true, namely that if the strong projection method works for every subset of a frame and for every indexing of the elements, then the frame should be a Riesz frame. However, this is not true, as the following example shows:

Example 1: Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and define

$$\{f_i\}_{i=1}^{\infty} := \{e_i, e_i + \frac{1}{2^i} e_1\}_{i=2}^{\infty}.$$

Clearly $\{f_i\}_{i=1}^{\infty}$ is a frame, but not a Riesz frame. Let $\{h_i\}_{i=1}^{\infty}$ be a permutation of any subfamily of $\{f_i\}_{i=1}^{\infty}$. We want to prove that the strong projection method works for $\{h_i\}_{i=1}^{\infty}$, and by Theorem 1 this amounts to showing that $\{h_i\}_{i=1}^{\infty}$ is a conditional Riesz frame. Given $n \in N$, write

$$\{h_i\}_{i=1}^{n} = \{e_j\}_{j \in \Delta_1} \cup \{e_j + \frac{1}{2^j} e_1\}_{j \in \Delta_2}.$$

Let $\mathcal{H}_1 = span\{e_j\}_{j \in \Delta_1}, \mathcal{H}_2 = span\{e_j + \frac{1}{2^j} e_1\}_{j \in \Delta_2}$ and denote the frame operator for $\{h_i\}_{i=1}^{n}$ by S_n. First, assume that $\Delta_1 \cap \Delta_2 = \emptyset$. It is easy to check that $\{e_j + \frac{1}{2^j} e_1\}_{j \in \Delta_2}$ is a Riesz basis for $\mathcal{H}_2$ with lower bound $1/2$. Since $\{e_j\}_{j \in \Delta_1}$ is an orthonormal basis for $\mathcal{H}_1$ and $\mathcal{H}_1 \perp \mathcal{H}_2$, the family $\{h_i\}_{i=1}^{n}$ is a Riesz basis for its span with lower bound $1/2$, and so $\|S_n^{-1}\| \leq 2$. Now assume that $\Delta_1 \cap \Delta_2 \neq \emptyset$, and let m be the smallest number in the intersection. Then, $span\{h_i\}_{i=1}^{n} = span\{\{e_1\}, \{e_i\}_{i \in \Delta_1 \cup \Delta_2}\}$. Let $f = a_1 e_1 + \sum_{i \in \Delta_1 \cup \Delta_2} a_i e_i$ satisfy

$$\|f\|^2 = |a_1|^2 + \sum_{i \in \Delta_1 \cup \Delta_2} |a_i|^2 = 1.$$

Now, if $|a_1|^2 \geq \frac{1}{2^2}$, then

$$\sum_{i=1}^{n} | < f, h_i > |^2 \geq | < f, e_m > |^2 + | < f, e_m + \frac{1}{2^m} e_1 > |^2$$

$$= |a_m|^2 + |a_m - \frac{-a_1}{2^m}|^2 \geq \frac{|a_1|^2}{2^{2m+2}} \geq \frac{1}{2^{2m+4}}.$$

(The next to last inequality follows from the observation that for any two complex numbers a, b we have, $|a|^2 + |a-b|^2 \geq (max\{|a|, |a-b|\})^2 \geq |\frac{b}{2}|^2$) Otherwise, if $|a_1|^2 \leq \frac{1}{2^2}$, then

$$\sum_{i=1}^{n} | < f, h_i > |^2 = \sum_{i \in \Delta_1} |a_i|^2 + \sum_{i \in \Delta_2} |a_i + \frac{1}{2^i} a_1|^2 \geq$$

$$\sum_{i \in \Delta_1} |a_i|^2 + (\sqrt{\sum_{i \in \Delta_2} |a_i|^2} - \sqrt{\sum_{i \in \Delta_2} |\frac{a_1}{2^i}|^2})^2 =$$

$$\sum_{i \in \Delta_1 \cup \Delta_2} |a_i|^2 + \sum_{i \in \Delta_2} |\frac{a_1}{2^i}|^2 - 2\sqrt{\sum_{i \in \Delta_2} |a_i|^2} \sqrt{\sum_{i \in \Delta_2} |\frac{a_1}{2^i}|^2} \geq$$

$$\frac{3}{4} - 2|a_1| \sqrt{\sum_{i=1}^{\infty} \frac{1}{2^{2i}}} \geq \frac{3}{4} - \frac{1}{\sqrt{3}}.$$

Putting all together, we have that $\|S_n^{-1}\| \leq 2^{2m+4}$ for all n, so $\{h_i\}_{i=1}^{\infty}$ is a conditional Riesz frame.

We consider it as important to understand the difference between Riesz frames and conditional Riesz frames. It is shown in [C3] that every Riesz frame contains a Riesz basis (see [CC] for an extension). We now present an example of a conditional Riesz frame not containing a Riesz basis:

Example 2: Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and consider the frame

$$\{f_i\}_{i=1}^{\infty} := \{e_1, e_1 + e_2, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} e_2, e_2 + e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, e_3 + e_4, \cdots\}$$

Consider the first n elements of the frame and let

$$\mathcal{H}_n = \{f_i\}_{i=1}^n = span\{e_1, e_1+e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, e_2+e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, e_3+e_4,$$

$$\cdots, e_{m-1}+e_m, \frac{1}{\sqrt{m}}e_m, \frac{1}{\sqrt{m}}e_m, \cdots, \frac{1}{\sqrt{m}}e_m\} = span\{e_i\}_{i=1}^m$$

where we have $\leq m$ terms of the form $\frac{1}{\sqrt{m}}e_m$. Now consider an element $f \in \mathcal{H}_n$. Write $f = \sum_{i=1}^m a_i e_i$ and suppose that $\|f\| = 1$. If $|a_m| \geq \sqrt{\frac{2}{3}}$ then

$$\sum_{i=1}^n |< f, f_i >|^2 \geq |< f, e_{m-1}+e_m >|^2 \geq (\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}})^2$$

and if $|a_m| \leq \sqrt{\frac{2}{3}}$ we have

$$\sum_{i=1}^n |< f, f_i >|^2 \geq \sum_{j=1}^{m-1} \sum_{i=1}^{j} |< f, \frac{1}{\sqrt{j}}e_j >|^2 = \sum_{i=1}^{m-1} |a_i|^2 \geq \frac{1}{3}.$$

So the frame is a conditional Riesz frame. But the frame does not contain a Riesz basis: a Riesz basis is norm-bounded below, so in order to span $\mathcal{H}$ any subsequence of this frame which is going to form a Riesz basis must contain, for some k, the set $\{e_i + e_{i+1}\}_{i=k}^{\infty}$. But this set is not part of any Riesz basis.

Example 2 also provides an example of a conditional Riesz frame, where not every subfamily is a frame sequence.

Our next example shows how bad things can go for non-Riesz frames:

Example 3: Let $f_0 = \sum_{n=1}^{\infty} \frac{1}{n} e_n$. Then $\{f_i\}_{i=0}^{\infty} := \{f_0\} \cup \{e_i\}_{i=1}^{\infty}$ has the following properties:

1) Every subfamily is a frame sequence, but $\{f_i\}_{i=0}^{\infty}$ is not a conditional Riesz frame.

2) $\{f_i\}_{i=0}^{\infty}$ has a subset which is an orthonormal basis, but $\{f_i\}_{i=0}^{\infty}$ has no permutation for which the projection method works.

First we prove that every subfamily is a frame sequence. Let Δ be a proper subset of the natural numbers. If $0 \notin \Delta$, then $\{f_i\}_{i\in\Delta}$ is orthonormal, so assume $0 \in \Delta$. Then

$$\mathcal{H}_\Delta := span\{f_i\}_{i\in\Delta} = span\{\{e_i\}_{i\in\Delta-\{0\}}, \sum_{i\notin\Delta} \frac{1}{i} e_i\}.$$

Let $f \in \mathcal{H}_\Delta,\ f = \sum_{i\in\Delta-\{0\}} a_i e_i + a \sum_{i\notin\Delta} \frac{1}{i} e_i$ and suppose that

$$\|f\|^2 = \sum_{i\in\Delta-\{0\}} |a_i|^2 + (|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|)^2 = 1.$$

Now,

$$\sum_{i\in\Delta} |< f, f_i >|^2 = \sum_{i\in\Delta-\{0\}} |a_i|^2 + |< \sum_{i\in\Delta-\{0\}} a_i e_i, f_0 > + a \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2|^2.$$

We may assume that $\sum_{i\in\Delta-\{0\}} |a_i|^2 \le 1/2$. If now

$$(\sum_{i\in\Delta-\{0\}} |a_i|^2)^{1/2} \ge \frac{|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2}{2\|f_0\|} = \frac{|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|}{2\|f_0\|}$$

then

$$1 = \sum_{i\in\Delta-\{0\}} |a_i|^2 + (|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|)^2 \le \sum_{i\in\Delta-\{0\}} |a_i|^2 + \frac{4\|f_0\|^2}{\| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2} \sum_{i\in\Delta-\{0\}} |a_i|^2.$$

So

$$\sum_{i\in\Delta-\{0\}} |a_i|^2 \ge \frac{1}{1 + \frac{4\|f_0\|^2}{\| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2}}.$$

Otherwise,

$$\sum_{i\in\Delta} |< f, f_i >|^2 \ge |< \sum_{i\in\Delta-\{0\}} a_i e_i, f_0 > + a \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2|^2$$

$$\ge (|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2 - (\sum_{i\in\Delta-\{0\}} |a_i|^2)^{1/2} \cdot \|f_0\|)^2 \ge (\frac{|a| \cdot \| \sum_{i\notin\Delta} \frac{1}{i} e_i\|^2}{2})^2 =$$

$$\frac{1}{4}|a|^2 \cdot \| \sum_{i \notin \Delta} \frac{1}{i} e_i \|^4 = \frac{1}{4}(1 - \sum_{i \in \Delta - \{0\}} |a_i|^2) \| \sum_{i \notin \Delta} \frac{1}{i} e_i \|^2 \geq \frac{1}{8} \| \sum_{i \notin \Delta} \frac{1}{i} e_i \|^2.$$

So $\{f_i\}_{i \in \Delta}$ is a frame sequence with lower bound $min\{\frac{1}{2}, \frac{1}{1+\frac{4\|f_0\|^2}{\| \sum_{i \notin \Delta} \frac{1}{i} e_i \|^2}}, \frac{1}{8} \| \sum_{i \notin \Delta} \frac{1}{i} e_i \|^2\}$.

Part 2) combined with Theorem 1 implies that $\{f_i\}_{i=0}^{\infty}$ is not a conditional Riesz frame, so we prove 2)

2) Let $\{f_{\sigma(k)}\}_{k=1}^{\infty}$ be a permutation of the frame and let S_n be the frame operator for $\{f_{\sigma(k)}\}_{k=1}^{n}$. Assume $f_{\sigma(m)} = f_0$. For any $n \geq m$, let

$$h_n = \frac{\sum_{k=n+1}^{\infty} \frac{1}{\sigma(k)} e_{\sigma(k)}}{\| \sum_{k=n+1}^{\infty} \frac{1}{\sigma(k)} e_{\sigma(k)} \|^2}.$$

Then

$$S_n h_n = \sum_{k=1}^{n} < h_n, f_{\sigma(k)} > f_{\sigma(k)} =< h_n, f_{\sigma(m)} > f_{\sigma(m)} =< h_n, f_0 > f_0 = f_0.$$

Hence, $S_n^{-1} f_0 = h_n$. But, $lim_{n \to \infty} \|h_n\| = \infty$, so the projection method does not work for $\{f_{\sigma(k)}\}_{k=1}^{\infty}$.

The operator approach in Theorem 1 makes it easy to prove that the strong projection method still works if we apply a bounded invertible operator to a conditional Riesz frame:

Proposition 3: *Let $U : \mathcal{H} \to \mathcal{K}$ be an isomorphism. If the strong projection method works for the frame $\{f_i\}_{i=1}^{\infty}$, then it also works for the frame $\{Uf_i\}_{i=1}^{\infty}$.*

Proof: It is well known that $\{Uf_i\}_{i=1}^{\infty}$ is a frame, and direct calculation shows that the frame operator is

$$R : \mathcal{K} \to \mathcal{K}, \ \mathcal{R} = \mathcal{USU}^*.$$

Let T_n denote the restriction of U to $\mathcal{H}_n$. The frame operator for $\{Uf_i\}_{i=1}^{n}$ is

$$R_n : span\{Uf_i\}_{i=1}^{n} \to span\{Uf_i\}_{i=1}^{n}, \ R_n = T_n S_n T_n^*.$$

Since $||T_n^{-1}|| \leq ||U^{-1}||$, we have

$$||R_n^{-1}|| = ||(T_n^*)^{-1} S_n^{-1} T_n^{-1}|| \leq ||U^{-1}||^2 \cdot ||S_n^{-1}||.$$

So if $\{f_i\}_{i=1}^{\infty}$ is a conditional Riesz frame, then so is $\{Uf_i\}_{i=1}^{\infty}$. Now use Theorem 1. **Q.E.D.**

In particular, the strong projection method works for a frame $\{f_i\}_{i=1}^{\infty}$ if and only if it works for the dual frame $\{S^{-1}f_i\}_{i=1}^{\infty}$.

2 A moment problem.

The principle of approximation using finite subsets of a frame can be used in many other contexts, of which we present one more here. Let again $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ and let $\{a_i\}_{i=1}^{\infty} \subseteq \ell^2(N)$. We ask whether there exists $f \in \mathcal{H}$ such that

$$< f, f_i >= a_i, \ \forall i \in N.$$

A problem of this type is called a *moment problem*. For the general theory we refer to [Y]. It is easy to construct examples where there is no solution f, but as shown in [C2] there always exists a unique element in $\mathcal{H}$ minimizing $\sum_{i=1}^{\infty} |a_i - < f, f_i > |^2$; this element is $f = \sum_{i=1}^{\infty} a_i S^{-1} f_i$. We call $\sum_{i=1}^{\infty} a_i S^{-1} f_i$ the *b.a.s.* (best approximation solution) to the moment problem.
Corresponding to a subset $\{a_i\}_{i=1}^{n}$ the unique element in $\mathcal{H}_n$ minimizing $\sum_{i=1}^{n} |a_i - < f, f_i > |^2$ is $\sum_{i=1}^{n} a_i S_n^{-1} f_i$. We say that $\sum_{i=1}^{n} a_i S_n^{-1} f_i$ is the *b.a.s. to the truncated moment problem.* In analogy with above we would like to find conditions implying that

$$(3) \quad \sum_{i=1}^{n} a_i S_n^{-1} f_i \to \sum_{i=1}^{\infty} a_i S^{-1} f_i \text{ for } n \to \infty, \ \forall \{a_i\}_{i=1}^{\infty} \in l^2(N).$$

Zwaan [Z] has shown that (3) is satisfied if $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis. Also, the problem is clearly related to the projection method: if (3) is satisfied (or just for all sequences with 1 in one entry, otherwise 0), then $S_n^{-1} f_i \to S^{-1} f_i$ for $n \to \infty, \forall i$, so the projection method works. Actually we can prove that (3) is satisfied if and only if $\{f_i\}_{i=1}^{\infty}$ is a conditional Riesz frame, i.e., if and only if the strong projection method works. We need the operator

(sometimes called the *pre-frame operator*)

$$T : \ell^2(N) \to \mathcal{H}, \quad \mathcal{T}\{\rfloor_\rangle\}_{\rangle=\infty}^{\infty} = \sum_{\rangle=\infty}^{\infty} \rfloor_\rangle \{_\rangle.$$

We denote the kernel of T by N_T. T is bounded and the adjoint operator $T^* : \mathcal{H} \to \ell^{\in}(\mathcal{N})$ is given by $T^* f = \{< f, f_i >\}_{i=1}^{\infty}$.

Theorem 4: *Let $\{f_i\}_{i=1}^{\infty}$ be a frame. Then the following are equivalent:*

(a) $\sum_{i=1}^{n} a_i S_n^{-1} f_i \to \sum_{i=1}^{\infty} a_i S^{-1} f_i$ *for* $n \to \infty, \ \forall \{a_i\}_{i=1}^{\infty} \in \ell^2(N)$.

(b) $S_n^{-1} \sum_{i=1}^{n} b_i f_i \to 0$ *for* $n \to \infty$
for all $\{b_i\}_{i=1}^{\infty} \in l^2(N)$ *such that* $\sum_{i=1}^{\infty} b_i f_i = 0$.

(c) $\{f_i\}_{i=1}^{\infty}$ *is a conditional Riesz frame.*

Proof:

$(a) \Leftrightarrow (b)$: Since ℓ^2 is the orthogonal sum of the range of T^* and the kernel of T, we can write any sequence $\{a_i\}_{i=1}^{\infty} \in \ell^2$ as

$$\{a_i\}_{i=1}^{\infty} = \{< g, f_i >\}_{i=1}^{\infty} + \{b_i\}_{i=1}^{\infty}$$

for some $g \in \mathcal{H}, \{\lfloor_\rangle\}_{\rangle=\infty}^{\infty} \in \mathcal{N}_\mathcal{T}$. Now, the b.a.s. to the truncated moment problem is

$$\sum_{i=1}^{n} a_i S_n^{-1} f_i = \sum_{i=1}^{n} < g, f_i > S_n^{-1} f_i + \sum_{i=1}^{n} b_i S_n^{-1} f_i = P_n g + S_n^{-1} \sum_{i=1}^{n} b_i f_i.$$

The b.a.s. to the moment problem is

$$\sum_{i=1}^{\infty} a_i S^{-1} f_i = \sum_{i=1}^{\infty} < g, f_i > S^{-1} f_i + S^{-1} \sum_{i=1}^{\infty} b_i f_i = g,$$

from which the result follows.

$(a) \Rightarrow (c)$: Define $Q : \ell^2 \to \mathcal{H}$ by,

$$Q\{a_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} a_i S^{-1} f_i,$$

and define $Q_n : \ell^2 \to \mathcal{H}$ by,

$$Q_n\{a_i\}_{i=1}^{\infty} = \sum_{i=1}^{n} a_i S_n^{-1} f_i.$$

(a) states that $Q_n \to Q$ in the strong operator topology. Hence $sup_n ||Q_n|| < \infty$. But, $Q_n^* f = \{< f, S_n^{-1} f_i >\}_{i=1}^{n}$, and $P_n f = \sum_{i=1}^{n} < f, S_n^{-1} f_i > f_i$, implies

$$S_n^{-1} P_n f = \sum_{i=1}^{n} < f, S_n^{-1} f_i > S_n^{-1} f_i = Q_n Q_n^* f.$$

Hence,

$$sup_n ||S_n^{-1}||_{\mathcal{H}_n} = sup_n \|S_n^{-1} P_n\| = sup_n \|Q_n Q_n^*\| \leq sup_n \|Q_n\|^2 < \infty.$$

$(c) \Rightarrow (b)$: Follows from

$$||S_n^{-1} \sum_{i=1}^{n} b_i f_i|| \leq ||S_n^{-1}||_{\mathcal{H}_n} || \sum_{i=1}^{n} b_i f_i||.$$

Q.E.D.

The problems considered here are very important for frames arising in applications, namely, wavelet frames and Weyl-Heisenberg frames. It seems to be difficult to decide when such frames are conditional Riesz frames. This question is under current investigation.

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