

THE ROAD TO EQUAL-NORM PARSEVAL FRAMES

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ABSTRACT. The construction of equal-norm Parseval frames is fundamental for many applications of frame theory. We present a construction method based on a system of ordinary differential equations, which generates a flow on the set of Parseval frames that converges to equal-norm Parseval frames. We developed this method to address a question posed by Vern Paulsen: How close is a nearly equal-norm, nearly Parseval frame to an equal-norm Parseval frame? The distance estimate derived here can be used to substantiate numerically found, approximate constructions of equal-norm Parseval frames. The estimate is valid for a fairly general class of frames — requiring that the dimension of the Hilbert space and the number of frame vectors is relatively prime. In addition, we re-phrase our distance estimate to show that certain projection matrices which are nearly constant on the diagonal are close in Hilbert-Schmidt norm to ones which have a constant diagonal.

1. INTRODUCTION

A family of vectors $\{f_j\}_{j \in J}$ is a *frame* for a Hilbert space \mathcal{H} if it provides a stable embedding of \mathcal{H} in $\ell^2(J)$ when each vector in \mathcal{H} is mapped to the sequence of its inner products with the frame vectors. Frames were defined by Duffin and Schaeffer [16] to address some deep questions in non-harmonic Fourier series. Traditionally, frames were most popular in signal processing [21], but today, frame theory has an abundance of applications in pure mathematics, applied mathematics, engineering, medicine and even quantum communication [8, 15, 21, 28, 1, 5].

Many of these applications give rise to design problems in frame theory, the construction of frames with certain desired properties. Digital transmissions of analog signals, for example, often rely on frames because of their built-in resilience to data loss [20, 19], and it has been shown that encoding with equal-norm Parseval frames has certain optimality properties for this purpose [11] (see also [25, 6]). Moreover, the

The first author was supported in part by NSF grant DMS-0807399, the second author by NSF DMS-0704216.

use of frames for compensating quantization errors has relied on equal-norm Parseval frames as well [4, 7]. Despite their popularity, we know only a few ways to construct such frames analytically [23, 3, 9, 13], mostly with the help of group actions. Success has been claimed for generating a special type of equal-norm Parseval frames with numerical methods [32], however, the analytic verification of convergence remains wanting. The use of *frame potentials* [3, 10] shows the *existence* of large numbers of equal-norm Parseval frames, but offers little control over additional properties (see [25, 13]). Finally, there is an algorithm due to Holmes and Paulsen [25] for turning a Parseval frame into an equal-norm Parseval frame in finitely many moves. Unfortunately, to the best of the authors' abilities, it cannot be combined with the numerical results to provide the existence of an equal-norm Parseval frame in the close vicinity of a nearly equal-norm and nearly Parseval frame, because it does not include a distance estimate. Here, the metric on the set of frames is induced by the norm on the Hilbert space when frames are viewed as vector-valued, square summable functions (see Section 2 for precise definitions).

The closest Parseval frame to a frame $\{f_j\}_{j \in J}$ is known [2, 9, 12, 26]. Also, the closest equal-norm frame to a given frame can be found easily [9]. However, despite a significant amount of effort, so far we knew very little about the closest equal-norm Parseval frame to a given frame. This question is known in the field as the *Paulsen Problem*. The main problem here is that finding a close equal-norm frame to a given frame involves a *geometric* condition while finding a close Parseval frame involves an *algebraic or spectral condition*.

We will present the first method for finding an equal norm Parseval frame in the vicinity of a given frame which gives quantitative estimates for the distance. The new technique we introduce is a system of vector-valued ODEs which induces a flow on the set of Parseval frames that converges to equal norm Parseval frames. We then bound the arc length traversed by a frame by an integral of the so-called frame energy. With an exponential bound on the frame energy, we derive a quantitative estimate for the distance between our initial, ϵ -nearly equal-norm and ϵ -nearly Parseval frame $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ for a d -dimensional real or complex Hilbert space and the equal-norm Parseval frame $\mathcal{G} = \{g_1, g_2, \dots, g_n\}$ obtained as the limit of the flow governed by the ODE system:

$$\left(\sum_{j=1}^n \|f_j - g_j\|^2 \right)^{1/2} \leq \frac{29}{8} d^2 n(n-1)^8 \epsilon.$$

We also show that the order of ϵ in this estimate cannot be improved.

For our method to work, we must assume that the dimension d of the Hilbert space and the number n of frame vectors are relatively prime. We will use a tensor product technique to show that if our main goal is to produce equal norm Parseval frames, this is not a serious restriction. Finally, we show that the Paulsen problem is equivalent to a fundamental problem in matrix theory, and so we find an answer for the corresponding case of this problem.

We believe that the techniques introduced in this paper will have application to other “nearness” questions in frame theory, in particular, to the famous equiangular tight frame problem [25, 31]. Finding and classifying such frames, or even the easier problem of finding equiangular lines through the origin in \mathbb{R}^n or \mathbb{C}^n , started in 1948 by Hanntjes [?, 14], still leaves a lot to be done. This type of equal-norm Parseval frames is particularly important because of their applications to signal processing [33, 31, 6, 34, 27] and to quantum information theory [35, 30, 17, 5].

2. PRELIMINARIES

In this section, we introduce the notation and terminology used throughout the paper.

Definition 2.1. *A family of vectors $\mathcal{F} = \{f_j\}_{j \in J}$ is a frame for a Hilbert space \mathcal{H} if there are constants $0 < A \leq B < \infty$ so that*

$$A\|x\|^2 \leq \sum_{j \in J} |\langle x, f_j \rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

We call the largest A and smallest B the *lower* and *upper* frame bounds respectively. If we can choose $A = B$ then \mathcal{F} is a *tight frame* and if $A = B = 1$ it is a *Parseval frame*. If all the frame vectors have the same norm, it is an *equal norm frame*. The *analysis operator* of the frame is the map $V : \mathcal{H} \rightarrow \ell^2(J)$ given by $(Vx)_j = \langle x, f_j \rangle$. Its adjoint is the *synthesis operator* which maps $a \in \ell^2(J)$ to $V^*(a) = \sum_{j \in J} a_j f_j$. The *frame operator* is the positive, self-adjoint invertible operator $S = V^*V$ on \mathcal{H} and the *Grammian* is the matrix G with entries $G_{j,k} = \langle f_j, f_k \rangle$ so that $G_{j,k} = (VV^*)_{k,j}$, $k, j \in \{1, 2, \dots, n\}$.

Definition 2.2. *(1) A frame $\{f_j\}_{j=1}^n$ for a d -dimensional real or complex Hilbert space \mathcal{H} is ϵ -nearly equal norm with constant c if*

$$(1 - \epsilon)c \leq \|f_j\| \leq (1 + \epsilon)c, \quad \text{for all } j \in \{1, 2, \dots, n\}.$$

(2) The frame is ϵ -nearly Parseval if the frame constants can be chosen as $A = 1 - \epsilon$ and $B = 1 + \epsilon$, so for all $x \in \mathcal{H}$,

$$(1 - \epsilon)\|x\|^2 \leq \sum_{j \in J} |\langle x, f_j \rangle|^2 \leq (1 + \epsilon)\|x\|^2.$$

If a frame satisfies either of these properties (1) or (2) with $\epsilon = 0$ then we say that it is an equal-norm frame or a Parseval frame, respectively.

Remark 2.3. The operators V and V^* allow us to state the above properties in an alternative fashion. If a frame $\{f_j\}_{j=1}^n$ is ϵ -nearly equal norm, then the diagonal entries of the Grammian, $G_{j,j} = (VV^*)_{j,j} = \|f_j\|^2$, are bounded above and below by $(1 + \epsilon)^2 c^2$ and $(1 - \epsilon)^2 c^2$, respectively. If a frame is ϵ -nearly Parseval, then the operator inequalities $(1 - \epsilon)I \leq V^*V \leq (1 + \epsilon)I$ hold between V^*V and the identity I on \mathcal{H} .

When frames are interpreted as a vector-valued functions, they carry a natural metric.

Definition 2.4. The ℓ^2 -distance between two frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ for a Hilbert space \mathcal{H} is defined by

$$\|\mathcal{F} - \mathcal{G}\| = \left(\sum_{j=1}^n \|f_j - g_j\|^2 \right)^{1/2}.$$

Two frames \mathcal{F} and \mathcal{G} are ϵ -close if $\|\mathcal{F} - \mathcal{G}\| \leq \epsilon$.

We can now state the main problem we address in this paper.

Problem 2.5 (V. Paulsen). Let \mathcal{H} be a real or complex Hilbert space of dimension d . Given $\epsilon > 0$ and an integer $n \geq d$, find the largest number $\delta > 0$ so that whenever $\{f_j\}_{j=1}^n$ is a δ -nearly equal-norm, δ -nearly Parseval frame for a Hilbert space \mathcal{H} , there is an equal-norm Parseval frame $\{g_j\}_{j=1}^n$ whose ℓ^2 -distance to $\{f_j\}_{j=1}^n$ is less than ϵ .

The existence of such a δ is assured by an argument of Don Hadwin.

Proposition 2.6 (D. Hadwin). Given a real or complex Hilbert space \mathcal{H} of dimension d and an integer $n \geq d$, then for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever a frame $\{f_j\}_{j=1}^n$ for \mathcal{H} is δ -nearly equal norm and δ -nearly Parseval, then $\{f_j\}_{j=1}^n$ is ϵ -close to an equal norm Parseval frame.

Proof. We proceed by way of contradiction. If the assertion is false, then there exists some $\epsilon > 0$ and a sequence $\{\delta_m\}_{m=1}^\infty$ converging to zero and a sequence of frames $\{f_j^{(m)} : 1 \leq j \leq n, m \in \{1, 2, \dots\}\}$ so

that each $\{f_j^{(m)}\}_{j=1}^n$ is δ_m -nearly equal norm and δ_m -nearly Parseval but for any equal-norm Parseval frame $\{g_j\}_{j=1}^n$ we have

$$\sum_{j=1}^n \|f_j^{(m)} - g_j\|^2 \geq \epsilon^2.$$

By compactness and switching to a subsequence we may assume that the sequence of frame vectors $\{f_j^{(m)}\}_{m=1}^\infty$ has a limit for each fixed $j \in \{1, 2, \dots, n\}$,

$$\lim_{m \rightarrow \infty} f_j^{(m)} = f_j, \quad .$$

By continuity of the spectrum of V^*V in the frame vectors and of the entries in VV^* , it follows that $\{f_j\}_{j=1}^n$ is an equal-norm Parseval frame and that its distance to $\{f_j^{(m)}\}_{j=1}^n$ goes to zero as $m \rightarrow \infty$ which is in contradiction with the assumption that the distance between each $\{f_j^{(m)}\}_{j=1}^n$ and any equal-norm Parseval frame was bounded below by $\epsilon > 0$. \square

The diagonal entries of VV^* and the operator inequalities for V^*V are not affected when the frame vectors are multiplied by unimodular constants, because then VV^* is simply conjugated by a diagonal unitary, and V^*V is invariant. Therefore, we can form equivalence classes of frames which share the same nearly-equal norm and nearly Parseval properties.

Definition 2.7. We define two frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ for a real or complex Hilbert space to be switching equivalent if the frame vectors f_j and g_j are collinear and $\|f_j\| = \|g_j\|$ for each $j \in \{1, 2, \dots, n\}$. Accordingly, we speak of switching a frame \mathcal{F} to a frame \mathcal{G} , also denoted $\mathcal{F}^{(c)}$, if we multiply each frame vector by a unimodular constant, $g_j = c_j f_j$ with $|c_j| = 1$ for $j \in \{1, 2, \dots, n\}$.

Note that unlike the (nearly) equal-norm or Parseval properties, the ℓ^2 -distance between two frames is not preserved when one of them is switched. We now define another distance for frames which does not depend on which particular representative of an equivalence class is chosen.

Definition 2.8. The Bures distance between two frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ for a real or complex Hilbert space \mathcal{H} is defined by

$$d_B(\mathcal{F}, \mathcal{G}) = \left(\sum_{j=1}^n (\|f_j\|^2 + \|g_j\|^2 - 2|\langle f_j, g_j \rangle|) \right)^{1/2}.$$

Two frames \mathcal{F} and \mathcal{G} are ϵ -close in the Bures distance if $d_B(\mathcal{F}, \mathcal{G}) \leq \epsilon$.

The Bures distance is only a pseudo-metric on the set of frames, because $d_B(\mathcal{F}, \mathcal{G}) = 0$ only implies $f_j = c_j g_j$ with $|c_j| = 1$ for all $j \in \{1, 2, \dots, n\}$. We have extended its usual definition for a pair of normalized vectors f and g in a real or complex Hilbert space, which assigns their Bures distance to be $\sqrt{2 - 2|\langle f, g \rangle|}$, to the setting of vector-valued functions. This way of extending the Bures distance is natural when it is viewed as the solution of a minimization problem.

Lemma 2.9. *Let \mathcal{H} be a Hilbert space over the field of real or complex numbers, hereafter denoted by \mathbb{F} . The value $d_B(\mathcal{F}, \mathcal{G})$ is the solution of the minimization problem*

$$d_B(\mathcal{F}, \mathcal{G}) = \min_{c \in \mathbb{T}^n} \left(\sum_{j=1}^n \|f_j - c_j g_j\|^2 \right)^{1/2},$$

where $\mathbb{T}^n = \{c \in \mathbb{F}^n : |c_j| = 1 \text{ for all } 1 \leq j \leq n\}$.

Proof. The equivalence between these two definitions of d_B is seen from the inequality

$$\|f_j - c_j g_j\|^2 = \|f_j\|^2 + \|g_j\|^2 - 2\Re \bar{c}_j \langle f_j, g_j \rangle \geq \|f_j\|^2 + \|g_j\|^2 - 2|\langle f_j, g_j \rangle|,$$

which is saturated (i.e. gives equality) when each c_j is chosen so that $\Re \bar{c}_j \langle f_j, g_j \rangle = |\langle f_j, g_j \rangle|$. Here, \bar{c}_j denotes the complex conjugate of c_j . \square

The Bures distance is therefore the quotient metric obtained from the ℓ^2 -metric when passing from frames to their equivalence classes. From the fact that equal-norm and Parseval properties are switching-invariant, we get an immediate consequence for the closeness of frames.

Corollary 2.10. *A frame $\mathcal{F} = \{f_j\}_{j=1}^n$ is ϵ -close to an equal-norm Parseval frame $\mathcal{G} = \{g_j\}_{j=1}^n$ in Bures distance if and only if it is ϵ -close to an equal-norm Parseval frame $\mathcal{G}' = \{g'_j\}_{j=1}^n$ in ℓ^2 -distance.*

Proof. The “only if” part follows from choosing the ℓ^2 -distance minimizing equal-norm Parseval frame \mathcal{G}' in the equivalence class of \mathcal{G} . For this frame,

$$\|\mathcal{F} - \mathcal{G}'\| = d_B(\mathcal{F}, \mathcal{G}') = d_B(\mathcal{F}, \mathcal{G}) \leq \epsilon.$$

The “if” part is clear from the inequality $d_B(\mathcal{F}, \mathcal{G}') \leq \|\mathcal{F} - \mathcal{G}'\|$. \square

As a final remark before the main part of the paper, we will see in Section 3.4 that the Paulsen Problem is equivalent to a problem in matrix theory.

Problem 2.11. *Let the field \mathbb{F} be either the real or complex numbers, and assume \mathbb{F}^n is equipped with the canonical inner product. Given $\epsilon > 0$, find the largest number $\gamma > 0$ so that whenever P is an orthogonal rank- d projection matrix on \mathbb{F}^n with nearly constant diagonal, meaning there is $c > 0$ such that*

$$(1 - \gamma)c \leq P_{j,j} \leq (1 + \gamma)c, \quad \text{for all } j \in \{1, 2, \dots, n\},$$

then there exists an orthogonal projection Q satisfying

- (1) $Q_{j,j} = \frac{d}{n}$ for all $j \in \{1, 2, \dots, n\}$, and
- (2) $(\sum_{j,k=1}^n |P_{j,k} - Q_{j,k}|^2)^{1/2} < \epsilon$.

3. CONSTRUCTION OF EQUAL-NORM PARSEVAL FRAMES

3.1. First steps towards an equal-norm Parseval frame. We begin by first finding the closest Parseval frame to a given nearly equal-norm and nearly Parseval frame.

Proposition 3.1. *Let $\{f_j\}_{j=1}^n$ be an ϵ -nearly Parseval frame for a d -dimensional Hilbert space \mathcal{H} , with frame operator $S = V^*V$, then $\{S^{-1/2}f_j\}_{j=1}^n$ is the closest Parseval frame to $\{f_j\}_{j=1}^n$ and*

$$\sum_{j=1}^n \|S^{-1/2}f_j - f_j\|^2 \leq d(2 - \epsilon - 2\sqrt{1 - \epsilon}) \leq d\epsilon^2/4.$$

Proof. It is known that $\{S^{-1/2}f_j\}_{j=1}^n$ is the closest Parseval frame to $\{f_j\}_{j=1}^n$ [2, 9, 12, 26]. We summarize the derivation of this fact.

The squared ℓ^2 -distance between $\{f_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ can be expressed in terms of their analysis operators V and W as

$$\begin{aligned} \|\mathcal{F} - \mathcal{G}\|^2 &= \text{tr}[(V - W)(V - W)^*] \\ &= \text{tr}[VV^*] + \text{tr}[WW^*] - 2\Re \text{tr}[VW^*] \end{aligned}$$

Choosing a Parseval frame $\{g_j\}_{j=1}^n$ is equivalent to choosing the isometry W . To minimize the distance over all choices of W , consider the polar decomposition $V = UP$, where P is positive and U is an isometry. In fact, $S = V^*V$ implies $P = S^{1/2}$, which means its eigenvalues are bounded away from zero.

Since P is positive and bounded away from zero, the term $\text{tr}[VW^*] = \text{tr}[UPW^*] = \text{tr}[W^*UP]$ is an inner product between W and U . Its magnitude is bounded by the Cauchy Schwarz inequality, and thus it has a maximal real part if $W = U$ which implies $W^*U = I$. In this case, $V = WP = WS^{1/2}$, or equivalently $W^* = S^{-1/2}V^*$, and we conclude $g_j = S^{-1/2}f_j$ for all $j \in \{1, 2, \dots, n\}$.

After choosing $W = VS^{-1/2}$, the ℓ^2 -distance is expressed in terms of the eigenvalues $\{\lambda_k\}_{k=1}^d$ of $S = V^*V$ by

$$\begin{aligned} \|\mathcal{F} - \mathcal{G}\|^2 &= \operatorname{tr}[S] + \operatorname{tr}[I] - 2\operatorname{tr}[S^{1/2}] \\ &= \sum_{k=1}^d \lambda_k + d - 2\sum_{k=1}^d \sqrt{\lambda_k}. \end{aligned}$$

If $1 - \epsilon \leq \lambda \leq 1 + \epsilon$ for all $j \in \{1, 2, \dots, n\}$, calculus shows that the maximum of $\lambda - 2\sqrt{\lambda}$ is achieved when $\lambda = 1 - \epsilon$.

Consequently,

$$\|\mathcal{F} - \mathcal{G}\|^2 \leq 2d - d\epsilon - 2d\sqrt{1 - \epsilon}.$$

Estimating $\sqrt{1 - \epsilon}$ by the first three terms in its decreasing power series gives the inequality $\|\mathcal{F} - \mathcal{G}\|^2 \leq d\epsilon^2/4$. \square

Remark 3.2. *Examining the proof shows that the first inequality in Proposition 3.1 saturates (i.e., equality holds) if $\{g_j\}_{j=1}^n$ is a Parseval frame and $f_j = \sqrt{1 - \epsilon} g_j$. This means, the inequality cannot be improved further.*

For the Paulsen problem, this implies that we cannot expect to find an estimate for the distance between an ϵ -nearly equal-norm, ϵ -nearly Parseval frame $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ in a d -dimensional Hilbert space \mathcal{H} and the closest equal-norm Parseval frame \mathcal{G} in the form

$$\|\mathcal{F} - \mathcal{G}\| \leq C\epsilon$$

where C is smaller than $\sqrt{d}/2$. In the following, we show that we can derive an estimate of the above form, where C depends on d and n .

We have an upper bound for the distance between a frame and the closest Parseval frame, and for sufficiently small ϵ , we have control over how much of the “nearly equal-norm” property we lose.

Proposition 3.3. *Fix $0 \leq \epsilon \leq 1/2$ and let $\{f_j\}_{j=1}^n$ be an ϵ -nearly equal-norm frame with constant c which is also an ϵ -nearly Parseval frame with frame operator $S = V^*V$, then $\{S^{-1/2}f_j\}_{j=1}^n$ is a Parseval frame and for all $j \in \{1, 2, \dots, n\}$ we have*

$$(1 - 3\epsilon)c^2 \leq \frac{(1 - \epsilon)^2}{1 + \epsilon}c^2 \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1 + \epsilon)^2}{1 - \epsilon}c^2 \leq (1 + 7\epsilon)c^2.$$

Proof. Since the frame operator $S = V^*V$ is by assumption bounded by

$(1 - \epsilon)I \leq S \leq (1 + \epsilon)I$ we have via the spectral theorem

$$\frac{1}{\sqrt{1 + \epsilon}}I \leq S^{-1/2} \leq \frac{1}{\sqrt{1 - \epsilon}}I.$$

This means that the image of any unit vector has norm between $1/\sqrt{1+\epsilon}$ and $1/\sqrt{1-\epsilon}$, and for the frame vectors with norm bounds $(1-\epsilon)c \leq \|f_j\| \leq (1+\epsilon)c$, we get

$$\frac{(1-\epsilon)^2}{1+\epsilon}c^2 \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1+\epsilon)^2}{1-\epsilon}c^2.$$

Further, convexity and elementary estimates give together with the assumption $\epsilon \leq 1/2$ the bounds

$$(1-3\epsilon)c^2 \leq \|S^{-1/2}f_j\|^2 \leq (1+7\epsilon)c^2.$$

□

Corollary 3.4. *Fix $0 \leq \epsilon \leq 1/2$ and let $\{f_j\}_{j=1}^n$ be an ϵ -nearly equal-norm frame with constant c which is also an ϵ -nearly Parseval frame with frame operator $S = V^*V$, then the norm of each vector $S^{-1/2}f_j$, $j \in \{1, 2, \dots, n\}$, is bounded by*

$$\frac{(1-\epsilon)^3 d}{(1+\epsilon)^3 n} \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1+\epsilon)^3 d}{(1-\epsilon)^3 n}.$$

Proof. By summing the square norms of the frame vectors, and using the fact that the Gramian and the frame operator have the same eigenvalues, except possibly for zero, we obtain

$$(1-\epsilon)d \leq \sum_{j=1}^n \|f_j\|^2 \leq (1+\epsilon)d.$$

The nearly-equal norm condition gives

$$(1-\epsilon)d \leq (1+\epsilon)^2 c^2 n$$

and

$$(1+\epsilon)d \geq (1-\epsilon)^2 c^2 n.$$

This bounds the value of c by

$$\frac{(1-\epsilon)d}{(1+\epsilon)^2 n} \leq c^2 \leq \frac{(1+\epsilon)d}{(1-\epsilon)^2 n}.$$

Now we combine this with the preceding proposition to obtain

$$\frac{(1-\epsilon)^3 d}{(1+\epsilon)^3 n} \leq \|S^{-1/2}f_j\|^2 \leq \frac{(1+\epsilon)^3 d}{(1-\epsilon)^3 n}.$$

□

In the next section, we turn the resulting nearly equal-norm Parseval frame $\{S^{-1/2}f_j\}_{j=1}^n$ into an equal-norm Parseval frame while measuring the distance between them.

3.2. On the road to an equal-norm Parseval frame. We begin with a dilation argument. We observe that if $\{f_j\}_{j=1}^n$ is a Parseval frame for a real or complex Hilbert space, then the Grammian $G = (\langle f_j, f_k \rangle)_{j,k=1}^n$ is an orthogonal projection matrix and we have the expression $G_{j,k} = \langle Ge_j, Ge_k \rangle = \langle V^*e_j, V^*e_k \rangle$ with the canonical orthonormal basis $\{e_j\}_{j=1}^n$ on $\ell^2(\{1, 2, \dots, n\})$ and V^* , the adjoint of the analysis operator of $\{f_j\}_{j=1}^n$.

Proposition 3.5. *Let G be the Grammian of a Parseval frame for a real or complex Hilbert space \mathcal{H} , then the system of ODEs*

$$\frac{d}{dt}e_j(t) = \sum_{k=1}^n (\|Ge_j(t)\|^2 - \|Ge_k(t)\|^2)e_k(t), \quad j \in \{1, 2, \dots, n\} \quad (3.1)$$

for the vector-valued functions $\{e_j : \mathbb{R}^+ \rightarrow \ell^2(\{1, 2, \dots, n\})\}$ with the canonical basis vectors as initial values $\{e_j(0)\}_{j=1}^n$ has a unique, global solution on \mathbb{R}^+ . Moreover, there exists $t \geq 0$ such that $e'_j(t) = 0$ for all $j \in \{1, 2, \dots, n\}$ if and only if there is a $c > 0$ such that $\|Ge_j(t)\| = c$ for all $j \in \{1, 2, \dots, n\}$.

Proof. To simplify terminology in the proof, we write \mathbb{F}^n instead of the Hilbert space $\ell^2(\{1, 2, \dots, n\})$, where \mathbb{F} stands for \mathbb{R} or \mathbb{C} , depending on whether the Hilbert space \mathcal{H} is real or complex. Moreover, we identify a family of vectors $\{e_j(t)\}_{j=1}^n$ in \mathbb{F}^n with a vector $(e_1(t), e_2(t), \dots, e_n(t)) \in \oplus_{j=1}^n \mathbb{F} \equiv \mathbb{F}^n$. With this identification, the system of ODEs for $\{e_j(t)\}_{j=1}^n$ combines to an ODE for a single vector-valued function $\mathcal{E} : \mathbb{R}^+ \rightarrow \mathbb{F}^n$. Since the velocity vector field of the combined ODE is smooth on any bounded set in \mathbb{F}^n , we have local existence and uniqueness of the solution in a sufficiently small neighborhood of $t = 0$.

We first prove that these local solutions preserve orthonormality of $\{e_j(t)\}_{j=1}^n$, and then conclude the existence of global solutions.

Since $\sum_{j=1}^n e_j(0) \otimes e_j^*(0) = I$ we only have to prove that

$$\frac{d}{dt} \sum_{j=1}^n e_j(t) \otimes e_j^*(t) = 0.$$

Denoting $de_j(t)/dt = e'_j(t)$ and dropping the argument of the vector-valued functions, we compute

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^n e_j \otimes e_j^* &= \sum_{j=1}^n (e'_j \otimes e_j^* + e_j \otimes (e'_j)^*) \\ &= \sum_{j,k=1}^n ((\|Ge_j\|^2 - \|Ge_k\|^2)e_k \otimes e_j^* + (\|Ge_j\|^2 - \|Ge_k\|^2)e_j \otimes e_k^*) \\ &= 0. \end{aligned}$$

The last step follows from swapping the summation indices in the second term.

Now we invoke that these local solutions are uniformly bounded, because $\{e_j(t)\}_{j=1}^n$ is orthonormal for each $t \geq 0$. This implies that the local solution stays inside the compact set $S^n = \{(e_1, e_2, \dots, e_n) : \|e_j\| = 1 \text{ for all } j\} \subset \mathbb{F}^{n^2}$. The existence of a unique global solution now follows from the boundedness of the velocity vector field on the compact set S^n , because otherwise the maximal domain $[0, a)$ for a solution would yield a limiting value at a inside S , which we could again use as initial value to find a local solution in the neighborhood of a , and then by the uniqueness of local solutions extend the domain $[0, a)$ to include a neighborhood of a , contradicting the maximality assumption. For more details on this argument, see [29, Section 2.4].

Finally, we observe that $e'_j(t) = 0$ for all $j \in \{1, 2, \dots, n\}$ implies by orthonormality that $\|Ge_j(t)\|^2 - \|Ge_k(t)\|^2 = 0$ for all j and k and thus the family $\{Ge_j\}_{j=1}^n$ is equal norm. Conversely, it follows directly from the definition of the ODE system that all orthonormal bases which G projects to an equal-norm family are fixed points. \square

By mapping the evolving orthonormal basis with the synthesis operator of a Parseval frame, we obtain a family of Parseval frames which solves a corresponding ODE system.

Proposition 3.6. *Let G be the Grammian of a Parseval frame for a real or complex Hilbert space \mathcal{H} , let $V : \mathcal{H} \rightarrow \ell^2(\{1, 2, \dots, n\})$ be the analysis operator of the frame, and consider the solution $\{e_j : \mathbb{R}^+ \rightarrow \ell^2(\{1, 2, \dots, n\})\}_{j=1}^n$ of the initial value problem given in the preceding proposition, then $f_j(t) = V^*e_j(t)$ defines a family of Parseval frames $\{f_j : \mathbb{R}^+ \rightarrow \mathcal{H}\}_{j=1}^n$ which satisfies the ODE system*

$$\frac{d}{dt} f_j(t) = \sum_{k=1}^n (\|f_j(t)\|^2 - \|f_k(t)\|^2) f_k(t), \quad j \in \{1, 2, \dots, n\}. \quad (3.2)$$

Conversely, each solution of this ODE system, with a Parseval frame $\{f_j(0)\}_{j=1}^n$ having analysis operator V as initial value, is globally defined and unique, and to each such solution corresponds a unique solution for the ODE (3.1) starting at the canonical basis of $\ell^2(\{1, 2, \dots, n\})$ such that $V^*e_j(t) = f_j(t)$ for all $t \geq 0$.

Proof. We use the two facts that (1) the projection of any orthonormal basis $\{e_j\}_{j=1}^n$ with the Grammian G is a Parseval frame for the range of G and that (2) the analysis operator V of a Parseval frame is an isometry, which implies that for any $t \geq 0$, $\mathcal{F}(t) = \{V^*e_j(t)\}_{j=1}^n$ is a Parseval frame for \mathcal{H} . Moreover, from the identity $\|Ge_j(t)\| = \|V^*e_j(t)\| = \|f_j(t)\|$ for all $j \in \{1, 2, \dots, n\}$ and from applying V^* to both sides of the ODE system (3.1), \mathcal{F} solves the ODE system (3.2).

Moreover, the initial value problem for (3.2) has a unique solution, which is seen by repeating the argument of the preceding proposition with the vector-valued function $\mathcal{F} : \mathbb{R}^+ \rightarrow \oplus_{j=1}^n \mathcal{H}$ instead of \mathcal{E} and with the sphere $S = \{(f_1, f_2, \dots, f_n) : \sum_{j=1}^n \|f_j\|^2 = d\} \subset \oplus_{j=1}^n \mathcal{H}$ instead of S^n . The set S is preserved under the flow because each $\{f_j(t)\}_{j=1}^n$ is a Parseval frame, so the trace of its Grammian is equal to its rank, $\sum_{j=1}^n \|f_j\|^2 = d$, independent of the choice of $t \geq 0$.

Since the solution of the initial value problem (3.2) is unique, fixing $\{f_j(t)\}_{j=1}^n$ and the canonical basis $\{e_j(t)\}_{j=1}^n$ at $t = 0$ yields that each solution of (3.2) can be lifted to a unique solution of (3.1). \square

We conclude that similar to the dilation of a single Parseval frame to an orthonormal basis, we can lift the entire solution of the ODE system (3.2) to an evolving orthonormal basis solving (3.1). The reason for introducing the dilation argument with the ODE system for the basis vectors is that the set of fixed points of (3.2) contains more than all equal-norm Parseval frames, see the example below.

Proposition 3.7. *Given a family of n vector-valued functions $\{f_j : \mathbb{R}^+ \rightarrow \mathcal{H}\}_{j=1}^n$, satisfying (3.2), with $\{f_j(0)\}_{j=1}^n$ a Parseval frame, then $f'_j(0) = 0$ for all $j \in \{1, 2, \dots, n\}$ if and only if the frame is equal-norm or the following zero-summing conditions hold:*

$$\sum_{j=1}^n f_j(0) = \sum_{j=1}^n \|f_j(0)\|^2 f_j(0) = 0.$$

Proof. From the ODEs, we can see that if

$$\frac{d}{dt} f_j = \sum_{k=1}^n (\|f_j\|^2 - \|f_k\|^2) f_k = 0,$$

then

$$\|f_j\|^2 \sum_{k=1}^n f_k = \sum_{k=1}^n \|f_k\|^2 f_k.$$

Hence, if

$$\frac{d}{dt} f_j = \frac{d}{dt} f_m = 0,$$

for $j \neq m \in \{1, 2, \dots, n\}$, then

$$\|f_j\|^2 \sum_{k=1}^n f_k = \|f_m\|^2 \sum_{k=1}^n f_k.$$

That is,

$$\|f_j\| = \|f_m\| \quad \text{or} \quad \sum_{k=1}^n f_k = 0.$$

Consequently, if $\frac{d}{dt} f_j = 0$ for all $j \in \{1, 2, \dots, n\}$ then the frame is equal norm or

$$\sum_{k=1}^n f_k = \sum_k \|f_k\|^2 f_k = 0.$$

Conversely, if the zero-summing conditions hold, then $\frac{d}{dt} f_j = 0$ follows for all $j \in \{1, 2, \dots, n\}$ directly from the definition of the ODE system (3.2). \square

Example 3.8. Given a real or complex Hilbert space \mathcal{H} of dimension d and an orthonormal basis $\{e_1, e_2, \dots, e_d\}$ for \mathcal{H} , we can construct a Parseval frame $\{f_j\}_{j=1}^{2d+1}$ by

$$f_j = \begin{cases} \frac{1}{\sqrt{2}} e_j, & 1 \leq j \leq d, \\ -\frac{1}{\sqrt{2}} e_{j-d}, & d+1 \leq j \leq 2d, \\ 0, & j = 2d+1. \end{cases}$$

It is straightforward to check that this frame satisfies the zero-summing conditions in the preceding proposition, and is thus a fixed point for the ODE (3.2), but it is not an equal-norm Parseval frame.

It has been observed numerically that using an example of this type as initial value and dilating the Parseval frame to an orthonormal basis leads to an oscillating behavior of the basis vectors evolving under the ODE system (3.1). Therefore, one cannot hope to use these ODEs alone to achieve convergence to equal-norm Parseval frames.

Definition 3.9. We define the frame energy of a frame $\mathcal{F} = \{f_j\}_{j=1}^n$ by

$$U(\mathcal{F}) = \sum_{j,k=1}^n (\|f_j\|^2 - \|f_k\|^2)^2.$$

We will show below that with an appropriate use of intermittent switching, the energy of Parseval frames obtained from piecewise solutions of the ODE (3.2) decreases rapidly (in fact, exponentially) in time. Together with the following arc length estimate, this amounts to showing a rate of convergence to an equal-norm Parseval frame.

Definition 3.10. Given a family of differentiable vector-valued functions $\mathcal{F} = \{f_j : \mathbb{R}^+ \rightarrow \mathcal{H}\}_{j=1}^n$ and $0 \leq t_1 \leq t_2$, the arc length traversed by the family between time t_1 and t_2 is defined by

$$s = \int_{t_1}^{t_2} \left(\sum_{j=1}^n \|f'_j(t)\|^2 \right)^{1/2} dt.$$

The arc length traversed by the vector-valued function \mathcal{F} evolving under (3.2) is bounded by an energy integral.

Theorem 3.11. The arc length traversed by the solution $\mathcal{F} : \mathbb{R}^+ \rightarrow \oplus_{j=1}^n \mathcal{H}$ of the ODE system (3.2) between time t_1 and t_2 is bounded by the energy integral

$$s \leq \int_{t_1}^{t_2} (U(\mathcal{F}(t)))^{1/2} dt.$$

Proof. We pass from the solution of (3.2) to the orthonormal basis $\mathcal{E} = \{e_j : \mathbb{R}^+ \rightarrow \mathbb{F}^{n^2}\}_{j=1}^n$ evolving under (3.1), giving $V^*e_j(t) = f_j(t)$, where V^* is the synthesis operator of $\{f_j(0)\}_{j=1}^n$. Denoting by G the Gramian, we have by orthonormality,

$$\left\| \frac{d}{dt} e_j \right\|^2 = \sum_{k=1}^n (\|Ge_j\|^2 - \|Ge_k\|^2)^2$$

where we have suppressed the explicit time dependence of the orthonormal basis vectors. Summing over all j gives

$$\sum_{j=1}^n \left\| \frac{d}{dt} e_j \right\|^2 = \sum_{j,k=1}^n (\|Ge_j\|^2 - \|Ge_k\|^2)^2 = U(\mathcal{F}(t)).$$

Finally, again using the Parseval property, $\|Gx\| = \|V^*x\|$ for each $x \in \ell^2(\{1, 2, \dots, n\})$, yields $\|f'_j\| = \left\| \frac{d}{dt} V^*e_j \right\| = \left\| \frac{d}{dt} Ge_j \right\| \leq \left\| \frac{d}{dt} e_j \right\|$, and we have $\sum_{j=1}^n \|f'_j(t)\|^2 \leq U(\mathcal{F}(t))$. Now the definition of arc length provides the desired estimate. \square

Proposition 3.12. *An alternative expression for the frame energy of a Parseval frame $\mathcal{F} = \{f_j\}_{j=1}^n$ is*

$$U(\mathcal{F}) = 2n \sum_{j=1}^n \|f_j\|^4 - 2d^2,$$

where d is the dimension of \mathcal{H} .

Proof. We use the antisymmetry of $\|f_j\|^2 - \|f_k\|^2$ in j and k to write

$$U(\mathcal{F}) = 2 \sum_{j,k=1}^n (\|f_j\|^2 - \|f_k\|^2) \|f_j\|^2.$$

Now we can sum over k . Since $\{f_k\}_{k=1}^n$ is Parseval, the square norms sum to $d = \dim(\mathcal{H})$. The result is

$$U(\mathcal{F}) = 2 \sum_{j=1}^n (n\|f_j\|^4 - d\|f_j\|^2).$$

Again, we can split the two terms into separate sums and carry out the sum over j for the second term to get d again. \square

Next, we give a closed expression for the time derivative of the frame energy while the frame \mathcal{F} evolves under (3.2).

Lemma 3.13. *If $\mathcal{F} = \{f_j : \mathbb{R}^+ \rightarrow \mathcal{H}\}$ is a solution of (3.2) with a Parseval frame $\{f_j(0)\}_{j=1}^n$ as initial value, then*

$$\frac{d}{dt}U(\mathcal{F}(t)) = 4n \sum_{j,k=1}^n \langle f_j(t), f_k(t) \rangle (\|f_j(t)\|^2 - \|f_k(t)\|^2)^2.$$

Proof. Defining $G_{j,k}(t) = \langle Ge_j(t), e_k(t) \rangle = \langle f_j(t), f_k(t) \rangle$ and proceeding with the lifted ODE

$$e_j'(t) = \sum_{k=1}^n (G_{j,j}(t) - G_{k,k}(t))e_k(t),$$

we have

$$\frac{d}{dt}(G_{j,j}(t))^2 = 2G_{j,j}(t) \sum_{k=1}^n G_{j,k}(t)(G_{j,j}(t) - G_{k,k}(t)).$$

Summing over j and antisymmetrizing $G_{j,j}(t)$ with $G_{k,k}(t)$ gives

$$\frac{d}{dt}U(\mathcal{F}(t)) = 4n \sum_{j,k=1}^n G_{j,k}(t)(G_{j,j}(t) - G_{k,k}(t))^2.$$

In terms of the frame vectors, this is precisely the claimed expression. \square

Definition 3.14. We define σ_n to be the uniform probability measure on the n -torus $\mathbb{T}^n = \{c \in \mathbb{F}^n : |c_j| = 1 \text{ for all } j\}$, where \mathbb{F} is \mathbb{R} or \mathbb{C} . In the complex case, these are all n -tuples of unimodular complex numbers and in the real case n -tuples of ± 1 's. We also denote diagonal unitaries $\{D(c)\}$, parametrized by the diagonal entries $(D(c))_{j,j} = c_j$, $|c_j| = 1$ for all $j \in \{1, 2, \dots, n\}$.

For later notational convenience, we define

$$W(\mathcal{F}) = 4n \sum_{j,k=1}^n \langle f_j, f_k \rangle (\|f_j\|^2 - \|f_k\|^2)^2. \quad (3.3)$$

We recall the definition of two frames being switching equivalent, meaning the two families consist of vectors that are pairwise collinear and of the same norm.

Remark 3.15. We observe that if $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ are switching equivalent, then $U(\mathcal{F}) = U(\mathcal{G})$.

Switching a frame amounts to conjugating the Grammian with a diagonal unitary matrix. (If the Hilbert space is over the reals, then this is just a diagonal matrix with eigenvalues ± 1 .) So while $U(\mathcal{F})$ is switching invariant, $W(\mathcal{F})$ generally depends on which representative of a switching-equivalence class is chosen.

We now use the switching dependence of W to our advantage.

Proposition 3.16. Given a Parseval frame $\mathcal{F} = \{f_j\}_{j=1}^n$, then there is a choice $c \in \mathbb{T}^n$ such that

$$W(\mathcal{F}^{(c)}) \leq 0.$$

Proof. Let G denote the Grammian of \mathcal{F} . For the switched frame $\mathcal{F}^{(c)}$, we have

$$W(\mathcal{F}^{(c)}) = 4n \sum_{j,k=1}^n c_j c_k^* G_{j,k} (G_{j,j} - G_{k,k})^2.$$

Integrating over the torus \mathbb{T}^n with respect to the switching invariant measure σ_n gives

$$\int_{\mathbb{T}^n} c_j^* c_k d\sigma_n(c) = \delta_{j,k}.$$

Thus we note, since terms with $j = k$ have a vanishing contribution in $W(\mathcal{F}^{(c)})$,

$$\int_{\mathbb{T}^n} W(\mathcal{F}^{(c)}) d\sigma(c) = 0.$$

Since the average is equal to zero, there must be a choice of c which gives $W(\mathcal{F}^{(c)}) \leq 0$. \square

Next, we compute a lower bound for the variance of $W(\mathcal{F}^{(c)})$.

Proposition 3.17. *For a fixed Parseval frame \mathcal{F} , the variance of $W(\mathcal{F}^{(c)})$ with respect to the probability measure σ on the torus $\{c \in \mathbb{T}^n\}$ is*

$$\int_{\mathbb{T}^n} (W(\mathcal{F}^{(c)}))^2 d\sigma(c) = 16n^2 \sum_{j,k} |G_{j,k}|^2 (G_{j,j} - G_{k,k})^4.$$

Proof. Similar to the preceding proposition, with the help of

$$\int_{\mathbb{T}^n} c_j c_k^* c_l c_m^* d\sigma(c) = \delta_{j,k} \delta_{l,m} + \delta_{j,m} \delta_{k,l}.$$

□

Let $n, d \in \mathbb{N}$ be relatively prime, and define

$$\eta = \min_{\substack{n_1 < n \\ d_1 < d}} \left| \frac{d}{n} - \frac{d_1}{n_1} \right|$$

then we have a lower bound

$$\eta \geq \frac{1}{n(n-1)}.$$

This follows immediately from the fact that since d, n are relatively prime, $dn_1 - d_1n$ is a non-zero integer. Since $n_1 < n$ and $d_1 < d$ we have:

$$\left| \frac{d}{n} - \frac{d_1}{n_1} \right| = \left| \frac{dn_1 - nd_1}{nn_1} \right| \geq \frac{1}{nn_1} \geq \frac{1}{n(n-1)}.$$

Lemma 3.18. *Let $n \geq 2$, η as defined above, and let $\mathcal{F} = \{f_j\}_{j=1}^n$ be a Parseval frame for a d -dimensional Hilbert space, then the variance of the random variable $W : c \mapsto W(\mathcal{F}^{(c)})$ on the torus \mathbb{T}^n equipped with the uniform probability measure σ_n is bounded below by*

$$\frac{16\eta}{(n-1)^7} (U(\mathcal{F}))^2 \leq \int_{\mathbb{T}^n} (W(\mathcal{F}^{(c)}))^2 d\sigma_n.$$

Proof. Without loss of generality we can number the frame vectors so that their norms decrease, $\|f_1\| \geq \|f_2\| \geq \dots \|f_n\|$. If $U(\mathcal{F})$ does not vanish then $\|f_1\| > \|f_n\|$ and there is at least one $j \in \{1, 2, \dots, n-1\}$ such that

$$\|f_j\|^2 - \|f_{j+1}\|^2 \geq (\|f_1\|^2 - \|f_n\|^2)/(n-1).$$

This means, if $j' \leq j$ and $j'' \geq j+1$, then also

$$\|f_{j'}\|^2 - \|f_{j''}\|^2 \geq (\|f_1\|^2 - \|f_n\|^2)/(n-1).$$

Thus we have partitioned the frame vectors into two sets, and the difference of square-norms between any pair of vectors containing one from each of these sets is bounded below by $(\|f_1\|^2 - \|f_n\|^2)/(n-1)$.

Therefore, the matrix A containing entries $A_{j,k} = (\|f_j\|^2 - \|f_k\|^2)^4$ is entry-wise bounded below by a matrix (block notation)

$$A' = \begin{pmatrix} 0 & \epsilon J \\ \epsilon J^* & 0 \end{pmatrix}$$

Where J is a block containing all 1's and $\epsilon = (\|f_1\|^2 - \|f_n\|^2)^4/(n-1)^4$.

If we form the corresponding blocks in the Grammian

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

then we know $0 \leq G_{11} \leq I$, meaning the eigenvalues of G_{11} are contained in the closed interval $[0, 1]$. Since G is an orthogonal projection, $G_{11} = G_{11}^2 + G_{12}G_{21}$ which means

$$\text{tr}[G_{12}G_{21}] = \text{tr}[G_{11} - G_{11}^2].$$

But that is exactly the squared Frobenius norm of the block G_{12} . Hence,

$$\sum_{j,k} |G_{j,k}|^2 A_{j,k} \geq 2\epsilon \text{tr}[G_{11} - G_{11}^2].$$

The smallest number of non-zero entries in A' is achieved when J contains only one row. If n and d are relatively prime and the vectors are sufficiently near equal-norm, then the diagonal entries of G_{11} are close to d/n and summing them does not give an integer. Therefore, not all eigenvalues are 0 or 1. In fact, a lower bound for the Hilbert Schmidt square-norm of G_{12} is $\text{tr}[G_{11} - G_{11}^2] \geq \eta/(2n-2)$. This is because at least one of the eigenvalues has distance $\eta/(n-1)$ from $\{0, 1\}$ and the function $x \mapsto x(1-x)$ is bounded below by $x \mapsto x/2$ on $[0, 1/2]$ and by $x \mapsto 1/2 - x/2$ on $[1/2, 1]$.

Consequently,

$$\sum_{j,k} |G_{j,k}|^2 A_{j,k} \geq \frac{(\|f_1\|^2 - \|f_n\|^2)^4 \eta}{(n-1)^4(n-1)}.$$

Using the equivalence of norms again,

$$\sum_{j,k=1}^n (\|f_j\|^2 - \|f_k\|^2)^2 \leq n(n-1)(\|f_1\|^2 - \|f_n\|^2)^2$$

and then applying Proposition 3.17

$$\frac{16\eta}{(n-1)^7}(U(\mathcal{F}))^2 \leq 16n^2 \sum_{j,k=1}^n |G_{j,k}|^2 A_{j,k} = \int_{\mathbb{T}^n} \left(\frac{d}{dt}U(\mathcal{F}^{(c)})\right)^2 d\sigma(c).$$

□

Next we will bound $|W(\mathcal{F}^{(c)})|$ by the frame energy.

Lemma 3.19. *For a fixed Parseval frame \mathcal{F} , the random variable $W : c \mapsto W(\mathcal{F}^{(c)})$ on the torus \mathbb{T}^n is bounded,*

$$|W(\mathcal{F}^{(c)})| \leq dU(\mathcal{F}).$$

Proof. Let B denote the matrix with entries $B_{j,k} = (\|f_j\|^2 - \|f_k\|^2)^2$, and $G^{(c)} = D(c)GD^*(c)$, then $W(\mathcal{F}^{(c)}) = \text{tr}[G^{(c)}B]$. Estimating the inner product between $G^{(c)}$ and B gives

$$|W(\mathcal{F}^{(c)})| = |\text{tr}[G^{(c)}B]| \leq \max_P \text{tr}[P|B|]$$

where the maximum is over all rank- d orthogonal projections P , and the spectral theorem defines $|B| = \sqrt{B^*B}$. According to Perron-Frobenius, the largest eigenvalue of $|B|$ is bounded by $\max_j \sum_{k=1}^n B_{i,j}$. Hence,

$$|W(\mathcal{F}^{(c)})| \leq d \max_j \sum_{k=1}^n B_{j,k} \leq d \sum_{j,k=1}^n B_{j,k}.$$

Finally, we observe that $U(\mathcal{F}) = U(\mathcal{F}^{(c)}) = \sum_{j,k=1}^n B_{j,k}$. □

To finish the quantitative bound on the distance from our initial Parseval frame to our equal norm Parseval frame, we will find an exponential upper bound on the frame energy. Theorem 3.11 will then give the needed quantitative upper bound on the arc length.

Lemma 3.20. *Let $W : \Omega \rightarrow [-a, a]$, $a > 0$ be a bounded random variable on a probability space, which induces a normalized Borel measure m on $[-a, a]$. If the expectation and variance of W are $\mathbb{E}[W] = \int_{-a}^a x dm(x) = 0$ and $\mathbb{E}[W^2] = \int_{-a}^a x^2 dm(x) = \sigma^2 > 0$, then the support of m contains a point in the set $\{x \in [-a, a] : x \leq -\sigma^2/a\}$.*

Proof. We consider the polynomial given by $p(x) = (x-a)(x+b)$, then

$$\mathbb{E}[p(W)] = \int_{-a}^a (x^2 + (b-a)x - ab) dm(x) = \sigma^2 - ab.$$

Choosing $b = \sigma^2/a$ gives $\mathbb{E}[p(W)] = 0$, and so

$$\text{supp } m \cap \{x \in [-a, a] : p(x) \geq 0\} \neq \emptyset.$$

The subset of $[-a, a]$ where p is non-negative is $[-a, -b]$. □

Now we are able to bound $W(\mathcal{F}^{(c)})$ from above by a strictly negative quantity.

Theorem 3.21. *Let $n \geq 2$, η as defined above, and let $\mathcal{F} = \{f_j\}_{j=1}^n$ be a Parseval frame for a d -dimensional Hilbert space, then there exists $c \in \mathbb{T}^n$ such that*

$$W(\mathcal{F}^{(c)}) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}).$$

Proof. We have that $a = dU(\mathcal{F})$ bounds the magnitude of $W(\mathcal{F}^{(c)})$ and its variance σ^2 is bounded below by $\sigma^2 \geq \frac{16\eta}{(n-1)^7} (U(\mathcal{F}))^2$. The preceding lemma then establishes that there is a choice for $\{c_j\}_{j=1}^n$ such that

$$W(\mathcal{F}^{(c)}) \leq -\frac{\sigma^2}{dU(\mathcal{F})} \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}).$$

□

Theorem 3.22. *Let \mathcal{H} be a real or complex Hilbert space of dimension d , and let $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ be an ϵ -nearly equal-norm Parseval frame, with $n \geq 2$ and n and d relatively prime, then there exists an equal-norm Parseval frame \mathcal{G} at ℓ^2 -distance*

$$\|\mathcal{F} - \mathcal{G}\| \leq U(\mathcal{F})^{1/2} \frac{n(n-1)^8 d}{8}$$

Proof. We let the frame \mathcal{F} serve as the initial value $\mathcal{F}(0)$ for the ODE system (3.2). Assuming that for each t , we pick $c(t)$ which yields the desired estimate for W , then naively integrating the differential inequality

$$\frac{d}{dt} U(\mathcal{F}^{(c(t))}(t)) = W(\mathcal{F}^{(c(t))}(t)) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}(t))$$

obtained in the preceding theorem gives

$$U(\mathcal{F}^{(c(t))}(t)) \leq U(\mathcal{F}(0)) e^{-16\eta t / (n-1)^7 d}.$$

However, we note that there is no guarantee that c is a measurable function. To achieve this, we relax the constant governing the exponential decay.

Choose $0 < \alpha < 1$. We know that for any Parseval frame there is at least one choice of c which gives

$$\frac{d}{dt} U(\mathcal{F}^{(c)}) \leq -\frac{16\eta}{(n-1)^7 d} U(\mathcal{F}) < -\frac{16\alpha\eta}{(n-1)^7 d} U(\mathcal{F}). \quad (3.4)$$

By the continuity of U and dU/dt in \mathcal{F} , we can cover the space of Parseval frames with open sets for which the strict inequality holds

with the choice of a corresponding c . To finish the argument we need to patch together the local flows in each open set.

We define a global flow by the appropriate choice of c in each subset. Upon exiting a set at time t , we choose one of the open sets of which the frame $\mathcal{G}(t)$ is an element and continue with the respective flow given by the corresponding choice of c in this subset. Since the cover is open, c is piecewise constant and right continuous.

In the complex case, we choose a countable number of c 's which are dense in the torus. By continuity of U and W , for any frame there is a choice in this countable set of c 's such that again the strict differential inequality (3.4) is satisfied. Moreover, the countable family of open sets corresponding to all c 's cover the space of all Parseval frames. By the Heine Borel property of the compact set of Parseval frames, there is a finite sub-cover and we can repeat the argument as in the real case.

We recall that switching affects the ℓ^2 -distance. Piecewise integrating the differential inequality, including switching when necessary, and using the inequality between arc length and exponentially decaying frame energy, we obtain $\{\mathcal{F}^{(c(t))}(t)\}_{t \in \mathbb{R}^+}$ which restricts to a sequence $\{\mathcal{F}^{(c(m))}(m)\}_{m=0}^\infty$ that is Cauchy in the Bures metric.

Passing to a subsequence converging to an accumulation point \mathcal{G}' then yields that the equal-norm Parseval frame \mathcal{G}' is within Bures distance

$$\begin{aligned} d_B(\mathcal{F}(0), \mathcal{G}') &\leq \int_0^\infty U(\mathcal{F}^{(c(t))}(t))^{1/2} dt \\ &\leq \int_0^\infty U(\mathcal{F}(0))^{1/2} e^{-8\eta\alpha t/(n-1)^7 d} dt \\ &= U(\mathcal{F}(0))^{1/2} \frac{(n-1)^7 d}{8\eta\alpha}. \end{aligned}$$

However, we recall that we can always choose \mathcal{G} in the equivalence class of \mathcal{G}' which minimizes the ℓ^2 -distance to \mathcal{F} , and obtain the same result for the ℓ^2 -distance.

To finish the proof, we recall $\eta > 1/n(n-1)$ and use the fact that the set of equal-norm Parseval frames is closed in the compact set of all Parseval frames. Therefore, choosing a sequence of values for α converging to one, we obtain a sequence of frames with an accumulation point within the desired ℓ^2 -distance. \square

Now, putting together the distances we computed above, and taking into account that in the first step we moved from our nearly equal-norm, nearly Parseval frame to the closest Parseval frame, we can give the distance estimate for the Paulsen Problem.

Theorem 3.23. *Let $n, d \in \mathbb{N}$ be relatively prime, $n \geq 2$, let $0 < \epsilon < \frac{1}{2}$, and assume $\mathcal{F} = \{f_j\}_{j=1}^n$ is an ϵ -nearly equal-norm and ϵ -nearly Parseval frame for a real or complex Hilbert space of dimension d , then there is an equal-norm Parseval frame $\mathcal{G} = \{g_j\}_{j=1}^n$ such that*

$$\|\mathcal{F} - \mathcal{G}\| \leq \frac{29}{8}d^2n(n-1)^8\epsilon.$$

Proof. After passing to the closest Parseval frame to the given frame, denoted by $\mathcal{G}(0) = \{S^{-1/2}f_j\}$, we have by the lower and upper bound for the norms of $\{S^{-1/2}f_j\}$ in Corollary 3.4 a bound for the frame energy

$$U(\mathcal{G}(0)) \leq \frac{d^2(n-1)}{n} \left(\frac{(1+\epsilon)^3}{(1-\epsilon)^3} - \frac{(1-\epsilon)^3}{(1+\epsilon)^3} \right)^2.$$

Using convexity and elementary estimates, we infer for $\epsilon < 1/2$ that

$$U(\mathcal{G}(0)) < 27^2d^2\epsilon^2.$$

Now using the preceding theorem, we obtain that there is an equal-norm Parseval frame \mathcal{G} at distance

$$\|\mathcal{G}(0) - \mathcal{G}\| \leq \frac{27}{8}d^2n(n-1)^8\epsilon.$$

To complete the proof, we use the triangle inequality,

$$d(\mathcal{F}, \mathcal{G}) \leq d(\mathcal{F}, \mathcal{G}(0)) + d(\mathcal{G}(0), \mathcal{G}) \leq \frac{\sqrt{d}}{2}\epsilon + \frac{27}{8}d^2n(n-1)^8\epsilon,$$

and then combine the two contributions after estimating $\sqrt{d}/2 \leq d^2/2 \leq d^2n(n-1)^8/4$. \square

3.3. Construction of equal-norm Parseval frames without the relative prime condition. We conclude with an observation which allows us to reduce the construction of equal-norm Parseval frames to the special case discussed in the previous section.

Lemma 3.24. *Given two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 over the real or complex numbers and equal-norm Parseval frames $\mathcal{F} = \{f_1, \dots, f_{n_1}\}$ and $\mathcal{G} = \{g_1, \dots, g_{n_2}\}$, then the family of vectors $\mathcal{F} \otimes \mathcal{G} = \{f_i \otimes g_j : 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$ is an equal-norm Parseval frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Proof. The Parseval property of $\mathcal{F} \otimes \mathcal{G}$ is equivalent to the identity

$$x = \sum_{i,j} \langle x, f_i \otimes g_j \rangle f_i \otimes g_j$$

for all $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$. From the Parseval property of both frames it is clear that this identity holds for any $x = a \otimes b$ with $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$. Linearity then establishes the result for general $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$.

The equal-norm property follows from

$$\|f \otimes g\| = \|f\| \|g\|$$

for any pair $(f, g) \in \mathcal{F} \times \mathcal{G}$ and from the equal-norm property of the individual frames. \square

Corollary 3.25. *The construction of an equal-norm Parseval frame of n vectors in a d -dimensional real or complex Hilbert space \mathcal{H} can be reduced to the case of d and n being relatively prime.*

Proof. If their greatest common divisor is not one, say $\gcd(n, d) = m$, then we can proceed as follows. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim(\mathcal{H}_1) = d/m$ and $\dim(\mathcal{H}_2) = m$. Now choose an orthonormal basis $\{e_1, e_2, \dots, e_m\}$ for \mathcal{H}_2 and construct an equal-norm Parseval frame of n/m vectors $\{f_1, f_2, \dots, f_{n/m}\}$ for \mathcal{H}_1 . The resulting family of tensor products $\{f_i \otimes e_j : 1 \leq i \leq n/m, 1 \leq j \leq m\}$ is an equal-norm Parseval frame for \mathcal{H} . \square

3.4. The Paulsen Problem in Matrix Theory. In this section we will show that the estimate for the special case of the Paulsen Problem provides a partial answer for Problem 2.11 in matrix theory.

Proposition 3.26. *If $\{f_j\}_{j \in I}, \{g_j\}_{j \in I}$ are frames for \mathcal{H} with analysis operators V_1, V_2 respectively, then*

$$\sum_{j \in I} \|V_1 f_j - V_2 g_j\|^2 < 2 (\|V_1\|^2 + \|V_2\|^2) \sum_{j \in I} \|f_j - g_j\|^2.$$

Proof. Note that for all $j \in I$,

$$V_1 f_j = \sum_{i \in I} \langle f_j, f_i \rangle e_i,$$

and

$$V_2 g_j = \sum_{i \in I} \langle g_j, g_i \rangle e_i.$$

Hence,

$$\begin{aligned} \|V_1 f_j - V_2 g_j\|^2 &= \sum_{i \in I} |\langle f_j, f_i \rangle - \langle g_j, g_i \rangle|^2 \\ &= \sum_{i \in I} |\langle f_j, f_i - g_i \rangle + \langle f_j - g_j, g_i \rangle|^2 \\ &\leq 2 \sum_{i \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{i \in I} |\langle f_j - g_j, g_i \rangle|^2. \end{aligned}$$

Summing over j gives

$$\begin{aligned}
\sum_{j \in I} \|V_1 f_j - V_2 g_j\|^2 &\leq 2 \sum_{j \in I} \sum_{i \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{j \in I} \sum_{i \in I} |\langle f_j - g_j, g_i \rangle|^2 \\
&= 2 \sum_{i \in I} \sum_{j \in I} |\langle f_j, f_i - g_i \rangle|^2 + 2 \sum_{j \in I} \|f_j - g_j\|^2 \\
&= 2 \|V_1\|^2 \sum_{i \in I} \|f_i - g_i\|^2 + 2 \|V_2\|^2 \sum_{j \in I} \|f_j - g_j\|^2 \\
&= 2(\|V_1\|^2 + \|V_2\|^2) \sum_{j \in I} \|f_j - g_j\|^2.
\end{aligned}$$

□

Corollary 3.27. *Let $\{f_j\}_{j \in I}, \{g_j\}_{j \in I}$ be Parseval frames for \mathcal{H} with analysis operators V_1, V_2 respectively. If*

$$\sum_{j \in I} \|f_j - g_j\|^2 < \epsilon^2,$$

then

$$\sum_{j \in I} \|V_1 f_j - V_2 g_j\|^2 < 4\epsilon^2.$$

Proof. The analysis operators V_1 and V_2 are isometries, so the preceding proposition simplifies to the desired estimate. □

Corollary 3.28. *Let \mathcal{H} be a Hilbert space having two Parseval frames $\mathcal{F} = \{f_j\}_{j=1}^n$ and $\mathcal{G} = \{g_j\}_{j=1}^n$ at ℓ^2 -distance $\|\mathcal{F} - \mathcal{G}\| \leq \epsilon$, then their Grammians G and Q satisfy*

$$\|G - Q\|_{HS} \equiv \left(\sum_{j,k=1}^n |G_{j,k} - Q_{j,k}|^2 \right)^{1/2} < 2\epsilon.$$

Corollary 3.29. *Let $n, d \in \mathbb{N}$ be relatively prime, $n \geq 2$, and let $0 < \epsilon < 1/2$. If G is a rank- d orthogonal $n \times n$ projection matrix over \mathbb{R} or \mathbb{C} and there is $c > 0$ such that the diagonal entries satisfy*

$$(1 - \epsilon)^2 c^2 \leq G_{j,j} \leq (1 + \epsilon)^2 c^2$$

for all $j \in \{1, 2, \dots, n\}$, then there is an orthogonal rank- d projection Q with diagonal $Q_{j,j} = \frac{d}{n}$ and

$$\|G - Q\|_{HS} \leq \frac{29}{4} d^2 n (n-1)^8 \epsilon.$$

Proof. The matrix G is the Grammian of a nearly-equal norm Parseval frame. Using the distance estimate in Theorem 3.23 and the preceding corollary, we obtain the desired estimate for the Hilbert-Schmidt distance. □

Acknowledgment. The authors would like to thank the referee for detailed suggestions which lead to substantial improvements in transparency and readability.

REFERENCES

- [1] D. M. Appleby, SIC-POVMs and the Extended Clifford Group, *J. Math. Phys.*, 46 (2005) 052107/1-29,
- [2] R. Balan, Equivalence relations and distances between Hilbert frames. *Proceedings Amer. Math. Soc.* 127 (8) (1999) 2353-2366.
- [3] J. J. Benedetto, and M. Fickus, Finite normalized tight frames. *Adv. Comput. Math.* 18 (2003) 357–385.
- [4] J. J. Benedetto, A. M. Powell, and O. Yilmaz, Sigma-Delta quantization and finite frames, *IEEE Trans. Inform. Theory* 52 (2006) 1990–2005.
- [5] B. G. Bodmann, D. W. Kribs and V. I. Paulsen, Decoherence-Insensitive Quantum Communication by Optimal C^* -Encoding, *IEEE Trans. Inform. Theory* 53 (2007) 4738-4749.
- [6] B. G. Bodmann, and V. I. Paulsen, Frames, graphs and erasures, *Linear Algebra Appl.* 404 (2005) 118–146.
- [7] B. G. Bodmann and V. I. Paulsen. Frame Paths and Error Bounds for Sigma-Delta Quantization. *Appl. Comput. Harmon. Anal.* 22 (2007) 176–197.
- [8] P. G. Casazza, Modern tools for Weyl-Heisenberg (Gabor) frame theory. *Advances in imaging and electron physics* 115 (2001) 1-127.
- [9] P. G. Casazza, Custom building finite frames. in: *Wavelets, Frames and Operator Theory* (College Park, MD, 2003), *Contemp. Math.*, vol. 345, pp. 61–86, Amer. Math. Soc., Providence, RI, 2004.
- [10] P. G. Casazza, M. Fickus, J. Kovačević, M. Leon, and J. Tremain, A physical interpretation of tight frames. In: Heil, C. (ed.) *Harmonic analysis and applications*, Birkhäuser, Boston, (2006) 51-76.
- [11] P. Casazza and J. Kovačević, Equal-norm tight frames with erasures. (English summary) *Frames. Adv. Comput. Math.* 18 (2003) 387–430.
- [12] P. Casazza and G. Kutyniok, A generalization of Gram Schmidt orthogonalization generating all Parseval frames. *Advances in Computational Mathematics*, 18 (2007) 65-78.
- [13] P. Casazza and N. Leonhard, Classes of finite equal norm Parseval frames. *Contemp. Math.* 451 (2008) 11-31.
- [14] P. G. Casazza, D. Redmond, and J. Tremain, Real equiangular frames, *Information Sciences and Systems (42nd Annual Conference on)*, Princeton, N.J. (2008) 715-720
- [15] O. Christensen *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.
- [16] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Soc.* 72 (1952) 341-366.
- [17] S. T. Flammia, On SIC-POVMs in Prime Dimensions, *J. Phys. A: Math. Gen.* 39 (2006) 13483-13493.
- [18] C. D. Godsil and A. D. Hensel, Distance regular covers of the complete graph, *J. Combin. Theory Ser. B* 56 (1992) 205–238.

- [19] V. K. Goyal, J. Kovačević, and J. A. Kelner. Quantized frame expansions with erasures. *Appl. Comp. Harm. Anal.* 10 (2001) 203–233.
- [20] V. K. Goyal and M. Vetterli and N. T. Thao, Quantized overcomplete expansions in \mathbb{R}^N : Analysis, Synthesis and Algorithms, *IEEE Trans. Inform. Theory*, 44 (1998) 16-31.
- [21] K. Gröchenig, *Foundations of time-frequency analysis*. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.
- [22] J. Haantjes, Equilateral point-sets in elliptic two- and three-dimensional spaces, *Nieuw Arch. Wisk.* 22 (1948) 355-362.
- [23] D. Han and D. Larson, *Bases, Frames and Group Representations*, vol. 697, *Memoirs, Amer. Math. Soc.*, Providence, RI, 2000.
- [24] S. G. Hoggar, 64 Lines from a Quaternionic Polytope, *Geom. Dedicata* 69 (1998) 287–289.
- [25] R. B. Holmes and V. I. Paulsen. Optimal frames for erasures. *Linear Algebra Appl.* 377 (2004) 31-51.
- [26] A. J. E. M. Janssen, Zak transforms with few zeroes and the tie. In *Advances in Gabor Analysis*, H.G. Feichtinger and T. Strohmer, eds., Birkhäuser, Boston (2002) 31-70.
- [27] D. Kalra, Complex equiangular cyclic frames and erasures, *Linear Algebra Appl.* 419 (2006) 373–399.
- [28] Kovačević, J and Chebira, A. An introduction to frames, in *Foundations and Trends in Signal Processing* (2008), NOW publishers.
- [29] L. Perko, *Differential equations and dynamical systems*, 2nd ed., Springer, New York, 1996.
- [30] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric Informationally Complete Quantum Measurements, *J. Math. Phys.* 45 (2004) 2171–2180.
- [31] Th. Strohmer and R. W. Heath, Jr., Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmon. Anal.* 14 (2003) 257-275,
- [32] J. A. Tropp, I. S. Dhillon, R. Heath Jr. and T. Strohmer, “Structured Tight Frames via an Alternating Projection Method,” *IEEE Trans. Information Theory* 51 (2005) 188-209.
- [33] L. Welch, Lower bounds on the maximum cross correlation of signals, *IEEE Trans. Inform. Theory*, IT-20 (1974) 397–399.
- [34] P. Xia, Sh. Zhou and G. B. Giannakis, Achieving the Welch Bound with Difference Sets, *IEEE Transactions on Information Theory* 51 (2005) 1900–1907.
- [35] G. Zauner, *Quantendesigns - Grundzüge einer nichtkommutativen Designtheorie*, Doctorial thesis, University of Vienna, 1999.

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