Contemporary Mathematics

Robustness of Fusion Frames under Erasures of Subspaces and of Local Frame Vectors

Peter G. Casazza and Gitta Kutyniok

Abstract. Fusion frames were recently introduced to model applications under distributed processing requirements. In this paper we study the behavior of fusion frames under erasures of subspaces and of local frame vectors. We derive results on sufficient conditions for a fusion frame to be robust to such erasures as well as results on the design of fusion frames which are optimally robust in the sense of worst case behavior of the reconstruction error.

1. Introduction

In the last 20 years, frames, i.e., systems, which provide robust, stable and usually non-unique representations of vectors, have been employed in numerous applications such as filter bank theory [**7**], sigma-delta quantization [**4**], signal and image processing [**8**], and wireless communications [**17**]. However, a large number of new applications have emerged where the set-up can hardly be modeled naturally by one single frame system. Many of these applications share the common property of requiring distributed processing such as all types of sensor networks [**19**].

The notion of *fusion frames*, introduced by the authors in [**10**] and [**11**], provides an extensive framework not only to model sensor networks (cf. [**12**]), but also to provide a means to improve robustness or develop feasible reconstruction algorithms. Interestingly, it can be regarded as a generalization of conventional frame theory, thereby going "beyond frame theory". Related approaches with different foci were undertaken by Aldroubi, Cabrelli, and Molter [**1**], Fornasier [**15**], and Sun [**25, 26**].

Some results on robustness of fusion frames under erasures have already been derived. Bodmann, Kribs, and Paulsen [**6**] and Bodmann [**5**] employed Parseval fusion frames under the term *weighted projective resolution of the identity* for optimal transmission of quantum states and for packet encoding, thereby also studying

2000 *Mathematics Subject Classification.* 94A12, 42C15, 68M10, 68Q85.

Key words and phrases. Distributed Processing, Erasures, Frames, Fusion Frames, Signal Reconstruction.

The first author was supported by NSF DMS 0704216.

The second author was supported by Preis der Justus-Liebig-Universität Gießen 2006 and from Deutsche Forschungsgemeinschaft (DFG) Heisenberg-Fellowship KU 1446/8-1.

erasures resilience. Further, in [**20**], one of the authors joint with Pezeshki, Calderbank, and Liu have studied fusion frames which are optimally resilient against noise and erasures for *random* signals and have shown that, surprisingly, these optimal fusion frames are in fact optimal Grassmannian packings. Further results concerning various aspects of fusion frames were derived in [**2, 3, 16, 24**]. Finally, we would like to mention an interesting application of fusion frames performed by Rozell, Goodman, and Johnson [**21, 22, 23**], who used fusion frames to study noise reduction in sensor networks as well as overlapping feature spaces of neurons in visual and hearing systems.

In this paper we focus on the study of robustness of fusion frames under erasures. One motivation comes from the theory of sensor networks. Generally speaking, a fusion frame is a weighted set of subspaces $\{(W_i, v_i)\}_{i\in I}$ in a Hilbert space $\mathcal{H}$, say, with controlled "overlaps", where each subspace W_i is spanned by a local frame $\{f_{ij}\}_{j\in J_i}$. Now the measurements taken by the sensors in a sensor network can be modeled as the inner products of a given signal $x \in \mathcal{H}$ with all local frame vectors, i.e., $\{\langle x, f_{ij}\rangle\}_{j\in J_i,\, i\in I}$. The subspaces model the groupings of the sensors, and the weights model the significance of each group. The reconstruction is now done first within the subspaces, i.e., within each group of sensors, and secondly the signal is reconstructed completely. The first step can be modeled using conventional frame theory outputting $\{v_i \pi_{W_i}(x)\}_{i\in I}$, where π_{W_i} denotes the orthogonal projection onto the subspace W_i. However for the second step fusion frame theory becomes essential, i.e., to reconstruct x from the collection $\{v_i \pi_{W_i}(x)\}_{i\in I}$. A detailed introduction to fusion frame theory is provided in Section 2.

Considering this two-step reconstruction procedure its robustness needs to be studied with respect to the following two scenarios:

- Due to the fact that single sensors might lose their ability to transmit, it becomes necessary to study the robustness of fusion frames under *erasures of local frame vectors*.
- Also the signal of one whole group of sensors might be delayed or completely lost, hence the *erasure of one of multiple subspaces of fusion frames* needs to be examined.

We will show that indeed sufficient conditions on the robustness of a fusion frame with respect to erasures of subspaces can be formulated in terms of weight conditions (Theorem 3.2). It will be also pointed out how this leads to an easy construction process for such fusion frames. We further study the design of fusion frames which behave optimally with respect to the worst case reconstruction error in the setting of Parseval fusion frames. For Parseval fusion frames with prescribed – but not necessarily equal – dimensions, we characterize optimality in terms of conditions on the weights (Theorem 3.6). Considering erasures of local frame vectors, sufficient conditions depend – as expected – on the properties of the frames inside the subspaces (Theorem 4.1), and optimality is characterized in terms of conditions on the norms of the local frame vectors (Theorem 4.3).

This paper is organized as follows. In Section 2 we briefly review the main definitions and notions related to fusion frames. In Section 3 and 4 we study erasures of whole subspaces and of local frame vectors, respectively. In both cases we derive results on sufficient conditions for robustness of a fusion frame with respect to those erasures as well as results on the design of optimal fusion frames for this problem.

2. Review of Fusion Frames

Let I be a countable index set, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in some Hilbert space $\mathcal{H}$, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a *fusion frame*, if there exist constants $0 < C \le D < \infty$ such that

$$C\|x\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 \le D\|x\|^2 \quad \text{for all } x \in \mathcal{H}, \tag{2.1}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D the *fusion frame bounds*. The family $\{(W_i, v_i)\}_{i\in I}$ is called a *C-tight fusion frame*, if in (2.1) the constants C and D can be chosen so that $C = D$, and a *Parseval fusion frame* provided that $C = D = 1$. The frame bound C of a tight fusion frame $\{(W_i, v_i)\}_{i=1}^n$ in a finite-dimensional Hilbert space $\mathcal{H}$ can be interpreted as the *redundancy*, since in [**11**] it was shown that $C = (\sum_{i=1}^n v_i^2 \dim W_i)/\dim \mathcal{H}$.

Often it will become essential to consider a fusion frame together with a set of local frames for its subspaces. Recall that $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$, if there are constants $0 < A \le B < \infty$ (called the *lower* and *upper* frame bound, respectively) so that for every $x \in \mathcal{H}$ we have

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{f_{ij}\}_{j\in J_i}$ be a frame for W_i for each $i \in I$. Then we call $\{(W_i, v_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ a *fusion frame system* for $\mathcal{H}$. The constants C and D are the associated *fusion frame bounds*, if they are the fusion frame bounds for $\{(W_i, v_i)\}_{i\in I}$, and A and B are the *local frame bounds*, if these are the common frame bounds for the *local frames* $\{f_{ij}\}_{j\in J_i}$ for each $i \in I$.

At this point we would also like to draw the reader's attention to the fact that fusion frames can indeed be regarded as a generalization of frames. Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with frame bounds A and B, then $\{(\mathrm{span}\{f_i\}, \|f_i\|)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds A and B. Although the proof is straightforward, we would nevertheless like to point out the reference [**11**, Prop. 2.14]

In frame theory an input signal is represented by a collection of *scalar* coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory equals $\ell^2(I)$. However, in fusion frame theory an input signal is represented by a collection of *vector* coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

$$\left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} = \{\{f_i\}_{i\in I} | f_i \in W_i \text{ and } \{\|f_i\|\}_{i\in I} \in \ell^2(I)\}.$$

Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T_{\mathcal{W}}$ is employed, which is defined by

$$T_{\mathcal{W}} : \mathcal{H} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \quad \text{with} \quad T_{\mathcal{W}}(x) = \{v_i \pi_{W_i}(x)\}_{i\in I}.$$

The *synthesis operator* $T_{\mathcal{W}}^*$ is given by

$$T_{\mathcal{W}}^* : \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \to \mathcal{H} \text{ with } T_{\mathcal{W}}^*(f) = \sum_{i\in I} v_i f_i, \; f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2},$$

and the *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(x) = T_{\mathcal{W}}^* T_{\mathcal{W}}(x) = \sum_{i\in I} v_i^2 \pi_{W_i}(x).$$

Interestingly, a fusion frame operator exhibits properties similar to a frame operator concerning invertibility. In fact, if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D, then the associated fusion frame operator $S_{\mathcal{W}}$ is positive and invertible on $\mathcal{H}$, and $C\,Id \le S_{\mathcal{W}} \le D\,Id$. This immediately leads to a reconstruction procedure as pointed out in the introduction. In fact, we have

$$x = \sum_{i\in I} v_i S_{\mathcal{W}}^{-1}(v_i \pi_{W_i}(x)) \quad \text{for all } x \in \mathcal{H},$$

see [**11**, Prop. 4.1].

We wish to mention that constructing "good" fusion frames is a subtle task. One approach to derive a fusion frame for $\mathbb{R}^d$ is to start with a frame $\{f_j\}_{j=1}^n$ for this space with frame bounds A and B, then split $\{1, \dots, n\}$ into k sets $J_1, \dots, J_k$, and define $W_i = \text{span}\{f_j\}_{j\in J_i}$, $1 \le i \le k$. Let C and D be a common lower and upper frame bound for the frames $\{f_j\}_{j\in J_i}$ for W_i, $1 \le i \le k$. Then $\{(W_i, 1, \{f_j\}_{j\in J_i})\}_{i=1}^k$ is a fusion frame system with fusion frame bounds C/B, D/A (cf. [**11**, Ex. 2.5]). In order for this process to work effectively, the *local frames* have to possess (uniformly) good lower frame bounds, since these control the computational complexity of reconstruction. However, it is known [**13**] that the problem of dividing a frame into a finite number of subsets each of which has good lower frame bounds is equivalent to one of the deepest and most intractable unsolved problems in mathematics: *the 1959 Kadison-Singer Problem*.

For more details on the basic theory of fusion frames we refer the reader to [**11**]. An introduction to general frame theory is provided by [**14**].

3. Erasures of Subspaces

In this section we study the erasure of subspaces of a fusion frame, since sometimes the connection to one whole group of sensors might fail for some period of time, or might be destroyed completely. We first study sufficient conditions for a fusion frame to be robust with respect to the erasure of subspaces. Secondly, we derive results on fusion frames optimally designed for those erasures.

3.1. Erasures of Subspaces of an Arbitrary Fusion Frame. We first recall the following general observation, which is [**10**, Prop. 3.6].

PROPOSITION 3.1. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame in a finite-dimensional Hilbert space and let $i_0 \in I$. Then $\{(W_i, v_i)\}_{i\in I,\, i\ne i_0}$ is either a fusion frame or* $\text{span}\{W_i\}_{i\in I, i\ne i_0} \subsetneq \mathcal{H}$.

Notice that this result does not hold for an infinite-dimensional Hilbert space $\mathcal{H}$. For instance, let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$, and let $W_1 = \text{span}\{e_{2i}\}_{i=1}^\infty$, $W_2 = \text{span}\,\{e_{2i} + \frac{1}{i} e_{2i+1}\}_{i=1}^\infty$, and $W_3 = \mathcal{H}$. If W_3 is deleted, it is easily checked that $\{W_1, W_2\}$ does not form a fusion frame, however it does satisfy $\text{span}\{W_j\}_{j=1}^2 = \mathcal{H}$.

Our main result in this subsection provides sufficient conditions on the weights for a subspace to be deleted yet still leave a fusion frame.

THEOREM 3.2. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds C and D, and let $J \subset I$. Then the following statements hold.*

(i) *If $\sum_{i\in J} v_i^2 > D$, then $\bigcap_{i\in J} W_i = \{0\}$.*
(ii) *If $\sum_{i\in J} v_i^2 = D$, then $\bigcap_{i\in J} W_i \perp \mathrm{span}\{W_i\}_{i\in I\setminus J}$.*
(iii) *If $c = \sum_{i\in J} v_i^2 < C$, then $\{(W_i, v_i)\}_{i\in I\setminus J}$ is a fusion frame with bounds $C - c$ and D.*

PROOF. To prove part (i), let $x \in \bigcap_{i\in J} W_i$. Then, using the hypothesis that $\sum_{i\in J} v_i^2 > D$ and the fact that $\pi_{W_i}(x) = x$ for $i \in J$, we compute

$$D\|x\|^2 < \left(\sum_{i\in J} v_i^2\right) \|x\|^2 \le \sum_{i\in J} v_i^2 \|\pi_{W_i}(x)\|^2 + \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i}(x)\|^2 \le D\|x\|^2.$$

Hence $x = 0$.

Now suppose that $\sum_{i\in J} v_i^2 = D$, and let again $x \in \bigcap_{i\in J} W_i$. Then

$$D\|x\|^2 = \sum_{i\in J} v_i^2 \|\pi_{W_i}(x)\|^2 \le \sum_{i\in J} v_i^2 \|\pi_{W_i}(x)\|^2 + \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i}(x)\|^2 \le D\|x\|^2.$$

Thus $\sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i}(x)\|^2 = 0$, and hence $x \perp \mathrm{span}\{W_i\}_{i\in I\setminus J}$. This proves claim (ii).

Finally we show (iii). For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{i\in I\setminus J} v_i^2 \|\pi_{W_i}(x)\|^2 &= \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 - \sum_{i\in J} v_i^2 \|\pi_{W_i}(x)\|^2 \\ &\ge C\|x\|^2 - \left(\sum_{i\in J} v_i^2\right) \|x\|^2 \\ &= (C - c)\|x\|^2 . \end{aligned}$$

The upper bound is obvious. This completes the proof of part (iii). □

First, we remark that the claim in part (iii) is sharp. Indeed, let $\{e_i\}_{i=1}^n$ be an orthonormal basis for an n-dimensional Hilbert space $\mathcal{H}$, and equip the set of subspaces $\{W_i\}_{i=1}^n$ defined by $W_i = \mathrm{span}\{e_i, e_{i+1}\}$, $i \in \{1, \ldots, n-1\}$ and $W_n = \mathrm{span}\{e_n, e_1\}$ with weights $v_i = \frac{1}{\sqrt{2}}$. Then $\{(W_i, v_i)\}_{i=1}^n$ is a Parseval fusion frame, and, by Theorem 3.2(iii), one subspace can be deleted yet leaving a fusion frame. However, obviously the deletion of some pairs of subspaces destroys the fusion frame property.

Secondly, we observe that Theorem 3.2 provides an easy construction process of fusion frames $\{(W_i, v_i)\}_{i\in I}$ which are robust to erasures of an arbitrary set of k subspaces W_i. One possibility would be to build a Parseval fusion frame $\{(W_i, v_i)\}_{i\in I}$ having the property that the sum of any k weights v_i^2 is less than one. The previous example possesses this property for $k = 1$.

Theorem 3.2 immediately yields the following corollary, which focusses on the erasure of one single subspace.

COROLLARY 3.3. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds C and D. Then the following statements hold.*

(i) *For all* $i \in I$, *we have* $v_i^2 \le D$.
(ii) *If there exists* $i_0 \in I$ *such that* $v_{i_0}^2 = D$, *then* $\mathrm{span}\{W_i\}_{i\in I, i\neq i_0} \subsetneq \mathcal{H}$.
(iii) *If there exists* $i_0 \in I$ *such that* $v_{i_0}^2 < C$, *then* $\{(W_i, v_i)\}_{i\in I,\, i\neq i_0}$ *is a fusion frame with bounds* $C - v_{i_0}^2$ *and* D.

It can be easily seen that the converse implications in (ii) and (iii) don't hold. Let us consider the following easy counterexample: Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$, $W_1 = W_2 = \mathrm{span}\{e_1, e_2\}$, $W_3 = \mathrm{span}\{e_3\}$, and $v_1 = v_2 = v_3 = 1$. The fusion frame $\{(W_i, v_i)\}_{i\in I}$ has bounds $C = 1$ and $D = 2$. Hence $v_3^2 < D$, but $W_3 \perp W_1$ and $W_3 \perp W_2$. Thus the converse implication in (ii) doesn't hold. Furthermore, $v_1^2 \ge C$, but $\mathrm{span}\{W_2, W_3\} = \mathcal{H}$, thereby giving a counterexample for the converse implication in (iii).

Corollary 3.4. *Let* $\{(W_i, v_i)\}_{i\in I}$ *be a tight fusion frame with bound* C, *and let* $i_0 \in I$. *Then the following conditions are equivalent.*
(i) $v_{i_0}^2 < C$.
(ii) $\{(W_i, v_i)\}_{i\in I,\, i\neq i_0}$ *is a fusion frame.*

It is worth noticing that this result does not hold for arbitrary non-tight fusion frames. Indeed, in a 2-dimensional Hilbert space $\mathcal{H}$, let A be arbitrarily large and define $W_1 = \{e_1, e_2\}$, $W_2 = \{e_1\}$ and $v_1 = 1$, $v_2 = A$. Then $\{W_i, v_i\}_{i=1}^2$ is a fusion frame with bounds $C = 1$ and $D = A+1$. In particular, $v_2 = A$ can be much larger than C. However, the deletion of W_2 still leaves a fusion frame.

3.2. Optimal Fusion Frames for Erasures of Subspaces. The previous considerations identified those subspaces whose erasure leaves a fusion frame, i.e., each vector can still be reconstructed perfectly by adapting the reconstruction strategy accordingly. However, since often we cannot control the erasures, we have to deal with erasures of subspaces which leave an incomplete set of subspaces. These cases will be examined in the following, where we restrict to Parseval fusion frames due to their advantageous reconstruction properties. Also we consider only finite fusion frames, since these are the fusion frames that actually occur in practise. More precisely, we will characterize those Parseval fusion frames which are optimal for erasure of one subspace in the sense that the maximal distance between original vectors and their reconstructed versions is minimal. In our study we will always employ the same reconstruction strategy, since the application might not be aware of an occurring erasure.

First, we make the meaning of optimality precise. In our setting we are interested in the worst case behavior of the reconstruction error. This definition of optimality under erasures has already been studied for frames in [**18**]. Moreover, Bodmann has derived results on non-weighted Parseval fusion frames which behave optimally under the erasure of one single subspace for a fixed number of subspaces, fixed and equal dimension of all subspaces, and fixed dimension of the Hilbert space [**5**, Thm. 13]. For this situation, he also studies multiple erasures [**5**].

Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$, $I = \{1, \ldots, n\}$ be a Parseval fusion frame with analysis operator $T_{\mathcal{W}}$. Define operators $D_{i_0} : \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}$, $1 \le i_0 \le n$ by $\{D_{i_0}(f)\}_i = \delta_{i,i_0} f_{i_0}$ for all $i \in I$, where $f = \{f_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}$, which simulate the erasure of the subspace W_{i_0}. Recalling that $x = T_{\mathcal{W}}^* T_{\mathcal{W}} x$ for all $x \in \mathcal{H}$ leads to the following definition.

DEFINITION 3.5. Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame with analysis operator $T_{\mathcal{W}}$. We define the associated 1-*erasure reconstruction error* $\mathcal{E}_1(\mathcal{W})$ to be

$$\mathcal{E}_1(\mathcal{W}) = \max\{\|T_{\mathcal{W}}^* D_i T_{\mathcal{W}}\| : 1 \leq i \leq n\}.$$

The following result gives a characterization of all Parseval fusion frames with a prescribed number of subspaces and prescribed – not necessarily equal – dimensions of the subspaces which behave optimally under one erasure. It is remarkable that the weights do not have to be equal at all. This differs significantly from the situation in frame theory, but can be explained easily. The reason for this phenomenon is that the weights need to be chosen is such way that the weighted subspaces have "equal size", i.e., the weights have to make up for the different dimensions of the subspaces.

THEOREM 3.6. *Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame in a finite-dimensional Hilbert space $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame $\mathcal{W}$ satisfies $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n$ is a Parseval fusion frame with $\dim \widetilde{W}_i = \dim W_i$ for all $1 \leq i \leq n.\}$.*

(ii) *We have*

$$v_i^2 = \frac{\dim \mathcal{H}}{n \cdot \dim W_i} \quad \textit{for all } 1 \leq i \leq n.$$

Moreover, let $x \in \mathcal{H}$ and let $\hat{x}$ denote the reconstructed vector using the original reconstruction formula. Then we have the following error bound

$$\|x - \hat{x}\| \leq \frac{\dim \mathcal{H}}{n \cdot \min\{\dim W_i : 1 \leq i \leq n\}} \|x\| \quad \textit{for all } x \in \mathcal{H}.$$

PROOF. Let $\mathcal{W} = \{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame with analysis operator $T_{\mathcal{W}}$. Fix $i \in \{1, \ldots, n\}$. Then we have

$$\|T_{\mathcal{W}}^* D_i T_{\mathcal{W}}\| = \sup_{\|x\|=1} \|T_{\mathcal{W}}^* D_i T_{\mathcal{W}} x\| = \sup_{\|x\|=1} \left\|v_i^2 \pi_{W_i}(x)\right\| = v_i^2 \sup_{\|x\|=1} \|\pi_{W_i}(x)\|.$$

Choosing $x \in W_i$ with $\|x\| = 1$ yields $\|\pi_{W_i}(x)\| = \|x\| = 1$, which is the maximum due to $\|\pi_{W_i}(x)\| \leq \|x\| = 1$ for all $x \in \mathcal{H}$. Thus

$$\mathcal{E}_1(\mathcal{W}) = \max\{v_i^2 : 1 \leq i \leq n\}.$$

Now let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for W_i for each $1 \leq i \leq n$. By [**10**, Thm. 3.2], the sequence $\{v_i e_{ij}\}_{i=1,\, j=1}^{n,\ \dim W_i}$ is a Parseval frame for $\mathcal{H}$. Employing [**9**, Sec. 2.3] yields that

$$1 = \frac{\sum_{i=1}^n \sum_{j=1}^{\dim W_i} \|v_i e_{ij}\|^2}{\dim \mathcal{H}} = \frac{\sum_{i=1}^n v_i^2 \dim W_i}{\dim \mathcal{H}},$$

hence

$$\sum_{i=1}^n v_i^2 \dim W_i = \dim \mathcal{H}.$$

This implies that there exists some $i \in \{1, \ldots, n\}$ with $v_i^2 \dim W_i \geq \frac{\dim \mathcal{H}}{n}$. Since the dimensions as well as the number of subspaces are fixed, we can conclude that $\mathcal{E}_1(\mathcal{W})$ is minimal if and only if

$$v_i^2 \dim W_i = \frac{\dim \mathcal{H}}{n} \quad \text{for all } 1 \leq i \leq n.$$

The moreover-part follows immediately from the arguments above, since the maximal error is bounded by $\mathcal{E}_1(\mathcal{W})\,\|x\|$. □

For varying dimensions of the subspaces, we obtain the following result which can be derived from Theorem 3.6 and its proof.

Corollary 3.7. *Let $\{(W_i, v_i)\}_{i=1}^n$ be a Parseval fusion frame in a finite-dimensional Hilbert space $\mathcal{H}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame $\{(W_i, v_i)\}_{i=1}^n$ satisfies $\mathcal{E}_1(\{(W_i, v_i)\}_{i=1}^n) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i)\}_{i=1}^n$ is a Parseval fusion frame in $\mathcal{H}$.$\}$.*

(ii) *We have*

$$\dim W_i = \dim \mathcal{H} \text{ and } v_i^2 = \frac{1}{n} \quad \text{for all } 1 \le i \le n. \tag{3.1}$$

Moreover, let $\hat{x}$ be the reconstructed vector $x \in \mathcal{H}$ under one erasure using the original reconstruction formula. Then we have the following error bound

$$\|x - \hat{x}\| \le \frac{1}{n}\,\|x\| \quad \text{for all } x \in \mathcal{H}. \tag{3.2}$$

As can be easily seen from (3.2), an increase in the number of subspaces improves the error bound for the reconstruction. Note that here we always employ the same reconstruction strategy, which explains the error bound. Allowing an adapted reconstruction the reconstruction error would equal zero with all subspaces equal to $\mathcal{H}$. Further note that provided we allow infinitely many subspaces according to (3.1) the optimal weights would equal zero, which implies that the question of optimal infinite fusion frame systems under erasures is the wrong question to ask.

4. Erasures of Local Frame Vectors

In this section we are concerned with erasures of local frame vectors, since usually some sensors die over time due to battery limitations. We first study sufficient conditions for a fusion frame to be robust with respect to the erasure of subspaces. Secondly, we derive results on fusion frames optimally designed for those erasures.

4.1. Erasures of Local Frame Vectors in an Arbitrary Fusion Frame. Our first result provides sufficient conditions on the weights for a prescribed number of local frame vectors to be deleted in each subspace yet still leave a fusion frame.

Theorem 4.1. *Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame with bounds C and D. For every $i \in I$, let $\{f_{ij}\}_{j\in J_i}$ be a Parseval frame for W_i which is robust to k_i-erasures leaving a frame with lower frame bound A_i. For each $i \in I$, let $L_i \subset J_i$ satisfy $|L_i| \le k_i$, and define the set of subspaces $\{\widetilde{W}_i\}_{i\in I}$ by $\widetilde{W}_i = \mathrm{span}\{f_{ij}\}_{j\in J_i\setminus L_i}$. Then $\{(\widetilde{W}_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with bounds $(\min_{i\in I} A_i)\,C$ and D.*

Proof. Let $x \in \mathcal{H}$ and observe that

$$\sum_{i\in I} v_i^2 \|\pi_{\widetilde{W}_i}(x)\|^2 \ge \sum_{i\in I} v_i^2 \sum_{j\in J_i\setminus L_i} |\langle \pi_{W_i}(x), f_{ij}\rangle|^2 \ge \sum_{i\in I} A_i v_i^2 \|\pi_{W_i}(x)\|^2$$
$$\ge (\min_{i\in I}\ A_i) \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 \ge (\min_{i\in I}\ A_i)\, C\,\|x\|^2.$$

Furthermore, D is obviously still an upper bound. □

This result – as Theorem 3.2 – can be used to explicitly construct fusion frames robust to a specified number of erasures. For this, first the subspaces have to be defined. Then for each subspace we choose a Parseval frame which is robust to a prescribed number of erasures. Here we might use [**9**] to even choose these Parseval frames to be equal-norm. Then the theorem provides us with fusion frame bounds for the fusion frame after the erasures happened.

4.2. Optimal Fusion Frames for Erasures of Local Frame Vectors. Here we are interested in fusion frame systems, which are optimally robust with respect to the erasure of one local vector. As in Section 3.2, we restrict our analysis to finite Parseval fusion frames and to local Parseval frames and only consider application of the same reconstruction strategy.

First we make the meaning of optimality precise. Again we are interested in the worst case behavior of the reconstruction error.

For this, let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames. Let T denote the analysis operator of the associated fusion frame, T_i the analysis operator for the local frames for all $1 \le i \le n$, and define a vector of matrices $(D_1, \ldots, D_n) \in \Pi_{i=1}^n M(m_i \times m_i, \mathbb{R})$ in such a way that there exists one $i_0 \in \{1, \ldots, n\}$ so that $D_{i_0} = (d_{k,l})_{1 \le k,l \le n}$ with $d_{k,l} = \delta_{k,j_0}\delta_{l,j_0}$ for some $j_0 \in \{1, \ldots, m_{i_0}\}$ and all other matrices are zero-matrices. This simulates the erasure of the vector $f_{i_0 j_0}$. We denote the set of admissible vectors of matrices by $\mathcal{D}$. Recalling that $x = \sum_{i=1}^n v_i^2 T_i^* T_i x$ for all $x \in \mathcal{H}$ leads to the following definition.

Definition 4.2. Let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames. Let T denote the analysis operator of the associated fusion frame, and T_i the analysis operator for the local frames for all $1 \le i \le n$. Then we define the associated 1-*erasure reconstruction error* $\mathcal{E}_1(\mathcal{W})$ to be

$$\mathcal{E}_1(\mathcal{W}) = \max\{\|\sum_{i=1}^n v_i^2 T_i^* D_i T_i\| : (D_1, \ldots, D_n) \in \mathcal{D}\}.$$

The following result gives a characterization of all Parseval fusion frames with a prescribed number of local frame vectors and prescribed – not necessarily equal – dimensions of the subspaces which behave optimally under the erasure of one local frame vector.

Theorem 4.3. *Let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ be a Parseval fusion frame system with local Parseval frames $\{f_{ij}\}_{j=1}^{m_i}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame system $\mathcal{W}$ satisfies $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n) : \{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^n$ is a Parseval fusion frame system with local Parseval frames satisfying $\dim \widetilde{W}_i = \dim W_i$ for all $1 \le i \le n.\}$.*

(ii) *We have*

$$\|f_{ij}\|^2 = \frac{\dim W_i}{m_i} \quad \text{for all } i \in I,\ j \in J_i.$$

Moreover, let $\hat{x}$ be the reconstructed vector $x \in \mathcal{H}$ under one erasure using the original reconstruction formula. Then we have the following error bound

$$\|x - \hat{x}\| \le \frac{\max\{\dim W_i : 1 \le i \le n\}}{\min\{m_i : 1 \le i \le n\}} \|x\| \quad \text{for all } x \in \mathcal{H}.$$

PROOF. Let T_i denote the analysis operator for $\{f_{ij}\}_{j=1}^{m_i}$, $i = 1, \ldots, n$. Fix the index of the subspace in which a local frame vector will be deleted and denote it by i_0. Further, let $j_0 \in \{1, \ldots, m_{i_0}\}$ denote the index of the vector being deleted. Then, for each $x \in \mathcal{H}$,

$$\sum_{i=1}^{n} v_i^2 T_i^* D_i T_i x = \sum_{i=1}^{n} v_i^2 T_i^* (\delta_{i,i_0} \delta_{j,j_0} \langle x, f_{ij} \rangle)_{j=1}^{m_i} = v_{i_0}^2 \langle x, f_{i_0 j_0} \rangle f_{i_0 j_0}.$$

Hence

$$\begin{aligned} \|\sum_{i=1}^{n} v_i^2 T_i^* D_i T_i\| &= \sup_{\|x\|=1} \|v_{i_0}^2 \langle x, f_{i_0 j_0} \rangle f_{i_0 j_0}\| \\ &= v_{i_0}^2 \|f_{i_0 j_0}\| \sup_{\|x\|=1} |\langle x, f_{i_0 j_0} \rangle| \\ &= v_{i_0}^2 \|f_{i_0 j_0}\|^2. \end{aligned}$$

Thus for $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^{n}$ we have that

$$\mathcal{E}_1(\mathcal{W}) = \max\{v_i^2 \|f_{ij}\|^2 : 1 \le i \le n,\, 1 \le j \le m_i\}.$$

By [**9**, Sec. 2.3], we obtain

$$\sum_{j=1}^{m_i} \|f_{ij}\|^2 = \dim W_i \quad \text{for all } 1 \le i \le n,$$

which implies that there exists some $i \in \{1, \ldots, n\}$ with $\|f_{ij}\|^2 \ge \frac{\dim W_i}{m_i}$ for all j. Since the dimensions as well as the number of local frame vectors are fixed, we can conclude that $\mathcal{E}_1(\mathcal{W})$ is minimal if and only if

$$\|f_{ij}\|^2 = \frac{\dim W_i}{m_i} \quad \text{for all } i \in I,\ j \in J_i.$$

The moreover-part follows immediately from the arguments above, since the maximal error is bounded by $\mathcal{E}_1(\mathcal{W})\,\|x\|$. □

For varying dimensions of the subspaces, we obtain the following result which can be derived from Theorem 4.3 and its proof.

COROLLARY 4.4. *Let $\mathcal{W} = \{(W_i, v_i, \{f_{ij}\}_{j=1}^{m_i})\}_{i=1}^{n}$ be a Parseval fusion frame system with local Parseval frames $\{f_{ij}\}_{j=1}^{m_i}$. Then the following conditions are equivalent.*

(i) *The Parseval fusion frame system $\mathcal{W}$ satisfies $\mathcal{E}_1(\mathcal{W}) = \min\{\mathcal{E}_1(\{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^{n}) : \{(\widetilde{W}_i, \widetilde{v}_i, \{\widetilde{f}_{ij}\}_{j=1}^{m_i})\}_{i=1}^{n}$ is a Parseval fusion frame system with local Parseval frames.$\}$.*

(ii) *We have*

$$\dim W_i = 1 \text{ and } \|f_{ij}\|^2 = \frac{1}{m_i} \quad \text{for all } i \in I,\ j \in J_i. \tag{4.1}$$

Moreover, let $\hat{x}$ be the reconstructed vector $x \in \mathcal{H}$ under one erasure using the original reconstruction formula. Then we have the following error bound

$$\|x - \hat{x}\| \le \frac{1}{\min\{m_i : 1 \le i \le n\}} \|x\| \quad \text{for all } x \in \mathcal{H}. \tag{4.2}$$

As can easily be seen from (4.2), an increase in the number of local frames in each subspace improves the error bound for the reconstruction. Therefore, provided we allow infinitely many local frame vectors, according to (4.1) the optimal norms would equal zero, which implies that the question of optimal fusion frame systems under erasures does not make much sense allowing infinitely many local frame vectors. In particular it is the wrong question to ask in an infinite-dimensional Hilbert space.

Acknowledgments

The majority of the research for this paper was performed while the second author was visiting the Department of Mathematics at the University of Missouri. This author thanks this department for its hospitality and support during this visit.

We are also indebted to B. Bodmann, S. Li, and C. Rozell for interesting discussions on related topics.

References

[1] A. Aldroubi, C. Cabrelli, and U. M. Molter, *Wavelets on irregular grids with arbitrary dilation matrices and frame atoms for* $L^2(\mathbb{R}^d)$, Appl. Comput. Harmon. Anal. **17** (2004), 119–140.

[2] M. S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. **308** (2005), 541–553.

[3] M. S. Asgari and A. Khosravi, *Frames of subspaces and Approximation of the inverse frame operator*, Houston J. Math. **33** (2007), 907–920.

[4] J. Benedetto, A. Powell, and O. Yilmaz, *Sigma-Delta quantization and finite frames*, IEEE Trans. Inform. Th. **52** (2006), 1990–2005.

[5] B. G. Bodmann, *Optimal linear transmission by loss-insensitive packet encoding*, Appl. Comput. Harmon. Anal. **22** (2007), 274–285.

[6] B. G. Bodmann, D. W. Kribs, and V. I. Paulsen, *Decoherence-insensitive quantum communications by optimal* C^**-encoding*, preprint (2006).

[7] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, *Frame-theoretic analysis of oversampled filter banks*, IEEE Trans. Signal Processing **46** (1998), 3256–3268.

[8] E. J. Candès and D. L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise* C^2 *singularities*, Comm. Pure and Appl. Math. **56** (2004), 216–266.

[9] P. G. Casazza and J. Kovaĉević, *Equal-norm tight frames with erasures*, Adv. Comput. Math. **18** (2003), 387–430.

[10] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, in "Wavelets, Frames and Operator Theory" (College Park, MD, 2003), Contemp. Math. **345**, Amer. Math. Soc., Providence, RI, 2004, 87–113.

[11] P. G. Casazza, G. Kutyniok, and S. Li, *Fusion Frames and Distributed Processing*, Appl. Comput. Harmon. Anal., to appear.

[12] P. G. Casazza, G. Kutyniok, S. Li, and C. J. Rozell, *Modeling Sensor Networks with Fusion Frames*, Wavelets XII (San Diego, CA, 2007), SPIE Proc. **6701**, SPIE, Bellingham, WA, 2007.

[13] P. G. Casazza and J.C. Tremain, *The Kadison-Singer Problem in Mathematics and Engineering*, Proc. Nat. Acad. of Sci. **103** (2006), 2032-2039.

[14] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[15] M. Fornasier, *Quasi-orthogonal decompositions of structured frames*, J. Math. Anal. Appl. **289** (2004), 180–199.

[16] P. Gavruta, *On the duality of fusion frames*, J. Math. Anal. Appl. **333** (2007), 871–879.

[17] R. W. Heath and A. J. Paulraj, *Linear dispersion codes for MIMO systems based on frame theory*, IEEE Trans. Signal Processing **50** (2002), 2429–2441.

[18] R. B. Holmes und V. I. Paulsen, *Optimal frames for erasures*, Linear Algebra Appl. **377** (2004), 31–51.

[19] S. S. Iyengar and R. R. Brooks, eds., *Distributed Sensor Networks*, Chapman & Hall/CRC, Baton Rouge, 2005.

[20] G. Kutyniok, A. Pezeshki, A. R. Calderbank, and T. Liu, *Robust Dimension Reduction, Fusion Frames and Grassmannian Packings*, Appl. Comput. Harmon. Anal., to appear.
[21] C. J. Rozell and D. H. Johnson, *Analysis of noise reduction in redundant expansions under distributed processing requirements*, in International Conference on Acoustics, Speech, and Signal Processing, Philadelphia, PA, 2005.
[22] C. J. Rozell and D. H. Johnson, *Analyzing the robustness of redundant population codes in sonsory and feature extraction systems*, Neurocomputing **69** (2006), 1215–1218.
[23] C. J. Rozell, I. N. Goodman, and D. H. Johnson, *Feature-based information processing with selective attention*, in International Conference on Acoustics, Speech, and Signal Processing, Toulouse, France, 2006.
[24] M. A. Ruiz and D. Stojanoff, *Frames of subspaces and operators*, preprint (2007).
[25] W. Sun, *G-frames and G-Riesz Bases*, J. Math. Anal. Appl. **322** (2006), 437–452.
[26] W. Sun, *Stability of G-frames*, J. Math. Anal. Appl. **326** (2007), 858–868.

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: `pete@math.missouri.edu`

Department of Statistics, Stanford University, Stanford, CA 94305, USA
E-mail address: `kutyniok@stanford.edu`