SPANNING AND INDEPENDENCE PROPERTIES OF FRAME PARTITIONS

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Abstract. We answer a number of open problems in frame theory concerning the decomposition of frames into linearly independent and/or spanning sets. We prove that Parseval frames with norms bounded away from 1 can be decomposed into a number of sets whose complements are spanning, where the number of these sets only depends on the norm bound. Further, we prove a stronger result for Parseval frames whose norms are uniformly small, which shows that in addition to the spanning property, the sets can be chosen to be independent, and the complement of each set to contain a number of disjoint, spanning sets.

1. Introduction

Over the last decades, frame theory has developed into a vibrant subject including contributions in time-frequency analysis [9, 19, 13, 10, 8, 11] and applications in engineering such as wireless communications or other types of signal and image processing techniques, see the survey papers [15, 16] and the many references therein. In pure mathematics, frame theory has opened up new approaches to one of the significant open problems in analysis today - the notoriously intractable 1959 Kadison-Singer Problem [3, 7, 6].

Formally, a *frame* is a family of vectors $\{f_i\}_{i\in I}$ in a real or complex Hilbert space $\mathbb{H}$ so that there are constants $0 < A \le B < \infty$ (called the lower and upper frame bounds, respectively) satisfying

$$A\|x\|^2 \le \sum_{i\in I} |\langle x, f_i\rangle|^2 \le B\|x\|^2, \quad \text{for all } x \in \mathbb{H}. \tag{1.1}$$

The frame is an *A-tight frame* if $A = B$, a *Parseval frame* if $A = B = 1$, an *equal norm frame* if $\|f_i\| = c$, for all $i \in I$, and a *unit norm frame* if $c = 1$. If we only require the upper frame bound B, we call this a B-Bessel sequence. Associated to a frame we have the *analysis operator* $T : \mathbb{H} \to \ell_2(I)$ given by $Tx = \sum_{i\in I}\langle x, f_i\rangle e_i$, where $\{e_i\}_{i\in I}$ is the natural orthonormal basis of $\ell_2(I)$. Its adjoint T^* is the *synthesis operator* given by $T\left(\sum_{i\in I} a_i e_i\right) = \sum_{i\in I} a_i f_i$. The *frame operator* $S = T^*T$ satisfies $Sx = \sum_{i\in I}\langle x, f_i\rangle f_i$ for $x \in \mathbb{H}$. It provides the *reconstruction formula*

$$x = \sum_{i\in I}\langle x, S^{-1}f_i\rangle f_i, \text{ for all } x \in \mathbb{H}. \tag{1.2}$$

The first author was supported by NSF Grant DMS-0807399, the second author by NSF 1008183: DTRA/NSF 1042701, the third author by NSF DMS-0600191 and the fourth author by NSF DMS-0354957.

We refer the reader to [8] for the requisite background in frame theory.

What makes frames so useful in practice is their *redundancy*. That is, in general a frame might have smaller subsets, each with a dense linear span in the space $\mathbb{H}$. Accordingly, every vector in the space has uncountably many representations with respect to a frame, not just the canonical choice given by Equation 1.2. The flexibility in choosing representations is key to many applications [15, 16]. One of the most important problems in frame theory today is therefore to understand redundancy and its role in these applications. This includes a long list of fundamental questions concerning the behavior of subsets of a frame such as how many (disjoint) spanning sets does a frame contain? Or, how many (disjoint) linearly independent subsets does it contain? Or, putting these together, how many disjoint linearly independent spanning sets does our frame contain? Can we partition any unit norm frame into a finite number of subsets with specialized properties such as each subset being *nearly tight* for its span, i.e. with $\frac{B}{A} \approx 1$? This innocent looking problem in frame theory is now known to be equivalent to the 1959 Kadison-Singer Problem [4, 7].

One of the first results concerning the decomposition of frames into linearly independent sets is in [3] where it is shown that a frame can be partitioned into $\lceil B \rceil$-linearly independent sets. In [7] it is shown that a unit norm tight frame $\{f_i\}_{i=1}^{KM}$ in an M-dimensional space can be partitioned into K linearly independent spanning sets. It has been an open question since [7] appeared whether there is a similar result for unit norm frames $\{f_i\}_{i=1}^{KM+r}$ for $0 < r < M$. This is one of the questions we will answer in this paper. Recently, an intuitive, quantitative measure for redundancy was given for finite frames [2]. It is interesting to note that it relates to independence and spanning properties of frame partitions which we explore here. We expect that such a systematic refinement of our understanding of redundancy will have an important impact on applications.

Underlying much of this paper is the classification theorem for Parseval frames (see e.g. [8]):

Theorem 1.1 (Classification Theorem for Parseval Frames)**.** *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ with analysis operator T. Then $\{f_i\}_{i\in I}$ is a Parseval frame for $\mathbb{H}$ if and only if T is an isometry and the orthogonal projection $P : \ell_2(I) \to T(\mathbb{H})$ satisfies $Tf_i = Pe_i$, for all $i \in I$. In particular, this projection is given by the Gram matrix $(\langle f_j, f_i\rangle)_{i,j\in I}$.*

Moreover, in this case, $\{(I-P)e_i\}_{i\in I}$ is a Parseval frame for its span called the Naimark-Complement of $\{f_i\}_{i\in I}$.

It is immediate that an orthogonal projection takes a frame with frame bounds A, B to a frame for its range with the same frame bounds. In particular, an orthogonal projection maps a Parseval frame to a Parseval frame.

We will concentrate on Parseval frames here since with respect to independence and spanning properties, this is really the general case in the sense that every frame $\{f_i\}_{i\in I}$ is isomorphic to the Parseval frame $\{S^{-1/2}f_i\}_{i\in I}$, where S is the frame operator. That is, $S^{-1/2}$ is an invertible operator mapping our frame to a Parseval frame and hence it maintains linearly independence properties, spanning properties and Riesz basic sequences. Many of our results rely on the assumption that the norms of the Parseval frame vectors are uniformly bounded away from 1. This is a necessary assumption since for a Parseval frame $\{f_i\}_{i\in I}$, Equation 1.1 quickly yields that if $\|f_j\| = 1$, then $f_j \perp$ *span* $\{f_i\}_{i\neq j}$.

The remainder of this paper is organized as follows: In Section 2, we will give a dichotomy between linearly independent subsets of a Parseval frame and spanning subsets of its Naimark Complement. This represents a new approach to these questions which previously concentrated only on finding linearly independent subsets of a frame. We will then show that Parseval frames whose vectors have norms uniformly bounded away from one can be partitioned into a finite number of subsets so that the complement of each subset spans the whole space. In Section 3, we will derive a non-trivial variation of the famous Rado-Horn Theorem and use it to show that every finite equal norm Parseval frame can be partitioned into linearly independent spanning sets with (perhaps) a left over linearly independent set. This answers a problem stemming from the results in [7].

2. Spanning Properties for Frame Partitions

In Proposition 2.3 we establish a dichotomy between independence properties of subsets of a finite Parseval frame and spanning properties of complementary subsets of its Naimark complement. This gives a new approach to the Kadison-Singer Problem which is complementary to the standard equivalences of the problem. In Theorem 2.8 we will apply this result to show that given a Parseval frame $\{f_n\}_{n\in\mathbb{N}}$ with $\|f_n\|^2 \le 1-\delta$ we can partition $\mathbb{N}$ into r-subsets (r only depending upon δ) $\{A_j\}_{j=1}^r$ so that span $\{f_n\}_{n\in A_k^c} = \mathbb{H}$, for all $1 \le k \le r$.

To establish our dichotomy, we will first classify when submatrices of the Gram matrix of a Parseval frame are one-to-one operators. For notation, given a family of vectors $\mathcal{F} = \{f_i\}_{i\in S}$ in a Hilbert space $\mathbb{H}$, where S is some index set, and a subset $B \subseteq S$, we write $\mathcal{F}_B = \{f_i\}_{i\in B}$ and let $\mathbb{H}_B$ denote the closed linear span of $\mathcal{F}_B$. Recall that by Theorem 1.1 and the discussion following it, orthogonal projections take Parseval frames to Parseval frames and $\{f_i\}_{i\in S}$ is a Parseval frame for $\mathbb{H}$ if and only if the Gram matrix $G = (\langle f_j, f_i\rangle)_{i,j\in S}$ is the matrix of a projection operator on $\ell^2(S)$.

We start with a simple decomposition of Gram matrices.

Proposition 2.1. *Let $\{f_i\}_{i\in S}$ be a Parseval frame for $\mathbb{H}$, let P be an orthogonal projection onto a closed subspace of $\mathbb{H}$ and let I denote the identity operator on $\mathbb{H}$. Then $G = (\langle f_j, f_i\rangle)_{i,j,\in S}$, $R = (\langle Pf_j, Pf_i\rangle)_{i,j\in S}$ and $Q = (\langle (I-P)f_j, (I-P)f_i\rangle)_{i,j\in S}$ are the matrices of projection operators on $\ell^2(S)$ with $G = R+Q$. Moreover, $P = I$ if and only if 1 is not an eigenvalue of Q.*

Proof. The equality $G = R+Q$ is immediate from $R = (\langle Pf_j, f_i\rangle)$ and $Q = (\langle (I-P)f_j, f_i\rangle)$ for each $i,j \in S$, and from the linearity of the inner product in the first entry. The fact that G, Q and R are matrices of projections follows from the fact that the vectors $\{f_i\}_{i\in S}$ form a Parseval frame and the above remarks.

For the final statement, note that $P = I$ if and only if $Q = 0$. But since Q is a projection, $Q = 0$ if and only if 1 is not an eigenvalue. □

Given a subset $B \subseteq S$, we let $D_B = (d_{i,j})_{i,j\in S}$ denote the bounded operator on $\ell^2(S)$ whose matrix is the diagonal matrix with $d_{i,i} = 1$ when $i \in B$ and $d_{i,j} = 0$ when $i \in B^c$ or $j \in B^c$, the complement of the set B. Now we will relate the spanning properties of a subset of a Parseval frame to 1 not being an eigenvalue of the corresponding submatrix of the Gram matrix.

Proposition 2.2. *Let $\{f_i\}_{i\in S}$ be a Parseval frame for $\mathbb{H}$, let $G = (\langle f_j, f_i\rangle)_{i,j\in S}$ denote its Gram matrix, and let $B \subseteq S$. Then $\mathbb{H}_B = \mathbb{H}$ if and only if 1 is not an eigenvalue of $D_{B^c}GD_{B^c}$.*

Proof. Let P denote the projection onto $\mathbb{H}_B$ and apply Proposition 2.1. Since for $j \in B, f_j \in \mathbb{H}_B$, we have that when $j \in B$ then $\langle f_j, f_i\rangle = \langle Pf_j, Pf_i\rangle$. More generally, the matrices G and R are equal in any entry (i, j) provided that $i \in B$ or $j \in B$. Thus, the matrix Q must be 0 in any such entry. Hence we obtain the operator inequalities $0 \le Q = D_{B^c}QD_{B^c} \le D_{B^c}GD_{B^c} \le D_{B^c}$.

Now, if 1 is not an eigenvalue of $D_{B^c}GD_{B^c}$, then these inequalities imply that 1 is not an eigenvalue of Q. Invoking the preceding proposition, we get $P = I$, and so $\mathbb{H}_B = \mathbb{H}$.

Conversely, assume that 1 is an eigenvalue of $D_{B^c}GD_{B^c}$. Write $G = VV^*$ where $V : \mathbb{H} \to \ell^2(S)$ is the analysis operator of the Parseval frame. Since $D_{B^c}GD_{B^c} = (V^*D_{B^c})^*(V^*D_{B^c})$, we have that $(V^*D_{B^c})(V^*D_{B^c})^* = V^*D_{B^c}V$ also has eigenvalue 1. By the Parseval property, V is an isometry, $V^*V = I$, necessarily $V^*D_BV = I - V^*D_{B^c}V$ has eigenvalue zero. Thus the range of V^*D_B is orthogonal to the corresponding eigenvectors. Since the closure of the range of V^*D_B is by definition $\mathbb{H}_B$, it is not equal to $\mathbb{H}$. □

Now we can give our complementarity principle between spanning and linear independence.

Proposition 2.3. *Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{e_j\}_{j\in S}$, let P be the orthogonal projection onto a closed subspace of $\mathbb{H}$, and let $B \subseteq S$. Then the linear span of $\{Pe_j\}_{j\in B}$ is dense in $P(\mathbb{H})$ if and only if the operator $(\langle (I - P)e_j, (I - P)e_i\rangle)_{i,j\in B^c}$ on $\ell^2(B^c)$ is one-to-one.*

Proof. Note that the set $\{Pe_j : j \in S\}$ is a Parseval frame for $P(\mathbb{H})$. Hence, the span of $\{Pe_j : j \in B\}$ is dense in $P(\mathbb{H})$ if and only if the matrix $Q = (\langle Pe_j, Pe_i\rangle)_{i,j\in B_c}$ does not have 1 as an eigenvalue. But since $I_{\ell^2(B^c)} - Q = (\langle (I - P)e_j, (I - P)e_i\rangle)_{i,j\in B^c}$, Q not having eigenvalue 1 is equivalent to the latter matrix having a trivial kernel. □

Recall that a family of vectors $\{f_i\}_{i\in I}$ is called an *ℓ_2-independent set* if whenever $\{a_i\}_{i\in I} \in \ell_2(I)$ and $\sum_{i\in I} a_if_i = 0$, then $a_i = 0$ for all $i \in I$. In this language, Proposition 2.3 becomes:

Corollary 2.4. *Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{e_j\}_{j\in S}$, let P be the orthogonal projection onto a closed subspace of $\mathbb{H}$, and let $B \subseteq S$. Then the linear span of $\{Pe_j\}_{j\in B}$ is dense in $P(\mathbb{H})$ if and only if $\{(I - P)e_i\}_{i\in B^c}$ is ℓ_2-independent.*

Proof. The operator $A = (\langle (I-P)e_j, (I-P)e_i\rangle)_{i,j\in B^c}$ satisfies for all $a = \{a_i\}_{i\in B^c} \in \ell_2(B^c)$:

$$A(a) = \left(\langle (I - P)e_j, \sum_{i\in B^c} a_i(I - P)e_i\rangle\right)_{i,j\in B^c}.$$

In particular, $\sum_{i\in B^c} a_i(I-P)e_i = 0$ if and only if $A(a) = 0$ if and only if $a = 0$. □

In the finite dimensional setting, Proposition 2.3 implies:

Corollary 2.5. *If P is an orthogonal projection on a finite dimensional Hilbert space $\mathbb{H}$, then $\{Pe_j\}_{j\in B}$ spans $P(\mathbb{H})$ if and only if the set $\{(I-P)e_j : j \in B^c\}$ is linearly independent.*

For the final results in this section, we will show how to partition a Parseval frame whose vectors are uniformly bounded away from 1 in norm into subsets whose complementary sets are spanning.

For these results, we will need two results from the literature which involve partitioning frames into linearly independent sets or into ℓ_2-independent sets. The first is due to Casazza, Christensen, Lindner and Vershynin [3].

Theorem 2.6. *If $\{f_n\}_{n\in\mathbb{N}}$ is a unit norm B-Bessel sequence, then $\{f_n\}_{n\in\mathbb{N}}$ can be partitioned into $r = \lceil B \rceil$ linearly independent sets.*

We also need the following result of Casazza, Kutyniok, Speegle and Tremain [6].

Theorem 2.7. *Every unit norm Bessel sequence which is finitely linearly independent is a union of two ℓ_2-independent sets.*

Theorem 2.8. *Let $0 < \delta < 1$, and set $r = 2\lceil \frac{1}{\delta} \rceil$. If $\{f_n\}_{n\in\mathbb{N}}$ is a Parseval frame for a Hilbert space $\mathbb{H}$, with $\|f_n\|^2 \le 1-\delta$ for all $n \in \mathbb{N}$, then there exists a partition of $\mathbb{N}$ into r disjoint sets, $A_1 \cup \cdots \cup A_r = \mathbb{N}$, such that $\mathbb{H}_{A_k^c} = \mathbb{H}$, for $k = 1, ..., r$.*

Proof. By the classification theorem for Parseval frames, we may assume that there is a projection P on $\ell_2(\mathbb{N})$ satisfying $Pe_n = f_n$ for all $n \in \mathbb{N}$, with $\{e_n\}_{n\in\mathbb{N}}$ the natural orthonormal basis of $\ell_2(\mathbb{N})$. By our assumption in the theorem, $\|(I-P)e_n\|^2 = 1 - \|Pe_n\|^2 \ge \delta$. It follows that $\{\frac{1}{\|(I-P)e_n\|}(I-P)e_n\}_{n=1}^{\infty}$ is a $\frac{r}{2}$-Bessel sequence. By Theorem 2.6, there is a partition $\{B_1, B_2, \ldots, B_{\frac{r}{2}}\}$ of $\mathbb{N}$ so that $\{(I-P)e_n/\|(I-P)e_n\|\}_{n\in B_j}$ is finitely linearly independent for all $j \in \{1, 2, \ldots, \frac{r}{2}\}$. Now, applying Theorem 2.7, we can partition each of the B_j into two sets with the corresponding vectors ℓ_2-independent. We call $\{A_j\}_{j=1}^{r}$ the resulting partition for $\mathbb{N}$. It follows, of course, that after removing the normalization each $\{(I-P)e_n\}_{n\in A_j}$ is also ℓ_2-independent. The theorem now follows from Corollary 2.4. □

Examining the proof of Theorem 2.8 we see that in the finite dimensional case we do not need the extra step of a second partition of the frame into ℓ_2-independent sets. So in this case we have:

Theorem 2.9. *Let $\delta > 0$. Suppose that $\{f_j : j \in J\}$ is a Parseval frame for a finite dimensional Hilbert space with $\|f_j\|^2 \le 1-\delta$ for all $j \in J$. Then, it is possible to partition J into $r = \lceil \frac{1}{\delta} \rceil$ sets $\{A_1, \ldots, A_r\}$ such that for each $1 \le j \le r$, the family $\{f_i : i \notin A_j\}$ spans the space.*

Note that r depends only on the norm bound $1-\delta$, and not the dimension of the space, and an explicit formula for r as a function of δ is provided. Theorem 2.9 shows the advantage we have gained by switching from working with linearly independent sets to spanning sets. In particular, it is not possible, in general, to get the partition in Theorem 2.9 to have the property that the $\{f_j\}_{j\in A_i}$ are linearly independent. The problem is that without a lower bound on the norms of the frame vectors there can be an arbitrarily large number of them. That is, there can be too many frame vectors to be able to partition them into R linearly independent sets for any R which depends only upon δ. However, we will see in the next section that it is possible to achieve a partition in which all sets but one are linearly independent and spanning, if the norms of the vectors are uniformly small.

3. Spanning and linear independence properties for Parseval frames with uniformly small norms

Until now, we have considered the problem of partitioning Parseval frames into spanning sets or linearly independent sets. Now we will examine the much deeper problem of partitioning a Parseval frame into linearly independent spanning sets. This is a fundamental problem in the field which until now has had only one case that has been answered. Namely, in [7] it is shown that a equal norm Parseval frame $\{f_i\}_{i=1}^{KM}$ for $\mathbb{H}_M$ can be partitioned into K-linearly independent spanning sets. This proof relies on the Rado-Horn Theorem and to prove our result, we will have to first strengthen the Rado-Horn Theorem itself. The main theorem in this section is:

Theorem 3.1. *Let $\{f_i\}_{i\in I}$ be an equal norm Parseval frame for an N dimensional Hilbert space $\mathbb{H}_N$ with $|I| = rN + k$ with $0 \le k < N$. Then there is a partition $\{I_i\}_{i=1}^{r+1}$ of I so that for $i \in \{2, \ldots, r+1\}$, $\{f_j\}_{j\in I_i}$ is a linearly independent spanning set and $\{f_j\}_{j\in I_1}$ is linearly independent.*

This theorem will follow immediately from Propositions 3.3 and 3.5 combined with Theorem 3.7. Our proof will produce a number of fundamental results giving good bounds on how many linearly independent sets or spanning sets we can divide frames into. Without the equal norm assumption, the theorem fails badly. For example, if $\{e_i\}_{i=1}^M$ is an orthonormal basis for $\mathbb{H}_M$, the family $f_1 = \frac{1}{\sqrt{M+1}}e_1$, $f_2 = \frac{1}{\sqrt{M+1}}e_1, \ldots, f_{M+1} = \frac{1}{\sqrt{M+1}}e_1$, and $f_j = e_j$ for all $j = M+2, \ldots, 2M$, is a Parseval frame for $\mathbb{H}_M$ with $2M$-vectors which only contains one linearly independent spanning set, while the theorem would require this to contain 2 linearly independent spanning sets if the vectors had been equal norm. If $r \ge 2$ then this result implies that each set of frame vectors has a complement which is spanning, which was already obtained in Section 3. Moreover, the complement of each set can then be partitioned into at least $r-1$ spanning sets.

Since the proof of Theorem 3.1 is quite long, we will divide it into three subsections as follows:

(1) We first show that a Parseval frame as in Theorem 3.1 can be partitioned into $r+1$ linearly independent sets.

(2) We next show that a Parseval frame as in Theorem 3.1 can be partitioned into r spanning sets. This will follow from some more general results presented in this section which are of independent interest.

(3) In the last subsection, we show that any frame which has one partition into $r+1$ linearly independent sets and another partition into r linearly independent spanning sets plus a remaining set, has a third partition which consists of r linearly independent spanning sets plus a linearly independent set. This result requires adapting the generalized Rado-Horn Theorem given in [5].

3.1. **Partitioning a frame into linearly independent sets.** In this subsection we will generalize the result of Casazza and Tremain [7] which partitions an equal norm Parseval frame $\{f_i\}_{i=1}^{KM}$ for an M dimensional Hilbert space into K linearly independent spanning sets. For this, we need a generalization of the Rado-Horn Theorem due to Casazza, Kutyniok and Speegle [5].

Theorem 3.2. *Let $\{f_i\}_{i\in I}$ be a finite collection of vectors in a vector space X and let $M \in \mathbb{N}$. The following conditions are equivalent:*

(1) There exists a partition $\{I_j\}_{j=1}^M$ of I so that for each j, $\{f_i\}_{i\in I_j}$ is linearly independent.

(2) For all $J \subset I$,

$$\frac{|J|}{dim\ span\ \{f_i\}_{i\in J}} \le M.$$

Moreover, in the case that the above conditions fail, there exists a partition $\{I_j\}_{j=1}^M$ of I and a subspace S of X such that the following three conditions hold.

(a) For all $1 \le j \le M$, $S = span\ \{f_i : i \in I_j,\ and\ f_i \in S\}$.

(b) For $J = \{i \in I : f_i \in S\}$,

$$\frac{|J|}{dim\ span\ \{f_i\}_{i\in J}} > M.$$

(c) For each $1 \le j \le M$,

$$\sum_{i\in I_j, f_i\notin S} \alpha_i f_i = 0, \quad implies \quad \alpha_i = 0, \quad for\ all\ i.$$

In particular, for each $1 \le j \le M$, $\{f_i : i \in I_j,\ f_i \notin S\}$ is linearly independent.

We can now give a generalization of the result of Casazza and Tremain [7] discussed above.

Proposition 3.3. *Let r, k, N be natural numbers with $0 < k < N$ and let $\{f_i\}_{i=1}^{rN+k}$ be an equal norm Parseval frame for an N-dimensional Hilbert space $\mathbb{H}_N$. Then $\{f_i\}_{i=1}^{rN+k}$ can be partitioned into $r+1$ linearly independent sets. If $k = 0$, then $\{f_i\}_{i=1}^{rN}$ can be partitioned into r linearly independent spanning sets.*

Proof. Since $\{f_i\}_{i=1}^{rN+k}$ is an equal norm Parseval frame, we have

$$N = \sum_{i=1}^{rN+k} \|f_i\|^2 = (rN+k)\|f_j\|^2, \quad \text{for all } j = 1, 2, \ldots, rN+k.$$

That is,

$$\|f_i\|^2 = \frac{N}{rN+k}, \quad \text{for all } i = 1, 2, \ldots, N.$$

We will apply Theorem 3.2(2). Choose $J \subset \{1, 2, \ldots, rN+k\}$. Let P be the orthogonal projection of $\mathbb{H}_N$ onto span $\{f_i\}_{i\in J}$. Since $\{Pf_i\}_{i\in J}$ is a Parseval frame for its span we have

$$\dim \operatorname{span}\ \{f_i\}_{i\in J} = \sum_{i=1}^{rN+k} \|Pf_i\|^2 \ge \sum_{i\in J} \|Pf_i\|^2 = \sum_{i\in J} \|f_i\|^2 = \frac{N|J|}{rN+k}.$$

That is,

$$\frac{|J|}{\dim \operatorname{span}\,\{f_i\}_{i\in J}} \le \frac{rN+k}{N}.$$

Hence,

$$\frac{|J|}{\dim \operatorname{span}\,\{f_i\}_{i\in J}} \le \begin{cases} r & \text{if } k=0,\\ r+1 & \text{if } 0<k<N.\end{cases}$$

The result now follows by Theorem 3.2 and the fact that in the case $k = 0$, we have partitioned an rN element set into r linearly independent sets in an N-dimensional Hilbert space $\mathbb{H}_N$, and hence, each must contain exactly N elements and so it must be a spanning set. □

3.2. **Partitioning a frame into spanning sets.** In this section we will give fairly accurate bounds on how many spanning sets we can divide a frame into in terms of the lower frame bound of the frame and the norms of the frame vectors. As a useful tool in this work, we first show that a frame which can be partitioned into r linearly independent sets or into r spanning sets, has the property that any partition of the frame into r linearly independent sets forces these sets to already be spanning.

Lemma 3.4. *Let $\{f_i\}_{i\in I}$ be a set of vectors in an N-dimensional Hilbert space $\mathbb{H}_N$ and let $I_j \subset I$, $j = 1, 2, \ldots r$ be linearly independent subsets. Assume that there is a partition of I into $\{A_j\}_{j=1}^r$ so that*

$$\operatorname{span}\,\{f_i\}_{i\in A_j} = \mathbb{H}_N, \quad \textit{for all } j = 1, 2, \ldots, r.$$

Then $\{I_j\}_{j=1}^r$ is a partition of I and

$$\operatorname{span}\,\{f_i\}_{i\in I_j} = \mathbb{H}_N, \quad \textit{for all } j = 1, 2, \ldots, r.$$

Proof. For all $j = 1, 2, \ldots, r$, the fact that $\{f_i\}_{i\in I_j}$ are linearly independent implies that the dimension of the span of $\{f_i\}_{i\in I_j}$ is $|I_j|$. Also, the fact that $\{f_i\}_{i\in A_j}$ spans $\mathbb{H}_N$ implies $|A_j| \ge N$. Now, we have

$$Nr \ge \sum_{j=1}^{r} \dim \operatorname{span}\,\{f_i : i\in I_j\} = \sum_{j=1}^{r} |I_j| = |I| = |\cup_{j=1}^r A_j| = \sum_{j=1}^{r} |A_j| \ge Nr.$$

Hence, $\sum_{j=1}^r \dim \operatorname{span}\,\{f_i : i \in I_j\} = Nr$ and $\sum_{j=1}^r |I_j| = |I|$. Thus, $\dim \operatorname{span}\,\{f_i : i \in I_j\} = N$ for every $j = 1, 2, \ldots, r$ and the sets $\{I_j\}_{j=1}^r$ form a partition. □

Next, we will give a reasonable lower bound on the number of spanning sets any frame contains in terms of the lower frame bound of the frame and the norms of the frame vectors.

Proposition 3.5. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}_N$ with lower frame bound $A \ge 1$, let $\|f_i\|^2 \le 1$ for all $i \in I$ and set $r = \lfloor A\rfloor$. Then there exists a partition $\{I_j\}_{j=1}^r$ of I so that*

$$\operatorname{span}\,\{f_i : i \in I_j\} = \mathbb{H}_N, \quad \textit{for all } j = 1, 2, \ldots, r.$$

In particular, the number of frame vectors in a unit norm frame with lower frame bound A is greater than or equal to$\lfloor A\rfloor N$.

Proof. We replace $\{f_i\}_{i\in I}$ by $\{\frac{1}{\sqrt{r}} f_i\}_{i\in I}$ so that our frame has lower frame bound greater than or equal to 1 and $\|f_i\|^2 \le \frac{1}{r}$,for all $i \in I$. Assume the frame operator for $\{f_i\}_{i\in I}$ has eigenvectors $\{e_j\}_{j=1}^N$ with respective eigenvalues $\lambda_1 \ge \lambda_2 \ge \ldots \lambda_N \ge 1$. We proceed by induction on N.

We first consider $N = 1$: Since

$$\sum_{i\in I} \|f_i\|^2 \geq 1, \quad \text{and} \quad \|f_i\|^2 \leq \frac{1}{r}, \tag{3.1}$$

it follows that $|\{i \in I : f_i \neq 0\}| \geq r$ and so we have a partition into r spanning sets.

Next, we assume the inductive hypothesis holds for any Hilbert space of dimension N and let $\mathbb{H}_{N+1}$ be a Hilbert space of dimension $N+1$. We check two cases:

Case I: Suppose there exists a partition $\{I_j\}_{j=1}^r$ of I so that $\{f_i\}_{i\in I_j}$ is linearly independent for all $j = 1, 2, \ldots, r$.

In this case,

$$N+1 \leq (N+1)\lambda_N \leq \sum_{j=1}^{N+1} \lambda_j = \sum_{i\in I} \|f_i\|^2 \leq |I|\frac{1}{r},$$

and hence,

$$|I| \geq r(N+1).$$

However, by linear independence, we have

$$|I| = \sum_{j=1}^{r} |I_j| \leq r(N+1).$$

Thus, $|I_j| = N+1$ for every $j = 1, 2, \ldots, r$ and so $\{f_i\}_{i\in I_j}$ are all spanning.

Case II: Our family cannot be partitioned into r linearly independent sets. In this case, let $\{I_j\}_{j=1}^r$ and a subspace $\emptyset \neq S \subset \mathbb{H}_{N+1}$ be given by Theorem 3.2. If $S = \mathbb{H}_{N+1}$, we are done. Otherwise, let P be the orthogonal projection onto the subspace S. Let

$$I_j' = \{i \in I_j : f_i \notin S\}, \quad I' = \cup_{j=1}^r I_j'.$$

Theorem 3.2(c) implies that $\{(I-P)f_i\}_{i\in I_j'}$ is linearly independent for all $j = 1, 2, \ldots, r$. To see this, note that the non-zero elements of $\{(I-P)f_i\}_{i\in I}$ are $\{(I-P)f_i\}_{i\in I'}$. Fix $1 \leq j \leq r$ and assume there are scalars $\{\alpha_i\}_{i\in I_j'}$ with $\sum_{i\in I_j'} \alpha_i(I-P)f_i = 0$.

This implies by Theorem 3.2(c) that $\sum_{i\in I_j'} \alpha_i f_i \in S$ and so $\alpha_i = 0$ for all $i \in I_j'$.

Now, $\{(I-P)f_i\}_{i\in I'}$ has a lower frame bound 1 in $(I-P)(\mathbb{H}_{N+1})$, dim $(I-P)(\mathbb{H}_{N+1}) \leq N$ and $\|(I-P)f_i\|^2 \leq \|f_i\|^2 \leq \frac{1}{r}$ for all $i \in I'$.

Applying the induction hypothesis, we can find a partition $\{A_j\}_{j=1}^r$ of I' with span $\{(I-P)f_i\}_{i\in A_j} = (I-P)(\mathbb{H}_{N+1})$ for all $j = 1, 2, \ldots, r$.

Now, we can apply Lemma 3.4 together with the partition $\{A_j\}_{j=1}^r$ to conclude span $\{(I-P)f_i\}_{i\in I_j'} = (I-P)(\mathbb{H}_{N+1})$, and hence

$$\text{span}\,\{f_i\}_{i\in I_j} = \text{span}\,\{S, \{(I-P)f_i\}_{i\in I_j'}\} = \mathbb{H}_{N+1}.$$

□

Note that we cannot expect to get any linear independence in Proposition 3.5 because our vectors can have arbitrarily small norms and hence there can be an arbitrarily large number of them. However, we can remove appropriate vectors from the last $r-1$-sets until they are linearly independent and spanning. Putting the removed vectors into the first set, we get a partition into a spanning set and $r-1$ linearly independent spanning sets.

As an immediate consequence, we have the main result of this section.

Corollary 3.6. *Let $\{f_i\}_{i\in I}$ be a Parseval frame for a finite dimensional Hilbert space $\mathbb{H}$ and r a natural number so that $\|f_i\|^2 \le \frac{1}{r}$ for every $i \in I$. Then there is a partition $\{I_j\}_{j=1}^r$ of I so that*

$$\text{span}\,\{f_i : i \in I_j\} = \mathbb{H}, \quad \textit{for all } j = 1, 2, \ldots, r.$$

3.3. **Partitioning a frame into linearly independent spanning sets.** In this subsection, we want to combine the results from the previous two subsections to get a partition of a frame into linearly independent spanning sets. This will require a refinement of the proof of the generalization of the Rado-Horn Theorem given in [5]. To help explain why we need such technical lemmas, let us first look at the main result we want to prove.

Theorem 3.7. *Let $\{f_i\}_{i\in I}$ be a finite collection of vectors in a finite dimensional vector space X. Assume*

(1) $\{f_i\}_{i\in I}$ can be partitioned into $r+1$-linearly independent sets, and

(2) $\{f_i\}_{i\in I}$ can be partitioned into a set and r linearly independent spanning sets.

Then there is a partition $\{I_i\}_{i=1}^{r+1}$ so that $\{f_j\}_{j\in I_i}$ is a linearly independent spanning set for all $i = 2, 3, \ldots, r+1$ and $\{f_i\}_{i\in I_1}$ is a linearly independent set.

The idea of the proof of Theorem 3.7 is the following. We will choose the partition of our frame into r linearly independent spanning sets so that the dimension of the span of the remaining set is a maximum. Then, we will show that this remaining set must be linearly independent. This will be done by showing that if this remaining set is not linearly independent we can get a contradiction to the Rado-Horn Theorem. But to get this contradiction, will require a delicate interchange of the vectors from the sets in the partition. It is this interchange result we will now develop.

We will be considering partitions which maximize dimensions in a very particular way.

Definition 3.8. Let $\{f_i\}_{i\in I}$ be a family of vectors. We say that a partition $\{I_1, \ldots, I_M\}$ of the index set I has the maximality property (MD) if whenever $\{J_i\}_{i=1}^M$ is any partition of I satisfying that for all $1 \le i \le M$, $\dim\text{span}\,\{f_j\}_{j\in J_i} \ge \dim\text{span}\,\{f_j\}_{j\in I_i}$, then $\dim\text{span}\,\{f_j\}_{j\in I_i} = \dim\text{span}\,\{f_j\}_{j\in J_i}$ for all $i = 1, 2, \ldots, M$.

A straightforward consequence of maximality is the following:

Lemma 3.9. *Let $\mathcal{F} = \{f_i : i \in I\}$ be a finite collection of vectors in a vector space. Let $M \in \mathbb{N}$ and $\{I_j : j = 1, \ldots, M\}$ be a partition of I satisfying property (MD). If $f_k \in I_p$ and $f_k = \sum_{l\in I_p, l\neq k} \alpha_l f_l$, then $f_k \in \text{span}(\mathcal{F}_{I_j})$ for all $1 \le j \le M$.*

Proof. Assuming the hypothesis of the lemma, if $f_k = \sum_{l\in I_p, l\neq k} \alpha_l f_l$, then removing f_k from I_p keeps $\dim\text{span}(\mathcal{F}_{I_p})$ constant. By property (MD), moving f_k into another I_j, $j \neq p$ cannot increase $\dim\text{span}(\mathcal{F}_{I_j})$, and the result follows. □

If there are linearly dependent sets in a partition having property (MD) then we can move suitable vectors from one set to another. The following definition will be used to help us keep track of which vectors are being moved.

Definition 3.10. Let $\{f_i : i \in I\}$ be a collection of vectors in a vector space and let $\{I_j : j = 1, \dots, M\}$ be a partition of I. We define a *chain of length one* to be a set $\{(a, b)\}$ with $a \in I_b$, $b \in \{1, 2, \dots, M\}$ and $f_a = \sum_{j \in I_b, j \neq a} \alpha_j f_j$ for some choice of constants $\{\alpha_j\}_{j \in I_b, j \neq a}$. We define a *chain of length* n to be a finite sequence $\{(a_1, b_1), \dots, (a_n, b_n)\}$, where $a_i \in I$ and $b_i \in \{1, \dots, M\}$, such that

- (a_1, b_1) is a chain of length one,
- for $2 \leq i \leq n$, $a_i \in I_{b_i}$ and $f_{a_i} = \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \neq a_i} \alpha_j f_j$ for some $\alpha \neq 0$, and
- $a_i \neq a_k$ for $i \neq k$.

A chain of length n starting with $a_1 \in L \subset I$ and ending at $a_n \in I$ is a *chain of minimal length starting in* L *and ending at* a_n if every chain starting in L and ending at a_n has length greater than or equal to n.

We recall the following lemma.

Lemma 3.11 (Casazza, Kutyniok, Speegle)**.** *Let $\{f_i : i \in I\}$ be a collection of vectors in a vector space, let $\{I_j : j = 1, \dots, M\}$ be a partition of I, and let $L \subset I_1$.*

If $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a chain of minimal length starting in L and ending at a_n, then for each $1 \leq i \leq n$, $\{(a_1, b_1), \dots, (a_i, b_i)\}$ is a chain of minimal length starting in L and ending at a_i.

Proof. By induction it suffices to show that $\{(a_1, b_1), \dots, (a_{n-1}, b_{n-1})\}$ is a chain of minimal length. Suppose, for the sake of contradiction, that there did exist a chain $\{(u_1, v_1), \dots, (u_k, v_k)\}$ such that $u_k = a_{n-1}$ and $k < n - 1$. Since $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a chain,

$$f_{a_n} = \alpha f_{a_{n-1}} + \sum_{j \in I_{b_n}, j \neq a_n} \alpha_j f_j$$

for some $\alpha \neq 0$. Therefore, either $\{(u_1, v_1), \dots, (u_k, v_k), (a_n, b_n)\}$ is a chain with length $k + 1 < n$ or $a_n = u_i$ for some $i \leq k$, either of which contradicts the minimality of n. □

Next, we show how to choose a partition having property (MD).

Lemma 3.12. *Let $\mathcal{F} = \{f_i : i \in I\}$ be a finite collection of vectors, and $M \in \mathbb{N}$. There exists among all the partitions of I into M non-empty subsets a partition $\{I_1, I_2, \dots I_M\}$ with the property (MD). This partition can be chosen so that $\mathcal{F}_{I_j}$ is linearly independent for all $2 \leq j \leq M$.*

Proof. The set of partitions of I into M sets has a partial ordering with respect to which two partitions $\{I_j\}_{j=1}^M$ and $\{J_j\}_{j=1}^M$ satisfy $\{I_j\}_{j=1}^M \leq \{J_j\}_{j=1}^M$ if $\dim \mathcal{F}_{I_j} \geq \dim \mathcal{F}_{J_j}$ for all $j \in \{1, 2, \dots, M\}$. I is a finite set, so there are maximal elements. By definition, these partitions have the property (MD).

Assume that there is a partition with property (MD) which contains more than one set for which the associated vectors are linearly dependent, say I_1 and I_2. We can then successively remove indices from I_2 and place them into I_1 if the associated vectors are linear combinations of others remaining in the set indexed by I_2. After finitely many such moves, $\mathcal{F}_{I_2}$ is linearly independent. Moreover, by Lemma 3.9, the span of $\mathcal{F}_{I_1}$ and $\mathcal{F}_{I_2}$ retain their dimensions, which means the maximality is preserved. □

If $\mathcal{F}_{I_2}, \ldots, \mathcal{F}_{I_M}$ are linearly independent, $L = \{i \in I_1 : f_i = \sum_{j \in I_1, j \neq i} \alpha_j f_j\}$, and $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ is a chain of minimal length starting in L, it follows that for each $1 \leq i < n$, $b_i \neq b_{i+1}$. In this case, we can track the changes in the partition as vectors are moved among the sets in a straightforward manner.

Definition 3.13. If $\mathcal{F}_{I_2}, \ldots, \mathcal{F}_{I_M}$ are linearly independent, then proceeding by induction, we can define

$$U_k^1 = I_k, \quad 1 \leq k \leq M,$$

and for $2 \leq i \leq n$,

$$\begin{aligned} U_k^i &= U_k^{i-1} \text{ for } k \neq b_{i-1}, k \neq b_i, \\ U_{b_i}^i &= U_{b_i}^{i-1} \cup \{a_{i-1}\}, \\ U_{b_{i-1}}^i &= U_{b_{i-1}}^{i-1} \setminus \{a_{i-1}\}. \end{aligned}$$

Lemma 3.14. *Let $\mathcal{F} = \{f_i : i \in I\}$ be a finite collection of vectors, and $\{I_1, I_2, \ldots I_M\}$ a partition with the property (MD) for which $\mathcal{F}_{I_2}, \ldots, \mathcal{F}_{I_M}$ are linearly independent. Let L be as above and assume that $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ is a minimal chain starting in L. For each $1 \leq i \leq n$, f_{a_i} can then be written as the sum*

$$f_{a_i} = \sum_{j \in I_{b_i}, j \notin \{a_p : 1 \leq p \leq n\}} \alpha_j f_j + \sum_{j \in U_{b_i}^i \cap \{a_p : 1 \leq p < i\}} \alpha_j f_j. \tag{3.2}$$

Proof. For the case $i = 1$, note that $a_1 \in L$ implies that $f_{a_1} = \sum_{j \in L, j \neq a_1} \alpha_j f_j$ for some choice of α_j. By Lemma 3.11 none of these $j \in L$ can be in $\{a_p : 1 \leq p \leq n\}$ since this would not be a chain of minimal length starting in L. Recalling that $b_i = 1$, the claim is proven for $i = 1$.

Proceeding by induction, let $i \in \{1, \ldots, n\}$ and we assume (3.2) is true for $1 \leq k < i$. We will show that it is also true for i. Note that

$$\begin{aligned} f_{a_i} &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \neq a_i} \alpha_j f_j \qquad (3.3) \\ &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i} \cap U_{b_i}^i, j \neq a_i} \alpha_j f_j + \sum_{j \in I_{b_i} \setminus U_{b_i}^i} \alpha_j f_j \\ &= \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i} \cap U_{b_i}^i, j \neq a_i} \alpha_j f_j + \sum_{j \in I_{b_i} \cap \{a_p : 1 \leq p < i-1\}} \alpha_j f_j, \qquad (3.4) \end{aligned}$$

where we have used in the last two lines that $I_{b_i} \cap \{a_p : 1 \leq p < i-1\} = I_{b_i} \setminus U_{b_i}^i$. Now, suppose for the sake of contradiction that there is a $j \in I_{b_i} \cap U_{b_i}^i$ such that $\alpha_j \neq 0$ and $j = a_p$ for some $p > i$. Then $\{(a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_p, b_i)\}$ is a chain starting in L, which contradicts the minimality of the chain $\{(a_1, b_1), \ldots, (a_n, b_n)\}$. So, using the induction hypothesis on each term in the last sum in (3.4) and combining terms, one obtains

$$f_{a_i} = \alpha f_{a_{i-1}} + \sum_{j \in I_{b_i}, j \notin \{a_p : 1 \leq p \leq n\}} \tilde{\alpha}_j f_j + \sum_{j \in U_{b_i}^i \cap \{a_p : 1 \leq p < i\}} \tilde{\alpha}_j f_j$$

with an appropriate choice of $\tilde{\alpha}_j$'s. □

Finally, we will identify the fundamental property needed from a chain of maximal length.

Lemma 3.15. *Let $\mathcal{F} = \{f_i : i \in I\}$ be a finite collection of vectors, and $\{I_j\}_{j=1}^M$ a partition of the index set I into $M \in \mathbb{N}$ non-empty sets which has the property (MD) and for which sets I_2, I_3, $\dots I_M$ index linearly independent sets. Moreover, let $L = \{i \in I_1 : f_i = \sum_{j\in I_1, j\neq i} \alpha_j f_j\}$, $L_0 = \{i \in I :$ there is a chain starting in L and ending at $i\}$, and $L_j = L_0 \cap I_j$ for $1 \leq j \leq M$. If $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a chain of minimal length starting in L and ending at a_n, then $f_{a_n} \in \operatorname{span}(\mathcal{F}_{L_m})$ for all $1 \leq m \leq M$.*

Proof. We show that, if $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a chain of minimal length starting in L and ending at a_n, then $f_{a_n} \in \operatorname{span}(\mathcal{F}_{L_m})$ for each $1 \leq m \leq M$.

For $n = 1$, fix $m \in \{1, \dots, M\}$, and observe that $a_1 \in L$. Hence, by Lemma 3.9, we can write $f_{a_1} = \sum_{l\in I_m} \alpha_l f_l$. For each l such that $\alpha_l \neq 0$, $(a_1, 1), (l, m)$ is a chain ending at l. Therefore, $f_{a_1} \in \operatorname{span}(\mathcal{F}_{L_m})$, as desired.

By Lemma 3.14 and the fact that $I_{b_i} \setminus \{a_p : 1 \leq p \leq n\} \subset U^k_{b_i}$ for all $1 \leq k \leq n$, we have that $f_{a_i} \in \operatorname{span}(\mathcal{F}_{U^i_{b_i}\setminus\{a_i\}})$. Therefore, $\dim\operatorname{span}(\mathcal{F}_{U^i_{b_i}}) = \dim\operatorname{span}(\mathcal{F}_{U^{i+1}_{b_i}})$. In particular, the partition $\{U^i_k : 1 \leq k \leq M\}$ satisfies property (MD).

By property (MD), Lemma 3.14, and Lemma 3.9, $f_{a_n} \in \operatorname{span}(\mathcal{F}_{U^n_m})$ for each $1 \leq m \leq M$. Therefore, for $m \neq b_n$, there exist $\alpha^{(0)}_j$ such that

$$\begin{aligned} f_{a_n} &= \sum_{j\in U^n_m} \alpha^{(0)}_j f_j = \sum_{j\in U^n_m \cap I_m} \alpha^{(0)}_j f_j + \sum_{j\in U^n_m \setminus I_m} \alpha^{(0)}_j f_j \\ &= \sum_{j\in U^n_m \cap I_m} \alpha^{(0)}_j f_j + \sum_{j\in\{a_p : b_{p+1}=m, 1\leq p<n-1\}} \alpha^{(0)}_j f_j. \end{aligned} \tag{3.5}$$

By definition of a chain, for each a_p such that $b_{p+1} = m$ and $1 \leq p < n - 1$,

$$f_{a_p} = \alpha^p f_{a_{p+1}} + \sum_{j\in I_m, j\neq a_{p+1}} \alpha^{(p)}_j f_j, \tag{3.6}$$

for some choice of $\alpha^{(p)}_j$ and some $\alpha^{(p)} \neq 0$.

Fix j_0 such that $\alpha^{(0)}_{j_0} \neq 0$ in (3.5). We show that $j_0 \in L_m$, which finishes the proof of the lemma. Clearly, if $j_0 \in \{a_1, \dots, a_n\}$, then we are done, so we assume that $j_0 \notin \{a_1, \dots, a_n\}$.

Case 1: There is some $1 \leq p < n - 1$ such that $b_{p+1} = m$ and $\alpha^{(p)}_{j_0} \neq 0$. Then, one can solve (3.6) for f_{j_0} to obtain

$$f_{j_0} = \beta f_{a_p} + \sum_{j\in I_m, j\neq j_0, j\neq a_p} \beta_j f_j$$

for some $\beta \neq 0$. Hence, $(a_1, b_1), \dots, (a_p, b_p), (j_0, m)$ is a chain and $j_0 \in L_m$.

Case 2: For each $1 \le p < n-1$ such that $b_{p+1} = m$, we have $\alpha_{j_0}^{(p)} = 0$. We have

$$\begin{aligned} f_{a_n} &= \sum_{j \in U_m^n \cap I_m} \alpha_j^{(0)} f_j + \sum_{j \in \{a_p : b_{p+1} = m, 1 \le p < n-1\}} \alpha_j^{(0)} f_j \\ &= \sum_{j \in U_m^n \cap I_m} \alpha_j^{(0)} f_j + \sum_{p \in \{p : b_{p+1} = m, 1 \le p < n-1\}} \alpha_{a_p}^{(0)} f_{a_p} \\ &= \sum_{j \in U_m^n \cap I_m} \alpha_j^0 f_j + \sum_{p \in \{p : b_{p+1} = m, 1 \le p < n-1\}} \alpha_{a_p}^{(0)} \big(\alpha^{(p)} f_{a_{p+1}} + \sum_{j \in I_m, j \ne a_{p+1}} \alpha_j^{(p)} f_j\big) \\ &= \alpha_{j_0}^{(0)} f_{j_0} + \sum_{j \in I_m, j \ne j_0} \tilde{\alpha}_j f_j, \end{aligned}$$

where the first equality is (3.5), the second equality is a re-indexing, the third equality follows from (3.6), and the last equality holds for some choice of $\tilde{\alpha}_j$ by combining sums, since $\alpha_{j_0}^{(p)} = 0$ for all $1 \le p < n-1$ such that $b_{p+1} = m$, and $j_0 \notin \{a_1, \ldots, a_n\}$. Therefore, $\{(a_1, b_1), \ldots, (a_n, b_n), (j_0, m)\}$ is a chain and $j_0 \in L_m$. □

Now we can prove the main result of this subsection.

Proof of Theorem 3.7:

We choose the partition $\{I_i\}_{i=1}^{r+1}$ of I that maximizes dim span $\{f_j\}_{j \in I_1}$ taken over all partitions so that the last r sets span X. If $\{J_i\}_{i=1}^{r+1}$ is a partition of I such that for all $1 \le i \le r+1$, dim span $\{f_j\}_{j \in J_i} \ge$ dim span $\{f_j\}_{j \in I_i}$, then dim span $\{f_j\}_{j \in I_i} =$ dim span $\{f_j\}_{j \in J_i}$ for all $i = 2, \ldots, r+1$ since dim span $\{f_j\}_{j \in I_i} = \dim X$, and $\dim \operatorname{span}\{f_j\}_{j \in I_1} = \dim \operatorname{span}\{f_j\}_{j \in J_1}$ by construction. This means, the chosen partition has the property (MD) and the properties asserted by Lemma 3.15. Suppose that this does not partition $\mathcal{F}_I$ into linearly independent sets, i.e. $\mathcal{F}_{I_1}$ is not linearly independent. As in Lemma 3.15, let $L = \{i \in I_1 : f_i = \sum_{j \in I_1, j \ne i} \alpha_j f_j\}$ be the index set of the "linearly dependent vectors" in I_1, $L_0 = \{i \in I :$ there is a chain starting in L ending at $i\}$, and $L_j = L_0 \cap I_j, 1 \le j \le r+1$.

Let $S = \operatorname{span}(\mathcal{F}_{L_0})$. By Lemma 3.15, $S = \operatorname{span}(\mathcal{F}_{L_j})$ for all $1 \le j \le r+1$. Moreover, for $1 \le j \le r+1$, $i \in L_j$ implies that $i \in I_j$ and $f_i \in S$. Therefore,

$$S \subset \operatorname{span}\{f_i : i \in L_j\} \subset \operatorname{span}\{f_i : i \in I_j, f_i \in S\} = S.$$

Let $J = \{i \in I : f_i \in S\}$. By construction, $L \subset J$. Let $d = \dim(S)$ and see that, by the preceding portion of this proof, $\dim \operatorname{span}(\mathcal{F}_J) = d$. Moreover,

$$|J| = |L_1| + \cdots + |L_M| = |L_1| + rd > d(r+1),$$

because L_1 is linearly dependent, since it contains L by virtue of chains of length one. Therefore, for $J = \{i \in I : f_i \in S\}$, $\frac{|J|}{\dim \operatorname{span}(\mathcal{F}_J)} > r+1$. This is in contradiction with assumption (1), which implies by the Rado Horn theorem that $|J|/d \le r+1$, and completes the proof of the theorem.

Acknowledgments. The authors would like to thank the American Institute of Mathematics for hospitality and the support of the Structured Quartet Research Ensemble on the Kadison Singer Problem. A special thanks to the referee for detailed and thoughtful suggestions which significantly improved the presentation of this paper.

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