SPECTRAL TETRIS FUSION FRAME CONSTRUCTIONS

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Abstract. Spectral tetris is a flexible and elementary method to construct unit norm frames with a given frame operator having all of its eigenvalues greater than or equal to two. One important application of this method is the construction of fusion frames. We provide a sufficient condition for using spectral tetris to construct a fusion frame with prescribed eigenvalues for its fusion frame operator and with prescribed dimensions for its subspaces. This condition is shown to be also necessary in the tight case. We then generalize spectral tetris to construct unit norm tight frames of redundancy less than two and use it to derive non-equidimensional tight fusion frames.

1. Introduction

A fusion frame is a sequence of subspaces of a Hilbert space along with a sequence of weights so that the sequence of weighted orthogonal projections onto these subspaces sums to an invertible operator on the space. Fusion frames—introduced in [5] and refined in [7]—have become a subject of interest in the field due to their applicability to problems in distributed processing, sensor networks and a host of other directions. Fusion frames provide resilience to noise and erasures due to, for instance, sensor failures or buffer overflows [1, 6, 8, 10], as well as robustness to subspace perturbations [7] which can happen because of imprecise knowledge of sensor network topology. For fusion frame applications, we generally need extra structure on the fusion frame such as prescribing the fusion frame operator or the dimensions of the subspaces—or both.

In this paper we address the question of how to efficiently construct fusion frames with prescribed dimensions of the subspaces and prescribed eigenvalues of the fusion frame operator. After reviewing some preliminaries, we in Section 3 will characterize for which sequences of eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2,\infty)$ and dimensions we can use the elementary spectral tetris method to construct a fusion frame having those eigenvalues for its fusion frame operator and having those dimensions for its subspaces. In Section 4 we extend spectral tetris to construct unit norm tight frames of redundancy smaller then 2, i.e. having all eigenvalues of the frame operator equal to $\lambda \in [1,2)$. We then use this construction to derive non-equidimensional tight fusion frames with all eigenvalues of the fusion frame operator equal to $\lambda \in [1,2)$,

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provided the dimension of the subspaces is bounded by some constant dependent on the dimension of the ambient space and the sum of the dimensions of the subspaces.

2. Preliminaries and notation

2.1. **Fusion frames.** The *synthesis operator* of a finite sequence $\{f_m\}_{m=1}^M \subseteq \mathbb{C}^N$ is $F\colon \mathbb{C}^M \to \mathbb{C}^N$ given by

$$Fg = \sum_{m=1}^M g(m) f_m,$$

i.e. F is the $N \times M$ matrix whose mth column is f_m. The sequence $\{f_m\}_{m=1}^M$ is a *frame* if its *frame operator* $S = FF^*$ satisfies $AI \leq S \leq BI$ for some positive constants A, B where I is the identity on $\mathbb{C}^N$. It is a *tight frame* if $A = B$, i.e. if

$$Af = \sum_{m=1}^M \langle f, f_m \rangle f_m$$

for all $f \in \mathbb{C}^N$ or equivalently if

$$\sum_{m=1}^M f_m(n)\overline{f_m(n')} = \begin{cases} A, & n = n', \\ 0, & n \neq n'. \end{cases}$$

In the case of a tight frame, the constant A equals M/N and is also called the *tight frame bound* or the *redundancy* of the frame. A *unit norm tight frame* is a tight frame $\{f_m\}_{m=1}^M$ for which $\|f_m\| = 1$ for all $m = 1, \dots, M$. Unit norm tight frames provide Parseval-like decompositions in terms of nonorthogonal vectors of unit norm. If $\{f_m\}_{m=1}^M$ is unit norm, the operators $f \mapsto \langle f, f_m \rangle f_m$ arising in the frame operator

$$Sf = \sum_{m=1}^M \langle f, f_m \rangle f_m$$

are rank-one orthogonal projections. Fusion frame theory is the study of sums of projections with weights and of arbitrary rank. In particular, a sequence $\{W_k, v_k\}_{k=1}^K$ of subspaces $\{W_k\}_{k=1}^K$ of $\mathbb{C}^N$ and weights $\{v_k\}_{k=1}^K$ is a *fusion frame* if the sequence $\{P_k\}_{k=1}^K$ of orthogonal projections onto those subspaces satisfies

$$AI \leq \sum_{k=1}^K v_k^2 P_k \leq BI$$

for some positive constants A, B. It is a *tight fusion frame* if $A = B$. Using the Horn-Klyachko compatibility inequalities [9] gives a characterization of the sequences of weights and dimensions of the subspaces for which tight fusion frames exist. In this paper we restrict ourselves to the case where all weights are equal to one and denote the fusion frame by $\{W_k\}_{k=1}^K$. In this case, the *fusion frame operator* is $S = \sum_{k=1}^K P_k$. If $\{f_{k,d}\}_{d=1}^{D_k}$ is an orthonormal basis of the range of P_k then

$$Sf = \sum_{k=1}^K P_k f = \sum_{k=1}^K \sum_{d=1}^{D_k} \langle f, f_{k,d} \rangle f_{k,d}$$

for all $f \in \mathbb{C}^N$. This shows that every fusion frame arises from a traditional frame that satisfies additional orthogonality requirements. To be precise, we say that a sequence $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k} \subseteq \mathbb{C}^N$ *generates* a fusion frame $\{W_k\}_{k=1}^K$ with $\dim W_k = D_k$ for $k = 1\ldots, K$ if $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k}$ is a frame for $\mathbb{C}^N$ and $\{f_{k,d}\}_{d=1}^{D_k}$ is orthonormal for $k = 1\ldots, K$.

2.2. **Spectral Tetris.** The term *spectral tetris* refers to a systematic method for constructing unit norm tight frames. This construction was introduced in [4] to generate unit norm tight frames in $\mathbb{R}^N$ for any dimension N and any number of frame vectors M provided that $M \geq 2N$. Choosing all weights to equal 1, [4] provides a complete characterization of triples (N, K, d) for which tight fusion frames of K subspaces of equal dimension d exist in $\mathbb{R}^N$ and gives an elegant algorithm to produce such tight fusion frames for most of the triples (N, K, d).

A straightforward extension to the construction of unit norm frames having a desired frame operator with eigenvalues $(\lambda_n)_{n=1}^N \in [2,\infty)$ satisfying $\sum_{n=1}^N \lambda_n = M$ was introduced in [2]. For convenience we review this construction in Table 2.2 and will refer to it as the *spectral tetris construction* (STC). We strongly recommend a glance at [4] or [2] for instructive examples on how the algorithm constructs the desired synthesis matrices one vector at a time. In STC, as in the rest of this paper, $\{e_n\}_{n=1}^N$ denotes the sequence of standard unit vectors of $\mathbb{C}^N$. A construction for equi-dimensional fusion frames having eigenvalues as above for their fusion frame operators is given in [2]. The sufficient condition for this construction to work is that the dimension d of the subspaces satisfies $\sum_{n=1}^N \lambda_n = dK$ where K is the number of subspaces and that the sequence of eigenvalues is bounded by $K - 3$.

In this paper, we use spectral tetris to construct fusion frames with given fusion frame operators and subspaces of not necessarily equal dimensions. We will say that a (fusion) frame has certain eigenvalues if its (fusion) frame operator has these eigenvalues.

Definition 2.1. *A frame constructed via the spectral tetris construction STC of Table 1 is called a spectral tetris frame. A fusion frame $\{W_k\}_{k=1}^K$ is called a spectral tetris fusion frame if there is a partition of a spectral tetris frame $\{f_{k,d}\}_{k=1,d=1}^{K\ \ D_k}$ such that $\{f_{k,d}\}_{d=1}^{D_k}$ is an orthonormal basis for W_k for every $k = 1, \ldots, K$.*

Aside from the fact that spectral tetris frames are easy to construct, their major advantage for applications is the sparsity of their synthesis matrices. This sparsity is dependent on the ordering of the given sequence of eigenvalues for which STC is performed. Note that the original form of the algorithm in [2] assumes the sequence of eigenvalues to be in decreasing order. This assumption, however, was made only for classification reasons, and it is easily seen that it can be dropped. The sparsest synthesis matrices are achieved if the sequence of eigenvalues $(\lambda_n)_{n=1}^N$ is ordered *blockwise*, i.e. if for any permutation π of $\{1, \ldots, N\}$ the set of partial sums $\{\sum_{j=1}^s \lambda_j \colon s = 1, \ldots, N\}$ contains at least as many integers as the set $\{\sum_{j=1}^s \lambda_{\pi(j)} \colon s = 1, \ldots, N\}$. It has been shown in [3] that spectral tetris frames are optimally sparse in the sense that given $M \geq 2N$ and a sequence of eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2,\infty)$, the synthesis matrix of the spectral tetris frame having these parameters is sparsest in the class of all synthesis matrices of unit norm frames that have these parameters, provided STC is run for the sequence $(\lambda_n)_{n=1}^N$ rearranged to be ordered blockwise. Note that for unit norm

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STC: Spectral Tetris Construction

Parameters:
    • Dimension $N \in \mathbb{N}$.
    • Number of frame elements $M \in \mathbb{N}$.
    • Eigenvalues $(\lambda_n)_{n=1}^N \geq 2$ such that $\sum_{n=1}^N \lambda_n = M$.
Algorithm:
    Set $k := 1$.
    For $j = 1, \ldots, N$ do
        Repeat
            If $\lambda_j < 1$ then
                $f_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
                $f_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
                $k := k + 2$.
                $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
                $\lambda_j := 0$.
            else
                $f_k := e_j$.
                $k := k + 1$.
                $\lambda_j := \lambda_j - 1$.
            end.
        until $\lambda_j = 0$.
    end.
Output:
    • Frame $\{f_k\}_{k=1}^M$.
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Table 1. The STC algorithm for constructing a frame with a desired frame operator.

tight frames all eigenvalues are equal, and as such questions of rearranging the order of the eigenvalues do not arise.

3. FUSION FRAMES WITH PRESCRIBED EIGENVALUES ≥ 2 AND PRESCRIBED DIMENSIONS

Let $M \geq N$ be natural numbers and $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$ such that $\sum_{n=1}^N \lambda_n = M$. Given a sequence of dimensions we ask the question of whether and how we can find a spectral tetris fusion frame for $\mathbb{R}^N$ whose subspaces have those prescribed dimensions and whose fusion frame operator has the eigenvalues $(\lambda_n)_{n=1}^N$.

To get started, consider the following example of integer eigenvalues $(\lambda_n)_{n=1}^6 = (4, 4, 3, 3, 2, 2)$. Given this sequence of eigenvalues, the spectral tetris frame in $\mathbb{C}^7$ consists only of standard unit vectors:

$$S = \{e_1, e_1, e_1, e_1, e_2, e_2, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_5, e_5, e_6, e_6\}.$$

The question we are asking above now takes the following form. We want to partition S into sets of pairwise orthonormal vectors, i.e. each set of the partition should not have more than one copy of any standard unit vector. What sizes can these sets have? We start by considering the partition $S = \bigcup_{n=1}^{4} P_n$, where

$$
\begin{aligned}
P_1 &= \{e_1, e_2, e_3, e_4, e_5, e_6\}, \\
P_2 &= \{e_1, e_2, e_3, e_4, e_5, e_6\}, \\
P_3 &= \{e_1, e_2, e_3, e_4\}, \\
P_4 &= \{e_1, e_2\}.
\end{aligned}
$$

The sets of this partition have sizes $6, 6, 4$ and 2. To get a different partition we cannot take any vector out of P_i and put it into P_j if $i > j$ as this would destroy the orthonormality of the sets. We can on the other hand take certain vectors out of P_i and put them into P_j if $i < j$ without destroying the orthonormality of the sets. By doing so, we can for example easily find a partition into orthonormal sets of sizes $6, 5, 4$ and 3. But, it is not possible to find a partition into orthonormal sets of sizes $6, 6, 5$ and 1. The sequence $6, 6, 4, 2$ majorizes the sequences of sizes of orthonormal sets which we can partition S into. Let us recall the notion of majorization. Given $a = (a_n)_{n=1}^N \in \mathbb{R}^N$, denote by $a^{\downarrow} \in \mathbb{R}^N$ the vector obtained by rearranging the coordinates of a in decreasing order. If $(a_n)_{n=1}^N, (b_n)_{n=1}^N \in \mathbb{R}^N$, we say $(a_n)_{n=1}^N$ *majorizes* $(b_n)_{n=1}^N$, denoted by $(a_n) \succeq (b_n)$, if $\sum_{n=1}^m a_n^{\downarrow} \geq \sum_{n=1}^m b_n^{\downarrow}$ for all $m = 1, \ldots, N-1$ and $\sum_{n=1}^N a_n = \sum_{n=1}^N b_n$. We will also use the notion of majorization between tuples of different length, by agreeing to add zero entries to the shorter tuple, in order to have tuples of the same length.

We can use the idea of the above example to construct spectral tetris fusion frames in the general case of real eigenvalues as the spectral tetris frames for real eigenvalues consist only of standard unit vectors or linear combinations of two successive standard unit vectors. As above, we will determine a sequence of numbers depending on the given eigenvalues $(\lambda_n)_{n=1}^N$ and check whether or not this sequence majorizes the given sequence of dimensions. Again, the sequence we are going to determine will be the sequence of dimensions of a certain fusion frame for $\mathbb{R}^N$ having the eigenvalues $(\lambda_n)_{n=1}^N$. We now introduce this fusion frame.

Definition 3.1. *Let $M \geq N$ be natural numbers and let $(\lambda_n)_{n=1}^N \subseteq [2, \infty)$ have the property that $\sum_{n=1}^N \lambda_n = M$. The fusion frame constructed by the algorithm RFF presented in Table 2 is called the reference fusion frame for the eigenvalues $(\lambda_n)_{n=1}^N$.*

An example of RFF is included in the remarks following the proof of Proposition 3.2. The vectors in the spanning sets $(S_n)_{n=1}^t$ that RFF produces are either standard unit vectors e_i for some $i = 1, \ldots, N$, which we will call singletons, or linear combinations of two consecutive standard unit vectors, i.e. $ae_i + be_{i+1}$ for some $a, b \in \mathbb{R}$ and some $i = 1, \ldots, N-1$. The vectors of the form $ae_i + be_{i+1}$ we will call doubletons starting at i. If for some $i = 1, \ldots, N-1$ and $a, b \in \mathbb{R}$ we have $e_i \in S_{n_1}$ and $ae_i + be_{i+1} \in S_{n_2}$, then $n_1 < n_2$. We will refer to this property of the spanning sets of the reference fusion frame by saying that RFF "picks singletons first."

We will now use the reference fusion frame for $(\lambda_n)_{n=1}^N$ to decide whether or not a fusion frame for $\mathbb{R}^N$ with a certain fusion frame operator and certain dimensions of the subspaces is

RFF: REFERENCE FUSION FRAME

Parameters:

- Eigenvalues $(\lambda_n)_{n=1}^N \subseteq [2,\infty)$ such that $\sum_{n=1}^N \lambda_n = M \in \mathbb{N}$.

Algorithm:

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Run STC for $(\lambda_n)_{n=1}^N$ and get frame $(f_i)_{i=1}^M$.
$t = \max_{j=1,\dots,N} |\mathrm{supp}\,(f_i(j))_{i=1}^M|$
$S_i = \emptyset$ for $i = 1,\dots,t$
$k = 0$
Repeat
    $k = k + 1$
    $j = \min\{1 \le r \le t\colon \mathrm{supp}\, f_k \cap \mathrm{supp}\, f_s = \emptyset \quad \forall f_s \in S_r\}$
    $S_j = S_j \cup \{f_k\}$
until $k = M$.
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Output:

- Fusion frame $(V_i)_{i=1}^t$, where $V_i = \mathrm{span} S_i$.

TABLE 2. The RFF algorithm for constructing the reference fusion frame.

constructible via spectral tetris. In case it is constructible the proof describes an algorithm to construct it.

Proposition 3.2. *Let $M \ge N$ be natural numbers, $(\lambda_n)_{n=1}^N \in [2,\infty)^N$ and let $(d_i)_{i=1}^D \in \mathbb{N}^D$ such that $\sum_{n=1}^N \lambda_n = \sum_{n=1}^D d_n = M$. Let $(V_n)_{n=1}^t$ be the reference fusion frame for $(\lambda_n)_{n=1}^N$. If $(\dim V_n) \succeq (d_n)$, then there exists a spectral tetris fusion frame $(W_n)_{n=1}^D$ for $\mathbb{R}^N$ with dim $W_n = d_n$ for $n = 1,\dots,D$ and eigenvalues $(\lambda_n)_{n=1}^N$.*

Proof. We show how to iteratively construct the desired fusion frame $(W_n)_{n=1}^D$ in case the majorization condition holds. For $i = 1,\dots,t$ let $W_i^0 = S_i$, where t and S_i for $i = 1,\dots,t$ are given by RFF for $(\lambda_n)_{n=1}^N$. We add empty sets if necessary to obtain a collection $(W_i^0)_{i=1}^D$ of D sets. If $\sum_{i=1}^D ||W_i^0| - d_i| = 0$ then the sets $(W_i^0)_{i=1}^D$ span the desired fusion frame. Otherwise, starting from $(W_i^0)_{i=1}^D$ we will construct the spanning sets of the desired fusion frame. Let

$$m = \max\left\{j \le D\colon d_j \ne |W_j^0|\right\}.$$

Note that $\sum_{i=1}^m |W_i^0| = \sum_{i=1}^m d_i$ by the choice of m and $\sum_{i=1}^{m-1} |W_i^0| > \sum_{i=1}^{m-1} d_i$ by the majorization assumption. Therefore $d_m > |W_m^0|$ and there exists

$$k = \max\left\{j < m\colon |W_j^0| > d_j\right\}.$$

Notice that $|W_m^0| < d_m \leq d_k < |W_k^0|$ implies $|W_m^0| + 2 \leq |W_k^0|$. If there is some element $w \in W_k^0$ which has disjoint support from every element in W_m^0, define $\{W_i^1\}_{i=1}^D$ by

$$W_i^1 = \begin{cases} W_k^0 \setminus \{w\} & \text{if } i = k, \\ W_m^0 \cup \{w\} & \text{if } i = m, \\ W_i^0 & \text{else.} \end{cases} \tag{1}$$

Now suppose there is no such element in W_k^0. Pick any $w_1 \in W_k^0$. Next, choose all the elements from W_m^0 whose support intersects the support of w_1. Next, choose all elements from W_k^0 whose support intersects the support of some element chosen so far. Continue by choosing all elements from W_m^0 whose support intersects the support of some element chosen so far. Continue until you cannot choose an element anymore. Let U_1 be the set of the chosen elements. No element of U_1 has a support which intersects the support of any element of $(W_k^0 \cup W_m^0) \setminus U_1$. As $|W_m^0| + 2 \leq |W_k^0|$, there exists some $w_2 \in (W_k^0 \cup W_m^0) \setminus U_1$. Construct U_2 by the same procedure as above with $w_2 \in (W_k^0 \cup W_m^0) \setminus U_1$ instead of $w_1 \in W_k^0$. If there is some element $w_3 \in (W_k^0 \cup W_m^0) \setminus (U_1 \cup U_2)$, continue to construct U_3 in the above fashion. In this way, we construct sets $U_1, \ldots, U_r$, say, until we use up all the elements of $W_k^0 \cup W_m^0$. For $i = 1, \ldots, r$ the number of elements in U_i chosen from W_k^0 and W_m^0 differs by at most one, as the elements of W_k^0, respectively W_m^0, have pairwise disjoint supports of sizes 1 or 2. As $|W_m^0| + 2 \leq |W_k^0|$ there is a set U_j that contains one element more from W_k^0 than from W_m^0. Define $\{W_i^1\}_{i=1}^D$ by

$$W_i^1 = \begin{cases} (W_k^0 \cup U_j) \setminus (U_j \cap W_k^0) & \text{if } i = k, \\ (W_m^0 \cup U_j) \setminus (U_j \cap W_m^0) & \text{if } i = m, \\ W_i^0 & \text{else.} \end{cases} \tag{2}$$

In both of the above cases (1) and (2) we have defined $\{W_i^1\}_{i=1}^D$ such that

$$\sum_{i=1}^D ||W_i^1| - d_i| < \sum_{i=1}^D ||W_i^0| - d_i|.$$

Note that $\{W_i^1\}_{i=1}^D$ satisfies the majorization condition in the sense that $(|W_n^1|) \succeq (d_n)$. Thus if the sets of $\{W_i^1\}_{i=1}^D$ do not span the desired fusion frame, we can repeat the above procedure with $\{W_i^1\}_{i=1}^D$ instead of $\{W_i^0\}_{i=1}^D$ and get $\{W_i^2\}_{i=1}^D$ such that $\sum_{i=1}^D ||W_i^2| - d_i| < \sum_{i=1}^D ||W_i^1| - d_i|$. Continuing in this fashion we will, say after repeating the process l times, arrive at $\{W_i^l\}_{i=1}^D$ such that $\sum_{i=1}^D ||W_i^l| - d_i| = 0$, i.e. the sets of $\{W_i^l\}_{i=1}^D$ span the desired fusion frame $(W_n)_{n=1}^D$. ☐

Intuition suggests that for a given choice of eigenvalues, the dimensions derived from RFF for these eigenvalues in blockwise order majorize the dimensions derived from RFF for the same eigenvalues in non-blockwise ordering. We do not investigate in this direction because even different blockwise orderings of the eigenvalues will in general lead to different sequences of dimensions of the reference fusion frame as the following example shows. Given the eigenvalues $\frac{5}{2}, \frac{10}{3}, \frac{13}{6}$, STC produces the synthesis matrix

$$F = [f_1, \dots, f_8] = \begin{bmatrix} 1 & 1 & \sqrt{\frac{1}{4}} & \sqrt{\frac{1}{4}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{4}} & -\sqrt{\frac{3}{4}} & 1 & \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{12}} & -\sqrt{\frac{7}{12}} & 1 \end{bmatrix}$$

and thus the reference fusion frame for $(\frac{5}{2}, \frac{10}{3}, \frac{13}{6})$ is

$$V = (\operatorname{span}\{f_1, f_5, f_8\}, \operatorname{span}\{f_2, f_6\}, \operatorname{span}\{f_3\}, \operatorname{span}\{f_4\}, \operatorname{span}\{f_7\}).$$

Running RFF for a different blockwise ordering of the same eigenvalues, say $\frac{10}{3}, \frac{5}{2}, \frac{13}{6}$, yields the synthesis matrix

$$G = [g_1, \dots, g_8] = \begin{bmatrix} 1 & 1 & 1 & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{5}{6}} & -\sqrt{\frac{5}{6}} & \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{7}{12}} & -\sqrt{\frac{7}{12}} & 1 \end{bmatrix}$$

and thus the reference fusion frame for $(\frac{10}{3}, \frac{5}{2}, \frac{13}{6})$ is

$$W = (\operatorname{span}\{g_1, g_6\}, \operatorname{span}\{g_2, g_7\}, \operatorname{span}\{g_3, g_8\}, \operatorname{span}\{g_4\}, \operatorname{span}\{g_5\}).$$

Note that RFF is in both cases run for a blockwise ordering of the given eigenvalues, yet the sequences of dimensions of the subspaces of the reference fusion frames V and W are different. We now show that for constructing tight fusion frames the majorization condition is also sufficient as questions of how to order the eigenvalues no longer arise.

Theorem 3.3. *Let $M \geq 2N$ be natural numbers and $(d_n)_{n=1}^D \in \mathbb{N}^D$ such that $\sum_{n=1}^D d_n = M$. Let $(V_n)_{n=1}^t$ be the reference fusion frame for $(\lambda_n)_{n=1}^N = (\frac{M}{N}, \dots, \frac{M}{N})$. Then there exists a tight spectral tetris fusion frame $(W_n)_{n=1}^D$ for $\mathbb{R}^N$ with $\dim W_n = d_n$ for $n = 1, \dots, D$ if and only if $(\dim V_n) \succeq (d_n)$.*

Proof. By Proposition 3.2 it remains to show that the majorization condition is necessary. Let $(S_n)_{n=1}^t$ be the spanning sets of $(V_n)_{n=1}^t$ given by RFF and $(W_n)_{n=1}^D$ another fusion frame with spanning sets $(F_n)_{n=1}^D$ consisting of the frame vectors that spectral tetris produces for the unit norm $\frac{M}{N}$-tight frame. To prove the theorem it suffices to define a sequence of spanning sets of fusion frames $(S_n^j)_{n=1}^{D_j}$, $j = 1, \dots, r$ for some $r \in \mathbb{N}$ such that $(S_n^0)_{n=1}^{D_0} = (F_n)_{n=1}^D$, $(S_n^r)_{n=1}^{D_r} = (S_n)_{n=1}^t$ and $(|S_n^{j+1}|) \succeq (|S_n^j|)$ for all $j = 0, \dots, r-1$. We show how to construct such a sequence successively, i.e. we describe how to construct $(S_n^{j+1})_{n=1}^{D_{j+1}}$ out of $(S_n^j)_{n=1}^{D_j}$. So let $(S_n^0)_{n=1}^{D_0} = (F_n)_{n=1}^D$, fix j and suppose $(S_n^0)_{n=1}^{D_0}, \dots, (S_n^j)_{n=1}^{D_j}$ are already constructed. Let

$$n_0 = \min\{n \leq D_j \colon S_n^j \neq S_n\}.$$

If no such n_0 exists, then $(S_n^j)_{n=1}^{D_j} = (S_n)_{n=1}^t$ and we are done, i.e. $j = r$. So, suppose there is such n_0. Find the minimal $m_0 \in \{1, \dots, N\}$ for which S_{n_0} contains a vector supported at m_0 but $S_{n_0}^j$ contains no or a different vector supported at m_0 or at which S_{n_0} contains no vector supported at m_0 but $S_{n_0}^j$ does contain a vector supported at m_0. We will go through all possible cases which could have created this situation and describe how to construct (S_n^{j+1})

out of (S^j_n) such that $(|S^{j+1}_n|) \succeq (|S^j_n|)$ and $S^{j+1}_{n_0}$ and S_{n_0} either both contain no vector supported at m_0 or the same vector supported at m_0 so that by iterating this procedure we will after a finite number of steps arrive at some (S^r_n) which is identical to (S_n).

It is not possible that S_{n_0} contains no vector supported at m_0 but that $S^j_{n_0}$ does. Indeed, $S^j_{n_0}$ cannot contain a doubleton starting at $m_0 - 1$ by the minimality of m_0 and it cannot contain a doubleton starting at m_0 or the singleton e_{m_0}, since then by its very definition RFF would have also picked just the same doubleton or e_{m_0} respectively S_{n_0}.

Thus, the first case is that S_{n_0} contains e_{m_0} but $S^j_{n_0}$ does not contain e_{m_0}. Note that by the minimality of m_0 the reason for this cannot be that $S^j_{n_0}$ contains a doubleton starting at $m_0 - 1$. So, the first of two subcases which could have created this situation is that $S^j_{n_0}$ does not contain any vector supported at m_0. Then there must be some $k \geq n_0 + 1$ such that $e_{m_0} \in S^j_k$ and we define (S^{j+1}_n) by letting $S^{j+1}_{n_0} = S^j_{n_0} \cup \{e_{m_0}\}$, $S^{j+1}_k = S^j_k \setminus \{e_{m_0}\}$ and $S^{j+1}_n = S^j_n$ for $n \neq n_0, k$. The second subcase that could have created this situation is that $S^j_{n_0}$ contains some doubleton $ae_{m_0} + be_{m_0+1}$ starting at m_0. Again, identify the $k \geq n_0 + 1$ for which $e_{m_0} \in S^j_k$. Now, one of three things can happen. First, if S^j_k contains the singleton e_{m_0+1}, define (S^{j+1}_n) by letting $S^{j+1}_{n_0} = (S^j_{n_0} \setminus \{ae_{m_0} + be_{m_0+1}\}) \cup \{e_{m_0}, e_{m_0+1}\}$, $S^{j+1}_k = (S^j_k \setminus \{e_{m_0}, e_{m_0+1}\}) \cup \{ae_{m_0} + be_{m_0+1}\}$ and $S^{j+1}_n = S^j_n$ for $n \neq n_0, k$. (To simplify our notation for the rest of the proof we introduce the following terminology for how we just constructed (S^{j+1}_n) . We say we constructed (S^{j+1}_n) from (S^j_n) by switching $ae_{m_0} + be_{m_0+1}$ with e_{m_0} and e_{m_0+1} between $S^j_{n_0}$ and S^j_k.) Second, if S^j_k contains no vector supported at m_0, define (S^{j+1}_n) by switching $ae_{m_0} + be_{m_0+1}$ with e_{m_0} between $S^j_{n_0}$ and S^j_k. Third, S^j_k contains a doubleton starting at $m_0 + 1$. This third subcase again has three subcases. The first being that $S^j_{n_0}$ contains e_{m_0+2}. In this subcase, we construct (S^{j+1}_n) by switching $ae_{m_0} + be_{m_0+1}$ and e_{m_0+2} with e_{m_0} and the doubleton starting at $m_0 + 1$ between $S^j_{n_0}$ and S^j_k. The second being that $S^j_{n_0}$ contains no vector supported at $m_0 + 2$. In this subcase, we construct (S^{j+1}_n) by switching $ae_{m_0} + be_{m_0+1}$ with e_{m_0} and the doubleton starting at $m_0 + 1$ between $S^j_{n_0}$ and S^j_k. The third being that $S^j_{n_0}$ contains a doubleton starting at $m_0 + 2$. This third subcase will again have three subsubcases namely that S^j_k contains e_{m_0+3}, no vector supported at $m_0 + 3$ or a doubleton starting at $m_0 + 3$. In the first two of those subsubcases we can again define (S^{j+1}_n) by switching and the third subsubcase will create three new further subcases the first two of which will lead to (S^{j+1}_n) by switching and the third one of which will create three new subcases. So we can continue our argument successively considering three new subcases. Eventually, we must arrive at a point where we can define (S^{j+1}_n) by switching, as we only deal with a finite number of vectors.

The second case is that S_{n_0} contains some doubleton $ce_{m_0} + de_{m_0+1}$ starting at m_0 but $S^j_{n_0}$ does not. In this case it is not possible for $S^j_{n_0}$ to contain e_{m_0} as RFF picks singletons first and thus we would have $e_{m_0} \in S_{n_0}$ rather than a doubleton starting at m_0. It is also not possible for $S^j_{n_0}$ to contain a doubleton starting at $m_0 - 1$ by the minimality of m_0. Hence, $S^j_{n_0}$ contains no vector supported at m_0. Thus, the first of three subcases which could have created the case we are in is that $S^j_{n_0}$ does not contain any vector supported at $m_0 + 1$. Now, there must be some $k \geq n_0 + 1$ such that $ce_{m_0} + de_{m_0+1} \in S^j_k$ and we define (S^{j+1}_n) by letting $S^{j+1}_{n_0} = S^j_{n_0} \cup \{ce_{m_0} + de_{m_0+1}\}$, $S^{j+1}_k = S^j_k \setminus \{ce_{m_0} + de_{m_0+1}\}$ and $S^{j+1}_n = S^j_n$ for

$n \neq n_0, k$. The second subcase that could have created this situation is that $e_{m_0+1} \in S^j_{n_0}$. Again, there must be some $k \geq n_0 + 1$ such that $ce_{m_0} + de_{m_0+1} \in S^j_k$ and we define (S^{j+1}_n) by switching e_{m_0+1} with $ce_{m_0} + de_{m_0+1}$ between $S^j_{n_0}$ and S^j_k. The third subcase is that $S^j_{n_0}$ contains a doubleton starting at $m_0 + 1$. This subcase will have three new subcases in the same fashion as above, two of which lead to the definition on (S^{j+1}_n) by switching and the third of which again leads to three new subcases. So again we can continue our argument successively considering three new subcases and eventually we must arrive at a point where we can define (S^{j+1}_n) by switching, as we only deal with a finite number of vectors. ☐

In the trivial case of integer eigenvalues we can run STC whenever all given eigenvalues are ≥ 1 to make the following observation.

Corollary 3.4. *If $(\lambda_n)_{n=1}^N \in \mathbb{N}^N$ and $(d_i)_{i=1}^D \in \mathbb{N}^D$ such that $\sum_{n=1}^N \lambda_n = \sum_{n=1}^D d_n = M \in \mathbb{N}$, where $M \geq N$, then a spectral tetris fusion frame with eigenvalues $(\lambda_n)_{n=1}^N$ and dimensions $(d_n)_{n=1}^D$ exists if and only if $(a_n) \succeq (d_n)$, where $a_n = \max\{r \colon \lambda_r \geq n\}$ for $n = 1, \ldots, \max_{i=1,\ldots,N} \lambda_i$.*

Proof. Note that in this case RFF produces the output $t = \max_{i=1,\ldots,N} \lambda_i$ and that $\dim V_i = a_i$ for $i = 1, \ldots, t$. ☐

4. SPECTRAL TETRIS FOR UNIT NORM TIGHT FUSION FRAMES OF REDUNDANCY < 2

Given $J \in \mathbb{N}$, let $\omega = \exp(\frac{2\pi i}{J})$. We define the discrete Fourier transform (DFT) matrix in $\mathbb{C}^{J\times J}$ by

$$D_J = \left(\omega^{ik}\right)_{i,k=0}^{J-1}.$$

The rows of D_J are pairwise orthogonal vectors. We do not normalize D_J by the factor $1/\sqrt{J}$, thus every entry of D_J has absolute value one.

The algorithm DFTST introduced in Table 3 is a variation of spectral tetris. It uses scaled $J \times J$ DFT-matrices to construct M-element unit norm tight frames for $\mathbb{C}^N$ in the case $N < M < 2N$. Here, J is the uniquely determined natural number for which $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$. Note that this implies $J \geq 3$. Before stating our main Theorem 4.2 about DFTST we will demonstrate in the following example what DFTST essentially does.

Example 4.1. *Let $N = 4$ and $M = 7$. Then $J = 3$, and DFTST will construct the synthesis matrix of a 7-element unit norm tight frame in $\mathbb{C}^4$ by using scaled copies of the DFT matrix D_3. We start to build the synthesis matrix from the upper left corner. In Lines 3) to 7) of the algorithm we determine whether there still have to be built 3 more columns, which is the case. Thus $K = 3$, $\omega = \exp(\frac{2\pi i}{3})$ and we start by building the first 3 columns using Lines 9) until 21) of the algorithm. Line 10) tells us to start by putting the first row of D_3, scaled so that this row square sums to the desired tight frame bound $7/4$.*

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

DFT SPECTRAL TETRIS (DFTST)

Parameters:

- Dimension $N \in \mathbb{N}$.
- Number of frame elements $M \in \mathbb{N}$, where $N < M < 2N$.
- $J \in \mathbb{N}$ such that $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$.

Algorithm:

```
$m = 0, n = 1, \lambda = 0$
Repeat
    If $J < M - m$ then
        $K = J$
    else
        $K = M - m$
    end.
    $w = \exp(\frac{2\pi i}{K})$
    For $r = m + 1, \ldots, m + K$ do
        $f_r = \sqrt{\frac{M - N\lambda}{NK}} \cdot e_n$
        Repeat
            $k = 1$
            $n = n + 1$
            If $1 - \|f_r\|^2 \geq \frac{M}{NK}$ then
                $f_r = f_r + \sqrt{\frac{M}{NK}} \cdot \omega^{k(r-m-1)} \cdot e_n$
            else
                $f_r = f_r + \sqrt{1 - \|f_r\|^2} \cdot \omega^{k(r-m-1)} \cdot e_n$
            end.
            $k = k + 1$
        until $\|f_r\| = 1$.
    end.
    $m = m + K$
    $\lambda = \sum_{r=1}^{m+K} |f_r(n)|^2$
until $m = M$.
```

Output:

- Unit norm tight frame $\{f_i\}_{i=1}^{M} \in \mathbb{C}^N$.

TABLE 3. The DFT Spectral Tetris algorithm for constructing a unit norm tight frame of redundancy < 2.

Next, in the inner repeat loop of the algorithm, we check whether we can put the second row of D_3*, scaled to square sum to* $7/4$*, below the first row whithout exceeding the column norm of* 1 *for the desired synthesis matrix. Here this is not the case and so we scale the second* *row of D_3 such that the column norm of the first 3 columns of the synthesis matrix is 1 when setting the remaining entries of the first 3 columns to be zero. Note that scaling the rows of D_3 does not affect their orthogonality and that at this point the second row of the synthesis matrix square sums to less than the desired $7/4$.*

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \cdot & \cdot & \cdot & \cdot \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Having constructed the first three columns, we run the outer repeat loop of the algorithm again, checking first whether we have 3 more columns to be constructed, which again is the case. The first row of the synthesis matrix already square sums to $7/4$ and therefore must have zeros as the remaining entries.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Line $10)$ of the algorithm tells us to scale the first row of D_3 so that, after putting it behind the last DFT row we used, this row of the synthesis matrix square sums to the desired $7/4$.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Lines $12)$ to $20)$ of the algorithm tell us again in which way to put scaled versions of the rows of D_3.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \cdot \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & \cdot \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & \cdot \end{bmatrix}.$$

Now rows 2 and 3 of the synthesis matrix are finished.

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & \cdot \end{bmatrix}.$$

We again check whether or not we have to construct 3 *more columns, which this time is not the case. Only one more column has to be constructed so we follow the above steps this time using* D_1 *instead of* D_3 *to obtain the synthesis matrix*

$$\begin{bmatrix} \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}}\cdot\omega & \sqrt{\frac{5}{12}}\cdot\omega^2 & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & \sqrt{\frac{2}{12}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{7}{12}} & \sqrt{\frac{7}{12}}\cdot\omega & \sqrt{\frac{7}{12}}\cdot\omega^2 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{12}} & \sqrt{\frac{3}{12}}\cdot\omega^2 & \sqrt{\frac{3}{12}}\cdot\omega^4 & 1 \end{bmatrix}.$$

We can think of the structure of the synthesis matrix constructed via DFTST as consisting of blocks which arise from DFT-matrices whose rows have been scaled by appropriate factors. If M, N and J are as in DFTST and J divides M, then it is built from $\frac{M}{J}$ blocks of size $J \times J$. If J does not divide M, the synthesis matrix is built of $\lfloor \frac{M}{J} \rfloor$ blocks of size $J \times J$ and one block of size $(M - \lfloor \frac{M}{J} \rfloor J) \times (M - \lfloor \frac{M}{J} \rfloor J)$ in the lower right corner.

Theorem 4.2. *Let* $M, N \in \mathbb{N}$ *such that* $N < M < 2N$. *Then DFTST constructs a unit norm tight frame* $(f_m)_{m=1}^M \in \mathbb{C}^N$ *with the property that* $\langle f_m, f_{m'} \rangle = 0$ *whenever* $m - m' \geq 2J$, *where* $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$.

Proof. We first show that the arguments of the square roots appearing in the algorithm are nonnegative and that during the course of the algorithm we always have $0 \leq \lambda \leq \frac{M}{N}$. For this, assume first that we are in the case where $K = J$ and that $0 \leq \lambda \leq \frac{M}{N}$ at the beginning of a certain run of the outer repeat loop of DFTST. The choice of J implies $J \geq 3$ and thus

$$0 \leq \frac{M - N\lambda}{NK} \leq \frac{2N - N\lambda}{NK} \leq \frac{(J+\lambda)N - N\lambda}{NK} = 1.$$

From this it follows that all the appearing square roots during this run of the outer repeat loop have nonnegative arguments and consequently that the updated λ at the end of this run of the outer repeat loop lies again in the interval $[0, \frac{M}{N}]$. Second, assume that we are in the case where $K = M - m$, i.e. the case of Line 5) and suppose at this moment the values of n and m are n_0 and m_0, respectively. By what we have just shown, we know that at the beginning of this last run of the outer repeat loop we have $0 \leq \lambda \leq \frac{M}{N}$, which shows that the square root in Line 10) has a nonnegative argument. The fact that the square root in Line 17) has a nonnegative argument follows from

$$m_0 = \sum_{m=1}^{m_0} \sum_{n=1}^{n_0} |f_m(n)|^2 = \sum_{n=1}^{n_0} \sum_{m=1}^{m_0} |f_m(n)|^2 = \sum_{n=1}^{n_0-1} \frac{M}{N} + \lambda = (n_0 - 1)\frac{M}{N} + \lambda,$$

i.e.

$$\lambda = m_0 - (n_0 - 1)\frac{M}{N},$$

as this implies

$$\frac{M - N\lambda}{N(M - m_0)} = \frac{n_0 M - N m_0}{N(M - m_0)} \leq \frac{N(M - m_0)}{N(M - m_0)} = 1.$$

We now show that while being in the case of Line 3) of DFTST, i.e. while $K = J$, that the maximum support s of the constructed columns is less than or equal to N. This ensures the inner repeat loop terminates before exceeding the dimension of the columns. Indeed, assume $s > N$, and let n_0 and m_0 be the values of n and m, respectively, at the beginning of the outer repeat loop that we are in. We have

$$\sum_{m=1}^{M}\sum_{n=1}^{N}|f_m(n)|^2 = \sum_{m=1}^{m_0}\sum_{n=1}^{N}|f_m(n)|^2 + \sum_{m=m_0+1}^{m_0+J}\sum_{n=1}^{N}|f_m(n)|^2 = \sum_{m=1}^{m_0}1 + \sum_{m=m_0+1}^{m_0+J}\sum_{n=1}^{N}|f_m(n)|^2$$
$$< m_0 + \sum_{m=m_0+1}^{m_0+J} 1 = m_0 + J < M$$

while interchanging the order of summation yields

$$\sum_{n=1}^{N}\sum_{m=1}^{M}|f_m(n)|^2 = \sum_{n=1}^{n_0-1}\sum_{m=1}^{M}|f_m(n)|^2 + \sum_{m=1}^{M}|f_m(n_0)|^2 + \sum_{n=n_0+1}^{N}\sum_{m=1}^{M}|f_m(n)|^2$$
$$= \sum_{n=1}^{n_0-1}\frac{M}{N} + \lambda + J\left(\frac{M-N\lambda}{NJ}\right) + \sum_{n=n_0+1}^{N} J\frac{M}{NJ} = \frac{NM}{N} = M,$$

a contradiction.

We next show that, still being in the case of Line 3), the inner repeat loop does not run more than $J-1$ times, i.e. it terminates before or exactly when we have used up all the rows of D_J. For this it suffices to show that

$$1 - \frac{M-N\lambda}{NJ} - (J-2)\frac{M}{NJ} \le \frac{M}{NJ}$$

and thus we have to show $(N-M)J + N\lambda < 0$. And indeed, using $\lambda < \frac{M}{N} \le \frac{J-1}{J-2}$ we have

$$\left(1-\frac{M}{N}\right)J + \lambda \le \left(1-\frac{M}{N}\right)J + \frac{M}{N} = J + \frac{M}{N}(1-J) \le J + \frac{(J-1)(1-J)}{J-2} < 0,$$

where we used the fact that $J \ge 3$.

Eventually DFTST will enter the case of Line 5) and this will be the last time the outer repeat loop of the algorithm runs. In this last run the final $M-m_0$ columns $f_{m_0+1},\dots,f_M$ of the synthesis matrix are constructed and we have to show that the maximum of the support of these vectors, which we denote by N_0, equals N, that their support has size of at most $M-m_0$ and that the last row of the synthesis matrix has norm M/N. We first show $N_0 = N$. Indeed, supposing $N_0 < N$ yields

$$M = \sum_{m=1}^{M}1 = \sum_{m=1}^{M}\sum_{n=1}^{N}|f_m(n)|^2 = \sum_{n=1}^{N}\sum_{m=1}^{M}|f_m(n)|^2 = N_0\frac{M}{N} < M,$$

a contradiction. Assuming $N_0 > N$ implies

$$M = N\frac{M}{N} = \sum_{n=1}^{N}\sum_{m=1}^{M}|f_m(n)|^2 = \sum_{m=1}^{M}\sum_{n=1}^{N}|f_m(n)|^2 < \sum_{m=1}^{M}1 = M,$$

again a contradiction. Next, we show that the last row of the constructed synthesis matrix has norm M/N. To see this, note that

$$\begin{aligned} M = \sum_{m=1}^{M} 1 = \sum_{m=1}^{M}\sum_{n=1}^{N} |f_m(n)|^2 = \sum_{n=1}^{N}\sum_{m=1}^{M} |f_m(n)|^2 \\ = \sum_{n=1}^{N-1}\sum_{m=1}^{M} |f_m(n)|^2 + \sum_{m=1}^{M} |f_m(N)|^2 = (N-1)\frac{M}{N} + \sum_{m=1}^{M} |f_m(N)|^2, \end{aligned}$$

and thus indeed

$$\sum_{m=1}^{M} |f_m(N)|^2 = \frac{M}{N}.$$

Finally, suppose the support of f_M, which coincides with the support of the other columns constructed in this run of the outer repeat loop, is of size $s > M - m_0$. Then

$$\sum_{n=1}^{N} |f_M(n)|^2 \geq s \cdot \frac{M}{N(M-m_0)} \geq \frac{M}{N} > 1$$

in contradiction to the fact that f_M was constructed to have norm 1.

Having shown the above about DFTST, it is now easy to check that the synthesis matrix it produces is that of a unit norm tight frame: It is clear by construction that each frame vector has norm one. The rows of the synthesis matrix are pairwise orthogonal since they are made of multiples of rows of DFT matrices, thus the frame operator of the constructed frame has zero entries off of its diagonal. The powers of ω coming up in the construction of the columns have absolute value 1. Therefore, rows of the synthesis matrix that are defined entirely within one run of the outer repeat loop have entries that square sum to $K \cdot \frac{M}{NK} = \frac{M}{N}$. The same holds for rows that are defined using two consecutive runs of the outer repeat loop. Their entries square sum to

$$\lambda + K \cdot \frac{M - N\lambda}{NK} = \frac{M}{N}.$$

Thus, the frame operator of the constructed frame is $\frac{M}{N}I$. To see the orthogonality property of the columns stated in the theorem we note the following. A column constructed in a certain run of the outer repeat loop has disjoint support from any column constructed in a run of the outer repeat loop that did not directly precede or follow it. In every run of the outer repeat loop, at most J vectors are constructed, which therefore yields the orthogonality statement of the theorem. □

We now show how to use the unit norm tight frames constructed via DFTST to derive non-equidimensional tight fusion frames. We can do this as long as the desired dimensions of the subspaces stay below a certain bound by making use of the fact that many of the frame vectors constructed via DFTST have disjoint support. We first need a technical lemma which will be used to shuffle the frame vectors.

Lemma 4.3. *Let $M, L \in \mathbb{N}$ such that $L < M$. Then there exists a permutation π of $\{1, \ldots, M\}$ such that for all $k, l \in \{1, \ldots, M\}$ with $0 < |k-l| \leq \lfloor \frac{M}{L} \rfloor - 1$ we have $|\pi(k) - \pi(l)| \geq L$.*

Proof. Let $[1], \ldots, [L]$ be the partition of $(1, \ldots, M)$ into cosets modulo L. These cosets have either $\lfloor \frac{M}{L} \rfloor$ or $\lfloor \frac{M}{L} \rfloor + 1$ elements. Write each coset as an increasing sequence, say $[s] = (a_{s,1}, a_{s,2}, \ldots, a_{s,n_s})$, and let

$$(r_1, \ldots, r_M) := (a_{L,1}, \ldots, a_{L,n_L}, a_{L-1,1}, \ldots, a_{L-1,n_{L-1}}, \ldots, a_{1,1}, \ldots, a_{1,n_1}).$$

Define π by $\pi(i) = r_i$ for $i = 1, \ldots, M$. Let k, l as in the assumption of the lemma and assume as we may that $k < l$. If $\pi(k)$ and $\pi(l)$ belong to the same coset modulo L, then $|\pi(k) - \pi(l)| \geq L$. If not, there exists $m \in \{1, \ldots, L-1\}$ such that $\pi(k) \in [m+1]$, say $\pi(k) = a_{m+1,j}$ and $\pi(l) \in [m]$, say $\pi(l) = a_{m,i}$. As the cosets have at least $\lfloor \frac{M}{L} \rfloor$ elements, we have $i < j$ and thus

$$|\pi(k) - \pi(l)| = a_{m+1,j} - a_{m,i} > a_{m+1,j} - a_{m+1,i} \geq L.$$

□

Corollary 4.4. *If $N < M < 2N$ and $J \in \mathbb{N}$ such that $\frac{J}{J-1} < \frac{M}{N} \leq \frac{J-1}{J-2}$, then for all $(k_i)_{i=1}^K \subset \mathbb{N}$ with $\sum_{i=1}^K k_i = M$ and $k_i \leq \lfloor \frac{M}{2J} \rfloor$ for $i = 1, \ldots, K$, there exists a tight fusion frame $(V_l)_{l=1}^K$ with $\dim V_l = k_l$ for $l = 1, \ldots, K$.*

Proof. Let $\{f_m\}_{m=1}^M$ be the unit norm tight frame constructed in Theorem 4.2 and consider $\{f_{\pi(m)}\}_{m=1}^M$ where π is the permutation of $\{1, \ldots, M\}$ given in Lemma 4.3 for $L = 2J$. Let $R = \lfloor \frac{M}{2J} \rfloor$. Then for any $m = 1, \ldots, M - R + 1$ the vectors of the set $\{f_{\pi(m+i)}\}_{i=0}^{R-1}$ are pairwise orthonormal by Theorem 4.2. Note that this set contains R elements. Thus we can define

$$V_l = \operatorname{span}\left\{ f_{\pi(i + \sum_{j=1}^{l-1} k_j)} : i = 1, \ldots, k_l \right\}$$

for $l = 1, \ldots, K$. □

The new tools introduced in this paper should have broad application to other construction problems for frames and fusion frames. We are currently exploring these possibilities.

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