SUMS OF HILBERT SPACE FRAMES

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Abstract. We give simple necessary and sufficient conditions on Bessel sequences $\{f_i\}$ and $\{g_i\}$ and operators L_1, L_2 on a Hilbert space $\mathbb{H}$ so that $\{L_1 f_i + L_2 g_i\}$ is a frame for $\mathbb{H}$. This allows us to construct a large number of new Hilbert space frames from existing frames.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [8] as a part of their research in non-harmonic Fourier series. Their work on frames was somewhat forgotten until 1986 when Daubechies, Grossmann and Meyer [13] brought it all back to life during their fundamental work on wavelets. Today, frame theory plays an important role not just in signal processing, but also in dozens of applied areas (See [1, 2]).

Holub [12] showed that if $\{x_n\}$ is any normalized basis for a Hilbert space $\mathbb{H}$ and $\{f_n\}$ is the associated dual basis of coefficient functionals, then the sequence $\{x_n + f_n\}$ is again a basis for $\mathbb{H}$.

In this paper we study cases in which new frames can be obtained from old ones. Throughout $\mathbb{H}$ denotes a separable Hilbert space and $\mathbb{H}_N$ is the Nth dimensional Hilbert space. A frame for $\mathbb{H}$ is a family of vectors $f_i \in \mathbb{H}$, for $i \in I$ for which there exist constants $A, B > 0$ satisfying:

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in \mathbb{H}$. A and B are called the **lower and upper frame bounds** respectively. If $A = B$, this is called an **A-tight frame**. And if $A = B = 1$, it is a **Parseval frame**. If we have just the upper inequality, we call $\{f_i\}$ a **B-Bessel sequence**.

If $\{f_i\}_{i\in I}$ is a B-Bessel sequence, we define its **analysis operator** as $T : \mathbb{H} \to \ell_2(I)$ by:

$$T(f) = \{\langle f, f_i\rangle\}_{i\in I}.$$

Evidently $\{f_i\}_i \in I$ is a frame if and only if T is invertible on $\mathbb{H}$.

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The adjoint of the analysis operator is the **synthesis operator** given by:

$$T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i.$$

If the analysis operator is bounded, we define the frame operator $S = T^*T$ and note that S is a positive, self-adjoint operator which is invertible on $\mathbb{H}$ if and only if $\{f_i\}$ is a frame for $\mathbb{H}$. If $\{f_i\}$ is a frame, then every $f \in H$ has a representation of the form

$$f = \sum_{i\in I} \langle f, S^{-1}f_i\rangle f_i = \sum_{i\in I} \langle f, f_i\rangle S^{-1}f_i = \sum_{i\in I} \langle f, S^{-1/2}f_i\rangle S^{-1/2}f_i., \tag{1.2}$$

For an introduction to frame theory we recommend [3, 6]. For an introduction to Gabor frames we recommend Gröchenig [9].

2. Beginnings

We want to observe that if we have any frame $\{f_i\}_{i\in I}$ for a Hilbert space $\mathbb{H}$ with frame bounds A, B and frame operator S, then for all real numbers a, $\{f_i + S^a f_i\}_{i\in I}$ is also a frame for $\mathbb{H}$ with frame operator $(I + S^a)^2 S$ and frame bounds

$$\|I + S^a\|^2 A, \quad \|I + S^a\|^2 B.$$

In particular, $\{f_i + Sf_i\}$, $\{f_i + S^{-1}f_i\}$ (i.e. The frame added to its cononical dual frame) and $\{f_i + S^{-1/2}f_i\}$ (i.e. The frame added to its canonical Parseval frame) are all frames for $\mathbb{H}$. We start with a well known result.

Proposition 2.1. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ with frame operator S, frame bounds $A \le B$ and let $L : \mathbb{H} \to \mathbb{H}$ be a bounded operator. Then $\{Lf_i\}_{i\in I}$ is a frame for $\mathbb{H}$ if and only if L is invertible on $\mathbb{H}$. Moreover, in this case the frame operator for $\{Lf_i\}$ is LSL^* and the new frame bounds are $\|L^{-1}\|^2 A$, $\|L\|^2 B$.*

Proof. If L is invertible on $\mathbb{H}$ then for each $f \in \mathbb{H}$,

$$\sum_{i\in I} |\langle f, Lf_i\rangle|^2 = \sum_{i\in I} |\langle L^*f, f_i\rangle|^2 \ge A\|L^*f\|^2 \ge \|L^{-1}\|^2 A\|f\|^2$$

and

$$\sum_{i\in I} |\langle f, Lf_i\rangle|^2 = \sum_{i\in I} |\langle L^*f, f_i\rangle|^2 \le B\|Lf\|^2 \le \|L\|^2 B\|f\|^2.$$

Thus, $\{Lf_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame bounds $\|L^{-1}\|^2 A$ and $\|L\|^2 B$

Conversely, if $\{Lf_i\}_{i\in I}$ is a frame for $\mathbb{H}$ then its frame operator is invertible on $\mathbb{H}$. But the frame operator of $\{Lf_i\}_{i\in I}$ is

$$\sum \langle f, Lf_i\rangle Lf_i = L\left(\sum \langle L^*f, f_i\rangle f_i\right) = LSL^*f.$$

which implies that L is invertible on $\mathbb{H}$. □

If K is an invertible operator on $\mathbb{H}$ then the ranges of the analysis operators for the given frame $\{f_i\}_{i\in I}$ and the frame $\{Kf_i\}_{i\in I}$ coincide.

Trying to add a frame $\{f_i\}$ to $\{Lf_i\}$ can be problematic in general since we could have $Lf_i = -f_i$. However, the following corollary of Proposition 2.1 shows that this is all that can really go wrong.

Corollary 2.2. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ and $L : \mathbb{H} \to \mathbb{H}$ is a bounded operator, then $\{f_i + Lf_i\}$ is a frame for $\mathbb{H}$ if and only if $I + L$ is invertible on $\mathbb{H}$. In this case, the frame operator for the new frame is $(I + L)S(I + L^*)$ and the frame bounds are*

$$\|I + L\|^{-2}A, \quad \|I + L\|^2 B.$$

In particular, if L is a positive operator (or just $L > -1$) then $\{f_i + Lf_i\}$ is a frame with frame operator $S + LS + SL^* + LSL^*$.

The above corollary shows that all of our earlier sums give new frames for $\mathbb{H}$.

Corollary 2.3. *If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$ and P is an orthogonal projection on $\mathbb{H}$, then for all $a \neq -1$ we have that $\{f_i + aPf_i\}_{i\in I}$ is a frame for $\mathbb{H}$.*

Proof. Apply Corollary 2.2 for $L = aP$. □

The reason we want to add frames together is to produce frames with better properties for particular applications. Let us look at a simple example. Recall that a frame is ϵ-**nearly Parseval**, where $0 < \epsilon < 1$, if its frame bounds A, B satisfy: $1 - \epsilon \le A \le B \le 1 + \epsilon$.

Example 2.4. *Let $\{f_m\}_{m=1}^M$ be an ϵ-nearly Parseval frame for $\mathbb{H}_N$ frame operator S. Let*

$$g_m = \frac{1}{2}(f_m + S^{-1}f_m), \quad \text{for all } m = 1, 2, \cdots, M.$$

Then $\{g_m\}_{m=1}^M$ is a frame with frame bounds $1, 1 + \frac{\epsilon}{4}$ which is "close" to $\{f_m\}$.

Proof. The frame operator for the frame $\{g_m\}$ is

$$S_0 = \left[\frac{1}{2}(I + S^{-1})\right]^2 S = \frac{1}{2}I + \frac{S + S^{-1}}{4}.$$

Let $\{e_n\}_{n=1}^N$ be an eigenbasis for S with respective eigenvalues $\{\lambda_n\}_{n=1}^N$. Then $\{e_n\}_{n=1}^N$ is an eigenbasis for S_0 with respective eigenvalues

$$\frac{1}{2} + \frac{1}{4}(\lambda_n + \lambda_n^{-1}) \ge 1.$$

Finally, we check how "close" the new frame is to the old frame.

$$\begin{aligned}
\sum_{m=1}^{M}\|f_m - g_m\|^2 &= \sum_{m=1}^{M}\|\frac{1}{2}(f_m - S^{-1}f_m)\|^2 \\
&= \sum_{m=1}^{M}\|\frac{1}{2}(I - S^{-1})f_m\|^2 \\
&= \sum_{m=1}^{M}\sum_{n=1}^{N}|\frac{1}{2}(1-\frac{1}{\lambda_n})|^2|\langle f_m, e_n\rangle|^2 \\
&= \sum_{n=1}^{N}|\frac{1}{2}(1-\frac{1}{\lambda_n})|^2\sum_{m=1}^{M}|\langle f_m e_n\rangle|^2 \\
&\le \left|\frac{1}{2}\frac{\epsilon}{1-\epsilon}\right|^2(1+\epsilon)^2 \\
&= \left(\frac{\epsilon}{2}\right)^2\left(\frac{1+\epsilon}{1-\epsilon}\right)^2.
\end{aligned}$$

□

The reason the above frame is interesting is that it is much closer to $\{f_i\}$ than the nearest Parseval frame which is $\{S^{-1/2}f_i\}$ [4] and its frame bounds are much better than the original.

3. Sums of Bessel Sequences

Now we want to show that a frame can be added to any of its alternate dual frames to yield a new frame. Recall, if $\{f_i\}_{i\in I}$ is a frame, the **canonical dual frame** is $\{S^{-1}f_i\}$ and satisfies the property that for all $f \in \mathbb{H}$, $f = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i$. A frame $\{g_i\}$ is called an **alternate dual frame** if for all $f \in \mathbb{H}$,

$$f = \sum_{i\in I}\langle f, f_i\rangle g_i.$$

We start by extending our earlier ideas.

Proposition 3.1. *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in $\mathbb{H}$ with analysis operators T_1, T_2 and frame operators S_1, S_2 respectively. Also let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$. The following are equivalent:*

(1) $\{L_1f_i + L_2g_i\}_{i\in I}$ is a frame for $\mathbb{H}$.
(2) $T_1L_1^* + T_2L_2^*$ is an invertible operator on $\mathbb{H}$.
(3) We have

$$S = L_1S_1L_1^* + L_2S_2L_2^* + L_1T_1^*T_2L_2^* + L_2T_2^*T_1L_1^* > 0.$$

Moreover, in this case, S is the frame operator for $\{L_1f_i + L_2g_i\}_{i\in I}$.

Proof. (1) $\Leftrightarrow$ (2): $\{L_1f_i+L_2g_i\}_{i\in I}$ is a frame if and only if its analysis operator T is invertible on $\mathbb{H}$ where

$$\begin{aligned} Tf &= \{\langle f, L_1f_i + L_2g_i\rangle\} \\ &= \{\langle L_1^*f, f_i\rangle + \langle L_2^*f, g_i\rangle\} \\ &= T_1L_1^*f + T_2L_2^*f. \end{aligned}$$

(2) $\Leftrightarrow$ (3): The frame operator for our family is

$$\begin{aligned} S &= (T_1L_1^* + T_2L_2^*)^*(T_1L_1^* + T_2L_2^*) \\ &= L_1S_1L_1^* + L_2S_2L_2^* + L_1T_1^*T_2L_2^* + L_2T_2^*T_1L_1^*. \end{aligned}$$

Our family of vectors is a frame if and only if $S > 0$. □

The following theorem enables one to get a frame from a combination of a known frame and a Bessel sequence.

Theorem 3.2. *Let $\{f_i\}_{i\in I}$ be a frame for a Hilbert space $\mathbb{H}$ with frame operator S_1 and let $\{g_i\}_{i\in I}$ be a Bessel sequence in $\mathbb{H}$ with frame operator S_2. Let T_1, T_2 be the analysis operators for $\{f_i\}_{i\in I}, \{g_i\}_{i\in I}$ respectively so that range$T_2 \subset$ rangeT_1. If the operator $R = T_1^*T_2$ is a positive operator, then $\{f_i + g_i\}_{i\in I}$ is a frame for $\mathbb{H}$ with frame operator $S_1 + R + R^* + S_2$.*

Proof. Let T_1, T_2 be the analysis operators for $\{f_i\}$, $\{g_i\}$ respectively. Letting $L_1 = I = L_2$ in Proposition 3.1 we see that the frame operator for $\{f_i + g_i\}_{i\in I}$ is

$$S_0 = S_1 + S_2 + T_1^*T_2 + T_2^*T_1 = S_1 + S_2 + R + R^*.$$

□

As an application of the theorem we have

Corollary 3.3. *If $\{f_i\}$ is a frame for $\mathbb{H}$ with frame operator S and $\{g_i\}$ is an alternate dual frame then $\{S^af_i + S^bg_i\}$ is a frame for $\mathbb{H}$ for all real numbers a, b.*

Proof. We let

$$L(f) = \sum_{i\in I}\langle f, S^bg_i\rangle S^af_i = S^{a+b}(f).$$

That is, $L \geq 0$. So $\{S^af_i + S^bg_i\}_{i\in I}$ is a frame by Theorem 3.2. □

We do not necessarily need $\{g_i\}$ to be an alternate dual frame above.

Theorem 3.4. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ such that*

$$\inf_{i\in I}\|f_i\| > 0 \tag{3.3}$$

and let S be the frame operator. If $\{g_i\}_{i\in I} \subseteq \mathbb{H}$ such that $f = \sum_{i\in I}\langle f, g_i\rangle f_i$ unconditionally for all $f \in \mathbb{H}$ then $\{S^a(f_i) + S^bg_i\}_{i\in I}$ is a frame for $\mathbb{H}$ for all real numbers a, b.

Proof. Since $\{f_i\}_{i\in I}$ is bounded, assuming

$$\sum_{i\in I}\langle f, g_i\rangle f_i,$$

converges unconditionally implies

$$\sum_{i\in I}|\langle f, g_i\rangle|^2 < \infty, \quad \text{for all } f\in \mathbb{H}.$$

By the Uniform Boundedness Principle, we have that $\{g_i\}_{i\in I}$ is a Bessel sequence. Since the assumption is that $R = I \geq 0$, we can apply Theorem 3.2 to conclude that $\{f_i + g_i\}_{i\in I}$ is a frame for $\mathbb{H}$. Also,

$$L(f) = \sum_{i\in I}\langle f, S^b g_i\rangle S^a f_i = S^{a+b}(f).$$

That is, $L \geq 0$. So $\{S^a f_i + S^b g_i\}_{i\in I}$ is a frame by Theorem 3.2. □

The assumption that the frame $\{f_i\}_{i\in I}$ satisfies 3.3 in Theorem 3.4 is necessary. For example, if $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathbb{H}$ let

$$f_{2i+1} = e_i, \quad f_{2i} = \frac{1}{i}e_i, \quad g_{2i} = ie_i, \quad g_{2i+1} = 0.$$

Then for all $f\in\mathbb{H}$,

$$\sum\langle f, g_i\rangle f_i = f,$$

but $\{f_i + g_i\}_{i\in I}$ is not a frame since it is not a Bessel system.

Also, the assumption that the convergence is unconditional in Theorem 3.4 is necessary. For example, let $\{h_i, h_i^*\}_{i\in I}$ be a Schauder basis for $\mathbb{H}$ which is a Bessel system but not a frame. Let $\{e_i\}_{i\in I_1}$ be an orthonormal basis for $\mathbb{H}$. Let

$$\{f_i\} = \{e_i\}_{i\in I}\cup\{h_i\}_{i\in I_1}, \quad \{g_i\} = \{0\}_{i\in I}\cup\{h_i^*\}_{i\in I_1}.$$

Then for all $f\in\mathbb{H}$,

$$\sum_{i\in I}\langle f, e_i\rangle g_i + \sum_{i\in I_1}\langle f, h_i\rangle g_i = 0 + f.$$

But $\{f_i + g_i\}$ is not a frame since it is not a Bessel system.

We can more carefully do "local addition" for our frames.

Proposition 3.5. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathbb{H}$ with frame operator S and frame bounds A and B. Let $\{I_1, I_2\}$ be a partition of I and let S_j be the frame operator for the Bessel sequences $\{f_i\}_{i\in I_j}$, $j = 1, 2$. Then*

$$\{f_i + S_1^a f_i\}_{i\in I_1}\cup\{f_i + S_2^b f_i\}_{i\in I_2},$$

is a frame for $\mathbb{H}$ for all real numbers a, b.

Proof. Let $a, b \in \mathbb{R}$. Note that, for each $f \in \mathbb{H}$

$$\begin{aligned}\left(\sum_{i\in I_1} |\langle f, f_i + S_1^a f_i\rangle|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i\in I_1} |\langle f, f_i\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{i\in I_1} |\langle f, +S_1^a f_i\rangle|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{B}\|f\| + \sqrt{B}\|S_1^a f\| \leq \sqrt{B}(1 + \|S_1^a\|)\|f\|.\end{aligned}$$

Similarly, we have

$$\left(\sum_{i\in I_2} |\langle f, f_i + S_2^b f_i\rangle|^2\right)^{\frac{1}{2}} \leq \sqrt{B}(1 + \|S_2^b\|)\|f\|.$$

Thus,

$$\{f_i + S_1^a f_i\}_{i\in I_1} \cup \{f_i + S_2^b f_i\}_{i\in I_2},$$

is a Bessel sequence.

On the other hand, the frame operator for $\{f_i + S_1^a f_i\}_{i\in I_1}$ is

$$(I + S_1^a)S_1(I + S_1^a) = S_1 + 2S_1^{1+a} + S_1^{1+2a} \geq S_1.$$

Similarly for $\{f_i + S^b f_i\}_{i\in I_2}$. Hence, the frame operator S_0 for our family satisfies:

$$S_0 \geq S_1 + S_2 = S > 0.$$

Therefore, $\{f_i + S_1^a f_i\}_{i\in I_1} \cup \{f_i + S_2^b f_i\}_{i\in I_2}$ is a frame for $\mathbb{H}$. □

4. Sums of Gabor Frames

For $x, y \in \mathbb{R}$ define the operators E_x and T_y on $L^2(\mathbb{R})$ by:

$$E_x f(t) = e^{2\pi i x t}, \quad T_y f(t) = f(t - y).$$

Let $g \in L^2(\mathbb{R})$ and $0 < ab \leq 1$. Then (g, a, b) denotes the family: $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. If this family forms a frame for $L^2(\mathbb{R})$ we call it a **Gabor frame** with **window function g**.

It is exceptionally difficult to add window functions for Gabor frames to build a new Gabor frame. Our earlier results work in part because the frame operator S for the Gabor frame (g, a, b) commutes with the operators E_{mb}, T_{na}. So, for example, $(g+S^c g, a, b)$ is always another Gabor frame. But even simple cases can become quite complicated in this setting. For example, just letting

$$g = \chi_{[0,1]}, \quad h = \chi_{[1,2]},$$

it is easily checked that $(g, 1, 1)$ and $(h, 1, 1)$ are frames (actually orthonormal bases) for $L^2(\mathbb{R})$ while $(g + h, 1, 1)$ and $(g + ih, 1, 1)$ are not frames [9]. Now let

$$g = \chi_{[0,1/2]} + i\chi_{[1/2,1]}.$$

Then $(g, 1, 1)$ is a Gabor frame while $(Re\ g, 1, 1)$ and $Im\ g, 1, 1)$ do not form frames. Even if g, h are real valued and form Gabor frames it is possible that

$(g+ih,a,b)$ does not form a Gabor frame. For example, if $g \geq 0$ and (g,a,b) is a Gabor frame then for $x \neq 0$ $(T_x g,a,b)$ certainly forms a Gabor frame. However, $(g+T_x g,a,b)$ cannot yield a Gabor frame as the following result shows.

Proposition 4.1. *For any g, and $|c|=1$, any x and $0 \neq y \in \mathbb{R}$, $(g + cE_y T_x g,a,b)$ does not form a frame.*

Proof. Since

$$\{E_{mb}T_{na}(g+cE_xT_yg)\} = \{(I+cT_xE_{x+y})(E_{mb}T_{na}g)\}.$$

So it suffices to observe that $(I+cT_xE_{x+y})$ is not an invertible operator on $L^2(\mathbb{R})$. To see this let,

$$f = \sum_{k=1}^{n}(-1)^k \chi_{[kx,(k+1)x)} c^k E^k_{(x+y)}.$$

Then $\|f\|^2 = nx$, while $\|(I+aT_xE_{(x+y)}f\|^2 = 2x$. □

Now we will see a case where we can produce a frame by summing Gabor frames. To simplify the proof, we first recall a few standard calculations in this area. The first comes from Walut's PhD thesis (See [11]).

Proposition 4.2. *If (g,a,b) is a Gabor frame then for all $f \in L^2(\mathbb{R})$ we have:*

$$\sum_{m,n}|\langle f, E_{mb}T_{na}g\rangle|^2 = b^{-1}\int_{\mathbb{R}}|f(t)|^2\sum_n |g(t-na)|^2 dt +$$

$$b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t-k/b)\cdot\sum_n g(t-na)\overline{g(t-na-k/b)}dt.$$

The next calculation is due to Casazza and Christensen [5]. This is not exactly what they proved in their theorem. However, their proof works line for line in this case.

Proposition 4.3. *If (g,a,b) is a Gabor frame for $L^2(\mathbb{R})$ and for $k \in \mathbb{Z}$ we let*

$$G_k(t) = \sum_{n\in\mathbb{Z}}[T_{na}g_2(t)\overline{T_{na+k/b}g_1(t)} - T_{na}g_1(t)\overline{T_{na+k/b}g_2(t)}],$$

then

$$\left|\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t-k/b)G_k(t)\right| dt \leq \int_{\mathbb{R}}|f(t)|^2\sum_{k\neq 0}|G_k(t)|dt.$$

Now we are ready to prove the main result concerning summing Gabor frames.

Theorem 4.4. *Let (g_j, a, b), $j = 1, 2$ be Gabor frames with frame bounds $A_j \leq B_j$ respectively and the functions g_1, g_2 are real valued. Assume*

$$\frac{1}{b}\sum_{k\neq 0}|\sum_n g_2(t-na)g_1(t-na-k/b)-g_1(t-na)g_2(t-na-k/b)| \leq (1-\epsilon)(A_1+A_2),$$

for some $0 < \epsilon < 1$. Then $(g_1 + ig_2, a, b)$ is a Gabor frame.

Proof. Applying Propositions 4.2 and 4.3 at the appropriate point we can calculate:

$$\sum_{m,n}|\langle f, E_{mb}T_{na}(g_1 + ig_2)|^2 = b^{-1}\int_{\mathbb{R}}|f(t)|^2\sum_n|(g_1 + ig_2)(t - na)|^2 dt+$$

$$b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t - k/b)\sum_n(g_1 + ig_2)(t - na)\overline{(g_1 + ig_2)}(t - na - k/b)dt =$$

$$b^{-1}\int_{\mathbb{R}}|f(t)|^2\sum_n|g_1(t - na)|^2 dt + b^{-1}\int_{\mathbb{R}}|f(t)|^2\sum_n|g_2(t - na)|^2 dt+$$

$$b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(f - k/b)g_1(t - na)g_1(t - na - k/b)dt+$$

$$b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t - k/b)g_2(t - na)g_2(t - na - k/b)dt+$$

$$b^{-1}i\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t - k/b)G_k(t)dt =$$

$$\sum_{m,n}|\langle f, E_{mb}T_{na}g_1\rangle|^2 + \sum_{m,n}|\langle f, E_{mb}T_{na}g_2\rangle|^2+$$

$$b^{-1}i\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t - k/b)G_k(t)dt \geq$$

$$A_1\|f\|^2 + A_2\|f\|^2 - b^{-1}\sum_{k\neq 0}\int_{\mathbb{R}}\overline{f(t)}f(t - k/b)|G_k(t)|dt \geq$$

$$(A_1 + A_2)\|f\|^2 - b^{-1}(1 - \epsilon)b(A_1 + A_2)\|f\|^2 \geq \epsilon(A_1 + A_2)\|f\|^2.$$

□

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